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NOTE

OPEN-INTERVAL GRAPHS VERSUS CLOSED-INTERVAL GRAPHS

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A graph $G = (V, E)$ is said to be represented by a family F of nonempty sets if there is a bijection $f: V \to F$ such that $uv \in E$ if and only if $f(u) \cap f(v) \neq \emptyset$. It is proved that if G is a countable graph then G can be represented by open intervals on the real line if and only if G can be represented by closed intervals on the real line, however, this is no longer true when G is an uncountable graph. Similar results are also proved when intervals are required to have unit length.

1. Introduction

All graphs in this paper are simple but possibly infinite. A countable graph is one in which the vertex set is finite or countably infinite, whereas an uncountable graph is one with uncountably many vertices.

A graph $G = (V, E)$ is called an *interval graph* if there is a bijection f from V to a set F of intervals on the real line such that $uv \in E$ if and only if $u \neq v$ and $f(u) \cap f(v) \neq \emptyset$. The graph G is then said to be represented by the intervals in F. If these intervals are required to have a property P then the graph is called a *P-interval graph*. For example, an open-interval graph, a unit-interval graph, a closed-unit-interval graph, etc.

As far as finite graphs are concerned, there is no difference between the open-interval graphs and the closed-interval graphs; between the open-unit interval graphs and the closed-unit-interval graphs. Well, how about infinite graphs?

We will prove three theorems.

Theorem 1. *Let G be a countable graph. Then G is a closed-interval graph if and only if G is an open-interval graph.*

Let $[\boldsymbol{R}]$ and $\langle \boldsymbol{R} \rangle$ denote the graphs on the same vertex set $\boldsymbol{R}$ (the set of all real

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numbers) having the edge sets

$$(xy: 0<|x-y|\leq 1\} \quad \text{and} \quad \{xy: 0<|x-y|<1\},$$

respectively. Note that $[\boldsymbol{R}]$ is a closed-unit-interval graph, and $\langle \boldsymbol{R} \rangle$ is an open-unit-interval graph.

Theorem 2. *$[\boldsymbol{R}]$ is not an open-interval graph, and $\langle \boldsymbol{R} \rangle$ is not a closed-interval graph.*

For a nonempty subset X of $\boldsymbol{R}$, $[X]$ denotes the subgraph of $[\boldsymbol{R}]$ induced by X. Similarly $\langle X \rangle$ denotes the induced subgraph of $\langle \boldsymbol{R} \rangle$.

A graph G is said to be *embeddable* in another graph H if G is isomorphic to an induced subgraph of H. Notice that any closed-unit-interval graph is embeddable in $[\boldsymbol{R}]$ and any open-unit-interval graph is embeddable in $\langle \boldsymbol{R} \rangle$. As usual, $\boldsymbol{Q}$ denotes the set of all rational numbers. Then the graph $[\boldsymbol{Q}]$ and $\langle \boldsymbol{Q} \rangle$ are not isomorphic, because $[\boldsymbol{Q}]$ has a pair of vertices having a unique common neighbor (e.g. 1 and 3 have the unique common neighbor 2), while $\langle \boldsymbol{Q} \rangle$ has no such pair. Nevertheless, $[\boldsymbol{Q}]$ and $\langle \boldsymbol{Q} \rangle$ are embeddable into each other.

Theorem 3. *Let X be a countable subset of $\boldsymbol{R}$. Then $[X]$ is embeddable in $\langle \boldsymbol{Q} \rangle$, and $\langle X \rangle$ is embeddable in $[\boldsymbol{Q}]$.*

2. Proof of Theorem 1

Let V be the vertex set of G and suppose that G is represented by closed-intervals $\{I_u : u \in V\}$. Let X be the set of all end-points of the intervals. Then X is a subset of the reals $\boldsymbol{R}$. Since V is countable, so is X, and the elements of X can be enumerated as $x_1, x_2, x_3, \ldots$.

Define functions $f_n : \boldsymbol{R} \to \boldsymbol{R}$ $(n = 1, 2, 3, \ldots)$ inductively in the following way.

$$f_1(x) = \begin{cases} x & \text{for } x \leq x_1, \\ x + \frac{1}{2} & \text{for } x > x_1, \end{cases} \qquad f_n(x) = \begin{cases} f_{n-1}(x) & \text{for } x \leq x_n, \\ f_{n-1}(x) + 1/2^n & \text{for } x > x_n \ (n \geq 2). \end{cases}$$

Then each f_n is monotone increasing and

$$0 \leq f_n(x) - f_{n-1}(x) \leq \frac{1}{2^n}.$$

Hence we can define $f : \boldsymbol{R} \to \boldsymbol{R}$ by $f(x) = \lim_{n\to\infty} f_n(x)$.

Now for each x_1 of X let

$$y_i = f(x_i), \qquad z_i = \inf\{f(x_i + \varepsilon) : \varepsilon > 0\}.$$

Then it is clear that $x_i < x_j$ implies $z_i < y_j < z_j$. We define open intervals J_u, $u \in V$

as follows:

If $I_u = [x_i, x_j]$, then let $J_u = (y_i, z_j)$. Then it follows easily that

$$I_u \cap I_v \neq \emptyset \text{ if and only if } J_u \cap J_v \neq \emptyset,$$

hence $\{J_u: u \in V\}$ represents G.

If $\{I_u: u \in V\}$ is a family of open intervals representing G then for $I_u = (x_i, x_j)$, let $J_u = [z_i, y_j]$. Then the family $(J_u: u \in V\}$ also represents G. □

3. Proof of Theorem 2

Suppose $[\boldsymbol{R}]$ is represented by open intervals $\{I_x: x \in \boldsymbol{R}\}$. Let $O_x = I_{x-1} \cap I_x$. Then since x is adjacent to $x-1$ in $[\boldsymbol{R}]$, O_x is a nonempty open interval. If $x < y$ then y is not adjacent to $x-1$ in $[\boldsymbol{R}]$, and hence we have

$$\emptyset = I_{x-1} \cap I_x \cap I_{y-1} \cap I_y = O_x \cap O_y.$$

Thus $\{O_x: X \in \boldsymbol{R}\}$ is an uncountable set of disjoint open intervals. This contradicts the fact that "any set of disjoint open intervals of $\boldsymbol{R}$ contains at most a countable number of elements."

Now suppose $\langle \boldsymbol{R} \rangle$ is represented by closed-intervals $\{J_x: x \in \boldsymbol{R}\}$. Since x and $x-1$ are not adjacent in $\langle \boldsymbol{R} \rangle, J_{x-1} \cap J_x = \emptyset$. Let O_x be the open interval between J_{x-1} and J_x. If $x-1 < y < x$, then y is adjacent to both $x-1$ and x, and hence $J_y \cap J_{x-1} \neq \emptyset, J_y \cap J_x \neq \emptyset$. This implies $O_x \subset J_y$, and hence $O_x \cap J_{y+n} = \emptyset$ for $n = \pm 1, \pm 2, \ldots$. Thus O_x contains no end-points of the intervals J_z, $z \in \boldsymbol{R}$. This implies $O_x \cap O_y = \emptyset$ for $x \neq y$. Hence $\{O_x: x \in \boldsymbol{R}\}$ is a set of disjoint open intervals, a contradiction. □

Remark. Since the Euclidean n-space $\boldsymbol{R}^n$ is separable, i.e., there is a countable subset everywhere dense in $\boldsymbol{R}^n$, it can be similarly proved that $[\boldsymbol{R}]$ cannot be represented by any family of open sets in $\boldsymbol{R}^n$. However, $\langle \boldsymbol{R} \rangle$ can be represented by closed subsets in $\boldsymbol{R}^2$: For each t of $\boldsymbol{R}$, let

$$C_t = \left\{(x, y) \in \boldsymbol{R}^2: \frac{1}{x} \leq y - t \leq 1 - \frac{1}{x}, x \geq 2\right\}.$$

Then $\{C_t: t \in \boldsymbol{R}\}$ represents $\langle \boldsymbol{R} \rangle$.

4. Proof of Theorem 3

Define $X' = \{x - \lfloor x \rfloor : x \in X\} \cup \{0, 1\}$. Then X' is a countable set. Arrange the elements of X' in a sequence $x_1 = 0, x_2 = 1, x_3, x_4, \ldots$. We assign inductively to these elements closed intervals $I(x_1), I(x_2), \ldots$, on the real line. Let $I(x_1) =$

$[-\frac{1}{3}, \frac{1}{3}]$, $I(x_2) = [\frac{2}{3}, \frac{4}{3}]$. Suppose that the intervals $I(x_i)$ are defined for all $i \leqslant n$ ($n \geqslant 2$) and satisfy that

$$I(x_i)\text{'s are disjoint} \quad \text{and} \quad x_i < x_j \text{ implies } I(x_i) < I(x_j), \tag{1}$$

where $I(x_i) < I(x_j)$ means that the interval $I(x_i)$ lies entirely to the left of $I(x_j)$. Let $x_a = \max\{x_i: x_i < x_{n+1}, i \leqslant n\}$, and $x_b = \min\{x_j: x_j > x_{n+1}, j \leqslant n\}$. Define $I(x_{n+1})$ to be the (closed) middle third of the open interval between $I(x_a)$ and $I(x_b)$. Then (1) is still satisfied. Hence we can define $I(x_{n+2})$ similarly, and so on.

Denote $x - \lfloor x \rfloor$ by x' and the midpoint of $I(x')$ by $m(x')$. Then $x' < 1$ and by the definition of $I(x')$, the length of $I(x')$ and $m(x')$ are rationals. We are going to define a map f from X to Q by

$$f(x) = \lfloor x \rfloor + m(x') - \text{'adjusting term' } g(x)$$

so that f induces an isomorphism from $[X]$ to $\langle f(X) \rangle$. Let

$$g(x) = \text{sign}(x)[\text{length of } I(x')]\left(\frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^{k+1}}\right),$$

where k is the absolute value of $\lfloor x \rfloor$, and sign$(x) = 1$ or 0 or -1 accordingly as $x > 0$ or $=0$ or <0. Since $1/4 + 1/4^2 + \cdots = \frac{1}{3}$, it is clear that $m(x') - g(x) \in I(x')$. Hence we have

$$m(x') < m(y') \text{ implies } 0 < (m(y') - g(y)) - (m(x') - g(x)) < 1. \tag{2}$$

Since $g(x)$ is a rational number, $f(x) = \lfloor x \rfloor + m(x') - g(x)$ is also a rational number. Now we show that for $x, y \in X$.

$$|x - y| \leqslant 1 \text{ if and only if } |f(x) - f(y)| < 1. \tag{3}$$

First suppose $0 < y - x < 1$. Then

$$(\lfloor y \rfloor - \lfloor x \rfloor = 1 \text{ and } y' < x') \text{ or } (\lfloor y \rfloor - \lfloor x \rfloor = 0 \text{ and } y' > x').$$

In either case it follows easily from (2) that $0 < f(y) - f(x) < 1$. Next, suppose $y - x = 1$. Then $\lfloor y \rfloor - \lfloor x \rfloor = 1$ and $y' = x'$. Since

$$x < 0 \rightarrow [\text{sign}(x) < 0 \text{ and } |\lfloor x \rfloor| > |\lfloor y \rfloor|] \rightarrow g(y) > g(x)$$

and

$$x > 0 \rightarrow [\text{sign}(y) > 0 \text{ and } |\lfloor y \rfloor| > |\lfloor x \rfloor|] \rightarrow g(y) > g(x),$$

we have $f(y) - f(x) = 1 - (g(y) - g(x)) < 1$.

Finally, suppose $y - x > 1$. Then

$$(\lfloor y \rfloor - \lfloor x \rfloor = 1 \text{ and } y' > x') \quad \text{or} \quad (\lfloor y \rfloor - \lfloor x \rfloor \geqslant 2).$$

In either case $f(y) - f(x) > 1$ follows easily from (2). Thus (3) holds and therefore f induces an isomorphism from $[X]$ to $\langle f(X) \rangle \subset \langle Q \rangle$. This proves the first part of the theorem. To prove the second part we need only to replace the definition of f by $f(x) = \lfloor x \rfloor + m(x') + g(x)$. □