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A Helly Type Theorem for Hypersurfaces

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It is proved that if $H_1,..., H_m$ are hypersurfaces of degree at most d in n-dimensional projective space and $P_1,..., P_m$ are points so that $P_i \notin H_i$ for all i and $P_i \in H_j$ for $1 \leqslant i < j \leqslant m$ then $m \leqslant \binom{n+d}{d}$.

1. INTRODUCTION

Let Γ be a commutative field (finite or infinite) and let $P = P(n, \Gamma)$ be the n-dimensional projective space over Γ. Then every point $\mathbf{x} \in P$ can be expressed by $n+1$ homogene coordinates $\mathbf{x} = (x_0,..., x_n)$, not all zero and $(x_0,..., x_n) = (\lambda x_0,..., \lambda x_n)$ for $0 \neq \lambda \in \Gamma$.

By a hypersurface of degree d we simply mean the set of all points $\mathbf{x} \in P$ with $p(\mathbf{x}) = 0$, where $p(\mathbf{x})$ is a homogenous polynomial of degree d in the $n+1$ variables $x_0, x_1,..., x_n$.

THEOREM 1. *Suppose that $H_1,..., H_m$ are hypersurfaces of degree at most d and $\mathbf{x}_1,..., \mathbf{x}_m$ are points in P so that*

(i) $\mathbf{x}_i \notin H_i$ $i = 1,..., m$ *and*

(ii) $\mathbf{x}_i \in H_j$ *for* $1 \leqslant i < j \leqslant m$ *hold.*

Then we have

$$m \leqslant \binom{n+d}{d}. \qquad (1)$$

We shall see later that (1) is best possible.

The next result is obviously analogous to Helly's theorem (cf. [DGK]) on convex sets.

COROLLARY 2. *Suppose that $H_1,..., H_m$ are hypersurfaces of degree at most d in P so that the intersection of any choice of $\binom{n+d}{d}$ of them is non-empty. Then $\bigcap_{i=1}^{m} H_i \neq \varnothing$.*

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One can use Theorem 1 to give a new proof of the following variation of a result of Lovász [L].

THEOREM 3. (Frankl [F], Kalai [K]). *Suppose that* $A_1, \ldots, A_m$ *are* $(a-1)$*-dimensional subspaces in* P *and* $B_1, \ldots, B_m$ *are at most b-element subsets of* P *so that* $A_i \cap B_i = \emptyset$ *for all i, but* $A_i \cap B_j \neq 0$ *for* $i < j$. *Then*

$$m \leqslant \binom{a+b}{b}. \tag{2}$$

Let us mention that the original proofs of (2) use more involved, multilinear techniques but they allow to obtain (2) for the case when each B_i is a $(b-1)$-dimensional subspace also.

2. PROOF OF THEOREM 1

Let $l_i(\mathbf{x}) = \sum_{j=0}^{n} c_j^{(i)} x_j = 0$ be the equation for some hyperplane which does not contain $\mathbf{x}_i$. Let $q_i(\mathbf{x})$ be the defining polynomial of H_i, and let d_i be the degree of $q_i(\mathbf{x})$. Set $p_i(\mathbf{x}) = q_i(\mathbf{x})\, l_i(\mathbf{x})^{d-d_i}$. Then $p_1(\mathbf{x}), \ldots, p_m(\mathbf{x})$ are homogenous polynomials in $n+1$ variables and of degree d. The vector space of such polynomials over Γ has dimension $\binom{n+d}{d}$. Thus (1) will follow if we show that $p_1(\mathbf{x}), \ldots, p_m(\mathbf{x})$ are linearly independent.

Suppose the contrary, i.e., suppose that $\sum_{j=1}^{m} \alpha_j p_j(\mathbf{x})$ is the zero-polynomial, and not all α_j are zero. Let i be the first index with $\alpha_i \neq 0$. Then

$$\sum_{j<i} \alpha_j p_j(\mathbf{x}_i) + \alpha_i p_i(\mathbf{x}_i) + \sum_{i<j} \alpha_j p_j(\mathbf{x}_i)$$
$$= \alpha_i p_i(\mathbf{x}_i) = \alpha_i q_i(\mathbf{x}_i)\, l_i(\mathbf{x}_i)^{d-d_i} \neq 0$$

a contradiction. ∎

Next, we prove Corollary 2.

Suppose that $H_1 \cap \cdots \cap H_m = \emptyset$ and let us choose a subset $I \subseteq \{1, 2, \ldots, m\}$ of minimal size with the property $\bigcap_{i \in I} H_i = \emptyset$. By the minimal choice of I for every $i \in I$ we can find $\mathbf{x}_i \in \bigcap_{j \in (I - \{i\})} H_j$. Then $\mathbf{x}_i \notin H_i$ but $\mathbf{x}_i \in H_j$ for all $i \neq j \in I$. Thus Theorem 1 applies and gives $|I| \leqslant \binom{n+d}{d}$, which contradicts our assumption that the intersection of any $\binom{n+d}{d}$ hypersurfaces is non-empty. ∎

To see that the bound $\binom{n+d}{d}$ is best possible take $n+d$ hyperplanes in general position in P. Then the intersection of any n of them is a single point. Let $P_1, \ldots, P_{\binom{n+d}{d}}$ be these points. Let H_i be the union of the d hyperplanes not containing P_i. Then H_i is a hypersurface of degree d (its equation is the product of d linear equations) and $P_j \in H_i$ holds for all $j \neq i$.

3. Proof of Theorem 3

From the assumptions it is clear that $n \geqslant a$. Let us first prove (2) for $n=a$ and deduce the $n>a$ case from this special case later.

If $n=a$, then A_i is a hyperplane, $1 \leqslant i \leqslant m$. Let $\mathbf{x}_i=(x_0,\ldots,x_a)$ be the point so that

$$A_i=\left\{\mathbf{y}_i=(y_0,\ldots,y_a)\colon \sum_{j=0}^{a} y_j x_j=0\right\}.$$

Also, for each $B_i=\{\mathbf{b}_i^{(1)},\ldots,\mathbf{b}_i^{(|B_i|)}\}$ define H_i by the polynomial equation $p_i(x_0,\ldots,x_a)=\prod_{t=1}^{|B_i|}\sum_{j=0}^{a} z_j^{(t)}x_j=0$, where $\mathbf{b}_i^{(t)}=(z_0^{(t)},\ldots,z_n^{(t)})$.

Note that $p_i(x)$ is homogenous of degree $|B_i|$ and H_i is simply the union of $|B_i|$ hyperplanes.

Now x_i and H_i, $i=1,\ldots,m$, satisfy the assumptions of Theorem 1, yielding $m \leqslant \binom{a+b}{b}$, as desired.

Suppose now that $n>a$ and the statement has been proved for $n-1$. First we claim that one may assume that Γ is infinite. Indeed, if Γ is finite, then let Γ^* be an infinite extension field of Γ and consider $P^*=P\otimes_\Gamma \Gamma^*$, the n-dimensional projective space over Γ^* in which P is embedded. Let A_i^* be the unique $(a-1)$-dimensional subspace of P^* containing A_i. Then $A_i^* \cap B_i=\emptyset$, all i and $A_i^* \cap B_j \neq \emptyset$ for $i<j$.

From now on Γ is infinite. For $1 \leqslant i \leqslant m$ and $\mathbf{y} \in B_i$ the subspace generated by A_i and $\mathbf{y}$ is a-dimensional. Since $n>a$ and Γ is infinite, the union of these, at most mb subspaces does not cover P: let $\mathbf{z}$ be a point not covered by the union. Let H be an arbitrary hyperplane in P, not containing $\mathbf{z}$.

Let $\tilde{A}_i$ be the projection of A_i from $\mathbf{z}$ on H and $\tilde{B}_i$ the projection of B_i on H. By the choice of $\mathbf{z}$ we have $\tilde{B}_i \cap \tilde{A}_i=\emptyset$ for $1 \leqslant i \leqslant m$. On the other hand $\tilde{A}_i \cap \tilde{B}_j \neq \emptyset$ for $1 \leqslant i<j \leqslant m$ is clearly maintained by the projection. Since $\dim H=n-1<n$, the inequality $m \leqslant \binom{a+b}{b}$ follows by induction. ∎

4. Concluding Remarks

The proof of Theorem 1 was inspired by the paper of Koornwinder [Ko]. The special case of Theorem 1 when each H_i is the union of d hyperplanes was used in [DF] to obtain bounds on the maximal number of vectors in $\mathbb{R}^n$ with given scalar products.

It would be interesting to find a proof without multilinear algebra for the following theorem of Füredi [Fü].

Suppose that t is a positive integer, $A_1,\ldots,A_m$ is a collection of $(a+t)$-element sets, $B_1,\ldots,B_m$ is a collection of $(b+t)$-element sets such that $|A_i \cap B_i| \leqslant t$, $1 \leqslant i \leqslant m$, and $|A_i \cap B_j| > t$ for $1 \leqslant i<j \leqslant m$. Then $m \leqslant \binom{a+b}{b}$.

The case $t=0$ corresponds to Theorem 3. It is easy to see that the bound $\binom{a+b}{a}$ is best possible: take $\mathscr{D}=\{D_1,..., D_m\}$ as all a-subsets of an $(a+b)$-element set X. Let T be a t-element set disjoint to X and set $A_i=D_i \cup T$. Finally define $B_i=(X-D_i) \cup T$.

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