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Note

On Subsets of Abelian Groups with No 3-Term Arithmetic Progression

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A short proof of the following result of Brown and Buhler is given: For any $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that if A is an abelian group of odd order $|A| > n_0$ and $B \subseteq A$ with $|B| > \varepsilon |A|$, then B must contain three distinct elements x, y, z satisfying $x + y = 2z$.

1. INTRODUCTION

Let N denote the set of positive integers, and for $n \in N$, let $[n]$ denote the set $\{1, 2, \ldots, n\}$. A well-known theorem of Roth [R] asserts that if $P \subseteq N$ contains no 3-term arithmetic progression, then P has upper density zero. That is, for every $\varepsilon > 0$, $|P \cap [n]| < \varepsilon n$ holds for all sufficiently large n.

Brown and Buhler [BB1] proved the following generalization of Roth's result.

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THEOREM 1. *For every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ with the following property. Suppose A is an abelian group of odd order, $|A| > n_0$. Then every subset $B \subset A$ with $|B| > \varepsilon |A|$ contains three distinct elements x, y, z with $x + y = 2z$.*

For a finite set X, define $\binom{X}{k} := \{F \subseteq X\colon |F| = k\}$. A family $\mathbf{F} \subseteq \binom{X}{k}$ is called a k-graph. It is called *linear* if $|F \cap G| \leqslant 1$ holds for all distinct $F, G \in \mathbf{F}$. Three distinct edges, F, G, H of a linear k-graph are said to form a *triangle* if the three intersections $F \cap G$, $G \cap H$, $H \cap F$ are all non-empty and distinct.

THEOREM 2 (Ruzsa–Szemerédi [RS]). *Suppose that $\mathbf{F}$ is a linear 3-graph on n vertices which contains no triangle. Then $|\mathbf{F}| = o(n^2)$.*

For a simple proof of Theorem 2, see [EFR]. Here we show that Theorem 1 follows easily from Theorem 2.

2. PROOF OF THEOREM 1

Suppose A is an abelian group of odd order and $B \subseteq A$ contains no three distinct elements x, y, z with $x + y = 2z$. Define $X = A \times [3]$ to be the $3|A|$-element set with general element (a, i), $a \in A$, $1 \leqslant i \leqslant 3$. Now define a 3-graph $\mathbf{F}$ as

$$\mathbf{F} := \{\{(a, 1), (a + b, 2), (a + 2b, 3)\} : a \in A, b \in B\}.$$

Clearly, $|\mathbf{F}| = |A|\,|B|$. Also, $\mathbf{F}$ is linear since any two elements of an edge uniquely determine the edge.

Suppose now to the contrary that $\mathbf{F}$ contains a triangle, say

$$\{(a_i, 1), (a_i + b_i, 2), (a_i + 2b_i, 3)\}, \qquad i = 1, 2, 3.$$

By symmetry, we may assume that

$$a_1 = a_2, \qquad a_1 + b_1 = a_3 + b_3, \qquad a_2 + 2b_2 = a_3 + 2b_3.$$

However, these equations imply

$$2b_2 - 2b_3 = a_3 - a_2 = a_3 - a_1 = b_1 - b_3,$$

i.e.,

$$2b_2 = b_1 + b_3.$$

By the choice of B, this implies $b_1 = b_2 = b_3$ and thus, $a_1 = a_2 = a_3$, a contradiction. Thus, $\mathbf{F}$ contains no triangle.

Hence, by Theorem 2,

$$|\mathbf{F}| = |A|\,|B| = o(|A|^2),$$

i.e.,

$$|B| = o(|A|) \qquad \text{as desired.}$$

Remark. The same proof can be used in the case when A is a d-dimensional affine space over $\mathrm{GF}(2^t)$, $t \geqslant 2$. For the definition of edges in the proof, one replaces $a + 2b$ by $a + \gamma b$ where $\gamma \neq 0, 1$ is an arbitrary element of $\mathrm{GF}(2^t)$. The conclusion then becomes: B contains three points on a line.

3. Some Lower Bounds

The most important special cases of Theorem 1 are when A is a cyclic group (corresponding to Roth's theorem) and when A is an affine space $A(d, q)$ of dimension d over $GF(q)$.

In both cases, stronger theorems are known. Szemerédi's theorem [S] asserts that sets with positive upper density contain arithmetic progressions of arbitrary length, while a recent result of Furstenberg and Katznelson [FK] implies that for any $\varepsilon > 0$ and any prime power q there exists $d_0 = d_0(\varepsilon, q)$ so that the following is true: Every subset $B \subseteq A(d, q)$ with $|B| > \varepsilon q^d$, $d > d_0$, contains all the points of some line in $A(d, q)$.

In view of [BB2] this implies the same statement if we replace lines by planes, spaces, etc.

Let $a_q(d)$ denote the maximum of $|B|$ where $B \subseteq A(d, q)$ contains no line. In the case of the integers, Behrend [B] showed that for every $\delta > 0$ and $n > n_0(\delta)$ there exists $B \subseteq [n]$ with $|B| > n^{1-\delta}$ so that B contains no 3-term arithmetic progression. We do not know if the corresponding statement holds for affine spaces.

Problem. Is it true that for every $\delta > 0$, $q \geqslant 3$ and $d > d_0(\delta, q)$, there exists $B \subseteq A(d, q)$ which contains no line and satisfies $|B| > (q - \delta)^d$.

It is easy to construct such a B with $|B| = (q-1)^d$; simply take

$$B = \{(b_1, ..., b_d) : b_i \in GF(q) \setminus \{0\},\ i = 1, ..., d\}.$$

To improve on this bound note that if $x \in GF(q)$ is fixed and $\{(b_1^{(i)}, ..., b_d^{(i)}),\ i = 1, ..., q\}$ forms a line then the sets $F_i = \{j : b_j^{(i)} = x\}$ form a *sunflower of size* q, that is, $F_1 \cap \cdots \cap F_q = F_j \cap F_{j'}$ holds for all $1 \leqslant j < j' \leqslant q$.

Let $f_q(d, r)$ denote the maximum of $|\mathbf{F}|$ where $\mathbf{F} \subseteq \binom{[d]}{r}$, and $\mathbf{F}$ contains no sunflower of size q.

PROPOSITION. *For all positive integrs n, r, d, one has*

$$a_q(n) \geqslant f_q(n, r)(q-1)^{n-r} \qquad \text{and} \qquad a_q(nd) \geqslant a_q(n)^d. \tag{2}$$

Proof. Let $\mathbf{F} \subseteq \binom{[n]}{r}$ be a family without sunflowers of size q which satisfies $|\mathbf{F}| = f_q(n, r)$.

Fix an element $x \in GF(q)$. For $\mathbf{b} = (b_1, ..., b_n) \in A(n, q)$, define $F(\mathbf{b}) := \{j : b_j = x\}$ and

$$B := \{\mathbf{b} \in A(n, q) : F(\mathbf{b}) \in \mathbf{F}\}.$$

Then B contains no line. To prove the second assertion, one simply notes that if B contains no line then

$$B \oplus \cdots \oplus B \subseteq A(n, q) \oplus \cdots \oplus A(n, q) = A(dn, q)$$

contains no line either.

If we knew the value of $f_q(n, r)$, then probably we could get fairly good lower bounds on $a_q(n)$.

Although this problem goes back to Erdös and Rado [ER], very little is known about $f_q(n, r)$.

For q odd, $n = 2q$ and $r = 2$, one can take two disjoint complete graphs on q vertices each. This shows $f_q(2q, 2) \geqslant q(q-1)$. Actually one has equality, but we do not need it. Using (1) we obtain

$$a_q(2dq) \geqslant (q-1)^{2dq} \left(\frac{q}{q-1}\right)^d.$$

Using the fact that there is a collection of 300 6-element subsets of [18] without a sunflower of size three, one obtains $a_3(18) \geqslant 300 \cdot 12^{12}$ and thus $a_3(d) \geqslant (2.179)^d$ for $d > d_0$.

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