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Erdös–Ko–Rado Theorem with Conditions on the Maximal Degree

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A family $\mathscr{F}$ of distinct k-element subsets of the n-element set X is called *intersecting* if $F \cap F' \neq \emptyset$ holds for all $F, F' \in \mathscr{F}$. Erdös, Ko, and Rado proved that for $n \geqslant 2k$ necessarily $|\mathscr{F}| \leqslant \binom{n-1}{k-1}$ holds, which is clearly best possible (take all k-sets through a fixed element). For a family $\mathscr{F}$, its maximum degree $d(\mathscr{F})$ is the maximum number of sets in $\mathscr{F}$ containing any particular element of X. For $3 \leqslant i \leqslant k+1$ define intersecting families $\mathscr{F}_i$ as follows. Let $x \notin I \in \binom{X}{i-1}$ and set $\mathscr{F}_i = \{F \in \binom{X}{k}: x \in F, F \cap I \neq \emptyset\} \cup \{F \in \binom{X}{k}: x \notin F, I \subset F\}$. The main result of the present paper is: if $\mathscr{F} \subset \binom{X}{k}$ is intersecting, $d(\mathscr{F}) \leqslant d(\mathscr{F}_i)$ and $n \geqslant 2k$, then $|\mathscr{F}| \leqslant |\mathscr{F}_i|$ holds.

1. INTRODUCTION

Let $n \geqslant k \geqslant t$ be positive integers. Let $X = \{1, 2, ..., n\}$ be an n-element set. Define $\binom{X}{k} = \{F \subset X: |F| = k\}$. A family $\mathscr{F} \subset \binom{X}{k}$ is called *t-intersecting* if $|F \cap F'| \geqslant t$ holds for all $F, F' \in \mathscr{F}$, *intersecting* will mean 1-intersecting.

Let us recall the following classical result.

ERDÖS–KO–RADO THEOREM [EKR]. *Suppose that $\mathscr{F} \subset \binom{X}{k}$ is intersecting and $n \geqslant 2k$. Then*

$$|\mathscr{F}| \leqslant \binom{n-1}{k-1} \quad \textit{holds.} \tag{1.1}$$

Hilton and Milner proved that for $n > 2k$ the only way to achieve equality in (1.1) is to take all k-sets through a fixed element. Note that for $n = 2k$, (1.1) is almost trivial and there are very many ways to attain equality.

Set $[i, j] = \{i, i+1, ..., j\}$ for $1 \leqslant i \leqslant j \leqslant n$. Define the families $\mathscr{F}_i$, $i = 3, 4, ..., k+1$

$$\mathscr{F}_i = \left\{F \in \binom{X}{k}: 1 \in F, F \cap [2, i] \neq \emptyset\right\} \cup \left\{F \in \binom{X}{k}: 1 \notin F, [2, i] \subset F\right\}.$$

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It is easy to check that $\mathscr{F}_i$ is intersecting,

$$|\mathscr{F}_i| = \binom{n-2}{k-2} + \cdots + \binom{n-i}{k-2} + \binom{n-i}{k-i+1}, \tag{1.2}$$

in particular,

$$|\mathscr{F}_3| = |\mathscr{F}_4|. \tag{1.3}$$

HILTON–MILNER THEOREM [HM]. *Suppose that* $\mathscr{F} \subset \binom{X}{k}$ *is intersecting,* $\bigcap \mathscr{F} = \varnothing$, $n > 2k$. *Then*

$$|\mathscr{F}| \leqslant |\mathscr{F}_{k+1}| \quad \textit{holds.} \tag{1.4}$$

Moreover, $\mathscr{F}$ *attains equality in* (1.4) *if and only if* $\mathscr{F}$ *is isomorphic to* $\mathscr{F}_{k+1}$ *or* $k=3$ *and* $\mathscr{F}$ *is isomorphic to* $\mathscr{F}_3$.

For a short proof see [FF1].

The aim of this article is to prove a sharpening of the Hilton–Milner Theorem.

Let $d(\mathscr{F})$ denote the maximum degree of $\mathscr{F}$, that is, the maximum number of elements of $\mathscr{F}$ containing any particular element of X. With this notation we have

$$d(\mathscr{F}_i) = \binom{n-2}{k-2} + \cdots + \binom{n-i}{k-2}. \tag{1.5}$$

THEOREM 1.1. *Suppose that* $n > 2k$, $3 \leqslant i \leqslant k+1$, $\mathscr{F} \subset \binom{X}{k}$, F *is intersecting with* $d(\mathscr{F}) \leqslant d(\mathscr{F}_i)$. *Then*

$$|\mathscr{F}| \leqslant |\mathscr{F}_i|. \tag{1.6}$$

Moreover, if $\mathscr{F}$ *attains equality in* (1.6) *then either* $\mathscr{F}$ *is isomorphic to* $\mathscr{F}_i$ *or* $i=4$ *and* $\mathscr{F}$ *is isomorphic to* $\mathscr{F}_3$.

Remark. To see that Theorem 1.1 sharpens the Hilton–Milner theorem just note that if $A \subset \binom{X}{k}$, $x \notin A$ then there are exactly $d(\mathscr{F}_{k+1})$ k-element sets through x and intersecting A. That is, if $\bigcap \mathscr{F} = \varnothing$ for an intersecting family $\mathscr{F} \subset \binom{X}{k}$ then necessarily $d(\mathscr{F}) \leqslant d(\mathscr{F}_{k+1})$ holds.

We shall prove the inequality part, (1.6) of Theorem 1.1 in Sections 2, 3, and 4. The characterization of equality is postponed until Section 6.

Let us mention that the statement of the theorem for $i=3$ was conjectured by Hilton [H] more than 10 years ago. This result was inspired by a recent paper of Anderson and Hilton [AH], who showed that (1.6) holds if $i=3$ and we make the stronger assumption $d(\mathscr{F}_3) \leqslant d(\mathscr{F}) \leqslant d(\mathscr{F}_4)$.

By now there are many asymptotic results (i.e., $n > n_0(k)$) sharpening the Erdös–Ko–Rado theorem via degree conditions, see, e.g., [F1; Fü1, Fü2, FF2]. The study of those problems was initiated by Erdös, Rothschild, and Szemerédi (cf. [E]), wo proved that for $\mathscr{F} \subset \binom{X}{k}$, $n > n_0(k)$, $d(\mathscr{F}) \leqslant 2\,|\mathscr{F}|/3$ and $\mathscr{F}$ intersecting, one has $|\mathscr{F}| \leqslant |\mathscr{F}_3|$.

For a family of sets $\mathscr{H}$ and a positive integer l, let us define the lth shadow of $\mathscr{H}$ by

$$\partial_l(\mathscr{H}) = \{G : |G| = l, \exists\, H \in \mathscr{H}, G \subset H\}.$$

Suppose that we know that $\mathscr{H} \subset \binom{X}{k}$ and $|\mathscr{H}|$ is fixed. What is the minimum of $|\partial_l(\mathscr{H})|$? This important problem was solved independently by Kruskal and Katona.

To state their result let us define the lexicographic order $<$ on $\binom{X}{k}$ by setting $A < B$ if $\min\{i : i \in A - B\} > \min\{i : i \in B - A\}$. This induces a linear order on $\binom{X}{k}$, $1 \leqslant k \leqslant n$.

For $0 \leqslant m \leqslant \binom{n}{k}$ let $\mathscr{S}(m, k)$ ($\mathscr{L}(m, k)$) be the collection of the smallest (largest) m sets in $\binom{X}{k}$, respectively.

KRUSKAL–KATONA THEOREM [Kr, K1]. *Suppose that $\mathscr{F} \subset \binom{X}{k}$, $0 \leqslant l \leqslant k$. Then $|\partial_l(\mathscr{F})| \geqslant |\partial_l(\mathscr{S}(|F|), k)|$ holds.*

For a short proof see [F2]. We will need the following simple numerical corollary (cf. [Kr, K1]).

WEAK KRUSKAL–KATONA THEOREM. *Suppose that $m \geqslant k > l > 0$, $\mathscr{F} \subset \binom{X}{k}$, $0 < |\mathscr{F}| \leqslant \binom{m}{k}$. Then $|\partial_l(\mathscr{F})|/|\mathscr{F}| \geqslant \binom{m}{l}/\binom{m}{k}$ with equality if and only if $|\mathscr{F}| = \binom{Y}{k}$ holds for some m-element set Y.*

Note that the uniqueness of the optimal families for $|\mathscr{F}| = \binom{m}{k}$ is not a consequence of the Kruskal–Katona theorem, but follows from the actual proof. For a complete description of the values of $|\mathscr{F}|$ for which there is a unique optimal family in the Kruskal–Katona theorem see [FG] and [M].

The relevance of shadows to intersection problems was first noted by Katona [K2]. In fact, the Kruskal–Katona theorem is equivalent to the following.

Call the families $\mathscr{A}$ and $\mathscr{B}$ *cross-intersecting* if $A \cap B \neq \emptyset$ holds for all $A \in \mathscr{A}$, $B \in \mathscr{B}$.

THEOREM 1.2 (Hilton [H]). *Suppose that $\mathscr{A} \subset \binom{X}{a}$, $\mathscr{B} \subset \binom{X}{b}$, $\mathscr{A}$ and $\mathscr{B}$ are cross-intersecting. Then the same holds for $\mathscr{L}(|\mathscr{A}|, a)$ and $\mathscr{L}(|\mathscr{B}|, b)$.*

Proof. Let $\mathscr{A}^c = \{X - A : A \in \mathscr{A}\}$. Observe that $\mathscr{A}$ and $\mathscr{B}$ are cross-intersecting if and only if $\partial_b(\mathscr{A}^c) \cap \mathscr{B} = \emptyset$. In particular, $|\partial_b(\mathscr{A}^c)| \leqslant \binom{n}{b} - |\mathscr{B}|$.

By the Kruskal–Katona theorem $\partial_b(\mathscr{S}(|\mathscr{A}|,\ n-a))\cap\mathscr{L}(|\mathscr{B}|,b)=\varnothing$. Now the statement follows from $\mathscr{L}(m,a)^c=\mathscr{S}(m,n-a)$. ∎

We need three more results about cross-intersecting families. The proofs will be given in Section 5.

PROPOSITION 1.3. *Let a, b, m be nonnegative integers, $m>a+b$, $a\geqslant b$. Suppose that $|Y|=m$, $\mathscr{A}\subset\binom{Y}{a}$ and $\mathscr{B}\subset\binom{Y}{b}$ are cross-intersecting. Then $|\mathscr{A}|+|\mathscr{B}|\leqslant\binom{m}{a}$ with equality holding if and only if $\mathscr{A}=\binom{Y}{a}$, $\mathscr{B}=\varnothing$ or $a=b$, $\mathscr{A}=\varnothing$ and $\mathscr{B}=\binom{Y}{b}$.*

Recall that a family $\mathscr{B}$ is *t-intersecting* if $|B\cap B'|\geqslant t$ holds for all $B, B'\in\mathscr{B}$.

PROPOSITION 1.4. *Let a, t, m be positive integers, $m>2a+t$. Suppose that $|Y|=m$, $\mathscr{A}\subset\binom{Y}{a}$, $\mathscr{B}\subset\binom{Y}{a+t}$, $\mathscr{A}$ and $\mathscr{B}$ are cross-intersecting, moreover, $\mathscr{B}$ is t-intersecting. Then $|\mathscr{A}|+|\mathscr{B}|\leqslant\binom{m}{a}$ holds, with equality if and only if either $\mathscr{B}\neq\varnothing$ and $\mathscr{A}=\binom{Y}{a}$ or for some $T\in\binom{Y}{t}$,*

$$\mathscr{B}=\left\{B\in\binom{Y}{a+t}: T\subset B\right\}\qquad\text{and}\qquad \mathscr{A}=\left\{A\in\binom{Y}{a}: A\cap T\neq\varnothing\right\}.$$

PROPOSITION 1.5. *Let r, v be positive integers, $v>2r$. Suppose that $Y=[1,v]$, $\mathscr{G}\subset\binom{Y}{r}$, $\mathscr{H}\subset\binom{Y}{r}$, $\mathscr{G}$, and $\mathscr{H}$ are cross-intersecting and $|\mathscr{H}|\leqslant|\mathscr{G}|\leqslant\binom{v-1}{r-1}+\binom{v-2}{r-1}$ holds. Then $|\mathscr{G}|+|\mathscr{H}|\leqslant 2\binom{v-1}{r-1}$ with equality holding if and only if either $|\mathscr{G}|=|\mathscr{H}|=\binom{v-1}{r-1}$ or $|\mathscr{G}|=\binom{v-1}{r-1}+\binom{v-2}{r-1}$ and $|\mathscr{H}|=\binom{v-2}{r-2}$.*

For a family $\mathscr{F}\subset\binom{X}{h}$ and disjoint sets A, $B\subset X$ define $\mathscr{F}(A\bar{B})=\{F-A: A\subset F\in\mathscr{F},\ B\cap F=\varnothing\}$. The families $\mathscr{F}(A)$, $\mathscr{F}(\bar{B})$ are defined analogously. If $A=\{i\}$, $B=\{j\}$ then we write simply $\mathscr{F}(i\bar{j})$ or $\mathscr{F}(\bar{j}i)$ to denote $\mathscr{F}(\{i\}\overline{\{j\}})$. Note that $\mathscr{F}(A\bar{B})$ is *always* considered as a family on the underlying set $X-(A\cup B)$. This is important in the definition of $\mathscr{L}(|\mathscr{F}(A\bar{B})|,h)$. Namely, $\mathscr{L}(|\mathscr{F}(A\bar{B})|,h)$ consists of the $|\mathscr{F}(A\bar{B})|$ lexicographically largest sets in $\binom{X-(A\cup B)}{h}$.

Another important tool for investigating intersecting families is an operation on families, called *shifting*.

For a family, $\mathscr{F}\subset\binom{X}{k}$ and $1\leqslant i<j\leqslant n$ one defines the (i,j)-shift S_{ij} by

$$S_{ij}(F)=\begin{cases}(F-\{j\})\cup\{i\} & \text{if } i\notin F,\ j\in F \text{ and } ((F-\{j\})\cup\{i\})\notin\mathscr{F}\\ F & \text{otherwise,}\end{cases}$$

$$S_{ij}(\mathscr{F})=\{S_{ij}(F): F\in\mathscr{F}\}.$$

This operation was introduced by Erdös, Ko, and Rado who proved that $|S_{ij}(\mathscr{F})|=|\mathscr{F}|$ and if $\mathscr{F}$ is intersecting then so is $S_{ij}(\mathscr{F})$.

Part of the difficulties in the proof of Theorem 1.1 stems from the fact that the (i, j)-shift changes the degrees of i and j and it might happen that $S_{ij}(\mathscr{F})$ fails to satisfy the degree condition.

However, the following properties of $\mathscr{G} = S_{ij}(\mathscr{F})$ are obvious.

Claim 1.6. $\mathscr{G}(ij) = \mathscr{F}(ij)$, $\mathscr{G}(\bar{i}\bar{j}) = \mathscr{F}(\bar{i}\bar{j})$, $|\mathscr{G}(l)| = |\mathscr{F}(l)|$ for $l \neq i, j$ and $|\mathscr{G}(j)| \leqslant |\mathscr{F}(j)|$.

2. The Proof of Theorem 1.1 for Large Maximal Degree

In this section we prove

Lemma 2.1. *Suppose that* $4 \leqslant i \leqslant k+1$, $n > 2k$, $\mathscr{F} \subset \binom{X}{k}$, $\mathscr{F}$ *is intersecting and one has*

$$d(\mathscr{F}_{i-1}) \leqslant d(\mathscr{F}) \leqslant d(\mathscr{F}_i). \tag{2.1}$$

Then (1.6) *holds, that is,* $|\mathscr{F}| \leqslant |\mathscr{F}_i|$.

Proof. Let 1 be the vertex of maximal degree in $\mathscr{F}$, i.e., $|\mathscr{F}(1)| = d(\mathscr{F})$. Let us consider the cross-intersecting families $\mathscr{F}(1)$ and $\mathscr{F}(\bar{1})$ on $[2, n]$. In view of Theorem 1.2, the families $\mathscr{L}_0 = \mathscr{L}(|\mathscr{F}(1)|, k-1)$ and $\mathscr{L}_1 = \mathscr{L}(|\mathscr{F}(\bar{1})|, k)$ are cross-intersecting. In view of (2.1) $\mathscr{L}_0$ contains all $(k-1)$-subsets of $[2, n]$ which intersect $[2, i-1]$. This forces $[2, i-1] \subset L$ for all $L \in \mathscr{L}_1$.

Consequently, the families $\mathscr{L}_0(\overline{[2, i-1]})$ and $\mathscr{L}_1([2, i-1])$ are cross-intersecting and satisfy

$$|\mathscr{L}_0(\overline{[2, i-1]})| = d(\mathscr{F}) - d(\mathscr{F}_{i-1}) \leqslant \binom{n-i}{k-2}, \qquad |\mathscr{L}_1([2, i-1])| = |\mathscr{L}_1|$$

If $|\mathscr{L}_1| < \binom{n-i}{k-i+1}$, then $|\mathscr{F}| < |\mathscr{F}_i|$ follows. Thus we may assume that $|\mathscr{L}_1([2, i-1])| \geqslant \binom{n-i}{k-i+1}$ and therefore this family contains all $(k-i+2)$-subsets of $[i, n]$ through i. Consequently, the cross-intersecting property implies $i \in L$ for all $L \in \mathscr{L}_0(\overline{[2, i-1]})$.

Thus $\mathscr{A} = \mathscr{L}_0(\overline{[2, i-1]}\, i)$ and $\mathscr{B} = \mathscr{L}_1([2, i-1]\, \bar{i})$ satisfy

$$|\mathscr{A}| = |\mathscr{L}_0(\overline{[2, i-1]})| = d(\mathscr{F}) - d(\mathscr{F}_{i-1}),$$

$$|\mathscr{B}| = |\mathscr{L}_1| - \binom{n-i}{k-i+1}.$$

To prove (1.6) we need to show

$$|\mathscr{A}|+|\mathscr{B}|\leqslant\binom{n-i}{k-2}.$$

This, however, is implied by Proposition 1.3 applied with $m=n-i$, $a=k-2$, $b=k-(i-2)$. ∎

3. The Proof of Theorem 1.1 for Shift-Resisting Families

Suppose that $\mathscr{F}\subset\binom{X}{k}$, $\mathscr{F}$ is intersecting, 1 is a point of maximal degree and 2 an arbitrary element of X, different from 1. In view of Lemma 2.1 we may assume that

$$|\mathscr{F}(1)|\leqslant\binom{n-2}{k-2}+\binom{n-3}{k-2}. \tag{3.1}$$

First we prove a simple inequality

$$|\mathscr{F}(\bar{1}2)|\leqslant\binom{n-3}{k-2}. \tag{3.2}$$

Since 1 has maximal degree, we have

$$|\mathscr{F}(\bar{1}2)|=|\mathscr{F}(2)|-|\mathscr{F}(12)|\leqslant|\mathscr{F}(1)|-|\mathscr{F}(12)|=|\mathscr{F}(1\bar{2})|.$$

Thus $|\mathscr{F}(1\bar{2})|\leqslant\binom{n-3}{k-2}$ would imply (3.2).

Suppose the contrary and apply Theorem 1.2 to the cross-intersecting families $\mathscr{F}(1\bar{2})$, $\mathscr{F}(\bar{1}2)$ defined on the underlying set $[3,n]$.

Now $\mathscr{L}(|\mathscr{F}(1\bar{2})|, k-1)$ contains all $(k-1)$-subsets of $[3,n]$ going through 3. Therefore every member of $\mathscr{L}(|\mathscr{F}(\bar{1}2)|, k-1)$ must contain 3, yielding (3.2).

Now we conclude the proof of Theorem 1.1 in an important special case.

Lemma 3.1. *Assume* (3.1). *Suppose that* $|\mathscr{F}(12)|+|\mathscr{F}(\overline{12})|\leqslant\binom{n-2}{k-2}$ and $|\mathscr{F}(\overline{12})|\leqslant\binom{n-4}{k-2}$. *Then* (1.6) *holds.*

Proof. Let $\mathscr{H}\subset(\binom{X}{k}-\mathscr{F})$ be an arbitrary family consisting of $|\mathscr{F}(\overline{12})|$ subsets containing [12].

Set $\mathscr{G}=\mathscr{H}\cup\mathscr{F}-\mathscr{F}(\overline{12})$. Then $|\mathscr{G}|=|\mathscr{F}|$, $\mathscr{G}$ is intersecting and

$$d(\mathscr{G})\leqslant d(\mathscr{F})+|\mathscr{H}|\leqslant\binom{n-2}{k-2}+\binom{n-3}{k-2}+\binom{n-4}{k-2}.$$

If $d(\mathscr{G}) \geqslant \binom{n-2}{k-2} + \binom{n-3}{k-2}$, then (1.6) follows from Lemma 2.1. Suppose the contrary and observe

$$|\mathscr{G}| = |\mathscr{G}(1)| + |\mathscr{G}(\bar{1}2)| \leqslant d(\mathscr{G}) + |\mathscr{G}(\bar{1}2)| = d(\mathscr{G}) + |\mathscr{F}(\bar{1}2)|.$$

Now (3.2) implies

$$|\mathscr{G}| \leqslant \binom{n-2}{k-2} + \binom{n-3}{k-2} + \binom{n-3}{k-2}. \quad \blacksquare$$

Next we are going to use Lemma 3.1 to show that $\mathscr{F}$ can be always shifted unless (1.6) is true.

Let $2 \leqslant j \leqslant n$ and consider the family $\mathscr{G} = S_{1j}(\mathscr{F})$. In view of Claim 1.6, 1 is a vertex of maximal degree in $\mathscr{G}$. Thus the only problem which could prevent us from replacing $\mathscr{F}$ by $\mathscr{G}$ is

$$d(\mathscr{G}) = |\mathscr{G}(1)| \geqslant \binom{n-2}{k-2} + \binom{n-3}{k-2}.$$

If, however, $|\mathscr{G}(1)| \leqslant \binom{n-2}{k-2} + \binom{n-3}{k-2} + \binom{n-4}{k-2}$, then Lemma 2.1 implies that (1.6) holds for $\mathscr{G}$ and thus for $\mathscr{F}$.

Thus we may assume

$$|\mathscr{G}(1)| > \binom{n-2}{k-2} + \binom{n-3}{k-2} + \binom{n-4}{k-2} \tag{3.3}$$

Using the obvious inequality $|\mathscr{G}(1j)| \leqslant \binom{n-2}{k-2}$, we obtain

$$|\mathscr{G}(1\bar{j})| > \binom{n-3}{k-2} + \binom{n-4}{k-2}. \tag{3.4}$$

The families $\mathscr{G}(1\bar{j})$ and $\mathscr{G}(\overline{1j})$ are cross-intersecting. By Theorem 1.2 so are $\mathscr{L}_0 = \mathscr{L}(|\mathscr{G}(1\bar{j})|, k-1)$ and $\mathscr{L}_1 = \mathscr{L}(|\mathscr{G}(\overline{1j})|, k)$ as well. It follows from (3.4) that $\mathscr{L}_0$ contains all $(k-1)$-sets of $[2, n] - \{j\}$ which go through at least one of the first two elements. This immediately implies

$$|\mathscr{G}(\overline{1j})| = |\mathscr{L}_1| \leqslant \binom{n-4}{k-2}.$$

Using Claim 1.6 we infer

$$|\mathscr{F}(\overline{1j})| \leqslant \binom{n-4}{k-2}. \tag{3.5}$$

CLAIM 3.2. $|\mathscr{F}(1\bar{j})| + |\mathscr{F}(\overline{1j})| \leqslant \binom{n-2}{k-2}$.

Proof. Set $Y = [2, n] - \{j\}$, $\mathscr{A} = \mathscr{F}(1\bar{j}) \subset \binom{Y}{k-2}$, $\mathscr{B} = \mathscr{F}(\overline{1j}) \subset \binom{Y}{k}$. Set also

$$\mathscr{C} = \mathscr{B}^c = \{Y - B \colon B \in \mathscr{B}\} \subset \binom{Y}{n-k-2}.$$

By (3.5) we have

$$|\mathscr{C}| \leqslant \binom{n-4}{k-2} = \binom{n-4}{n-k-2}.$$

By the Weak Kruskal–Katona Theorem we infer

$$|\partial_{k-2}(\mathscr{C})| \geqslant |\mathscr{C}|.$$

Since $\mathscr{A}$, $\mathscr{B}$ are cross-intersecting, $\mathscr{A} \cap \partial_{k-2}(\mathscr{C}) = \varnothing$ and thus

$$|\mathscr{A}| + |\mathscr{B}| = |\mathscr{A}| + |\mathscr{C}| \leqslant |\mathscr{A}| + |\partial_{k-2}(\mathscr{C})| \leqslant \binom{n-2}{k-2}$$

follows. ∎

Now (3.5) and Claim 3.2 ensures that $\mathscr{F}$ verifies the assumptions of Lemma 3.1. Thus (1.6) holds.

4. The Proof of Theorem 1.1 for Shifted Families

In view of Sections 2 and 3, to prove (1.6) we may assume that the intersecting family $\mathscr{F} \subset \binom{X}{k}$ verifies

$$d(\mathscr{F}) = |\mathscr{F}(1)| \leqslant \binom{n-2}{k-2} + \binom{n-3}{k-2} \tag{4.1}$$

and the same holds for

$$\mathscr{G} = S_{1j}(\mathscr{F}), \qquad 2 \leqslant j \leqslant n.$$

Let us replace $\mathscr{F}$ by $\mathscr{F}_2 = S_{12}(\mathscr{F})$ then $\mathscr{F}_2$ by $\mathscr{F}_3 = S_{13}(\mathscr{F})$, etc. By abuse of notation write $\mathscr{F} = \mathscr{F}_n = S_{1n}(\mathscr{F}_{n-1})$.

In view of Section 3, $\mathscr{F}$ verifies (4.1) and (3.2).

Claim 4.1. *$\mathscr{F}(\overline{12})$ is 2-intersecting.*

Proof. Suppose for contradiction that A, $B \in \mathscr{F}(\overline{12})$ and $A \cap B = \{j\}$ for some $3 \leqslant j \leqslant n$. Define $A_0 = (A - \{j\}) \cup \{1\}$. Since $A_0 \cap B = \varnothing$, $A_0 \notin \mathscr{F}$.

Since applying the $(1, i)$-shift never throws away sets containing 1 and adds only sets containing 1, we infer $A \in S_{1j}(\mathscr{F}_{j-1}) = \mathscr{F}_j$, $A_0 \notin \mathscr{F}_j$, which contradicts the definition of the $(1, j)$-shift. ∎

Apply Proposition 1.4 with $a=k-2$, $t=2$, $m=n-2$, $Y=[3,n]$, $\mathscr{A}=\mathscr{F}(12)$, and $\mathscr{B}=\mathscr{F}(\overline{12})$. We infer

$$|\mathscr{F}(12)|+|\mathscr{F}(\overline{12})|\leqslant\binom{n-2}{k-2}. \tag{4.2}$$

In view of Lemma 3.1 we may assume

$$|\mathscr{F}(\overline{12})|\geqslant\binom{n-4}{k-2}. \tag{4.3}$$

Apply now Theorem 1.2 to the cross-intersecting families $\mathscr{F}(1\bar{2})$ and $\mathscr{F}(\overline{12})$. Then (4.3) implies

$$|\mathscr{F}(1\bar{2})|\leqslant\binom{n-3}{k-2}+\binom{n-4}{k-2}. \tag{4.4}$$

Applying Proposition 1.5 with $Y=[3,n]$, $r=k-1$, $\mathscr{H}=\mathscr{F}(\bar{1}2)$, $\mathscr{G}=\mathscr{F}(1\bar{2})$ we infer

$$|\mathscr{F}(1\bar{2})|+|\mathscr{F}(\bar{1}2)|\leqslant 2\binom{n-3}{k-2}. \tag{4.5}$$

Summing (4.2) and (4.5) gives

$$|\mathscr{F}|\leqslant\binom{n-2}{k-2}+2\binom{n-3}{k-2}=|\mathscr{F}_3|. \quad \blacksquare$$

5. The Proof of Propositions 1.3, 1.4, and 1.5

Proof of Proposition 1.3. Set $\mathscr{C}=\{C\in\binom{Y}{a}: \exists\, B\in\mathscr{B},\ B\cap C=\emptyset\}$. Make a bipartite graph $G=G(\mathscr{C},\mathscr{B})$ with vertex set $\mathscr{C}\cup\mathscr{B}$ and $C\in\mathscr{C}$ and $B\in\mathscr{B}$ forming an edge if $B\cap C=\emptyset$.

The degree of every vertex $B\in\mathscr{B}$ is exactly $\binom{m-b}{a}$ while the degree of every vertex $C\in\mathscr{C}$ is at most $\binom{m-a}{b}$. Moreover, should equality hold for every $C\in\mathscr{C}$, then $\mathscr{C}\cup\mathscr{B}$ is a connected component of $G(\binom{Y}{a},\binom{Y}{b})$. Since this latter is connected for $n>a+b$, either $\mathscr{B}=\emptyset$ or $\mathscr{B}=\binom{Y}{b}$ follows in this case.

Otherwise we have the strict inequality $|\mathscr{C}|\binom{m-a}{b}>|\mathscr{B}|\binom{m-b}{a}$, or, equivalently, $|\mathscr{C}|>|\mathscr{B}|\binom{m}{a}/\binom{m}{b}\geqslant|\mathscr{B}|$. Since $\mathscr{A}$ and $\mathscr{B}$ are cross-intersecting, $\mathscr{A}\cap\mathscr{C}=\emptyset$, implying

$$|\mathscr{A}|+|\mathscr{B}|<|\mathscr{A}|+|\mathscr{C}|\leqslant\binom{m}{a}. \quad \blacksquare$$

For the proof of Proposition 1.4 we need the following special case of a result of Katona.

THEOREM 5.1 [K2]. *Suppose that* $\emptyset \neq \mathscr{C} \subset \binom{Y}{h}$ *is* l*-intersecting. Then* $|\partial_{h-l}(\mathscr{C})| \geqslant |\mathscr{C}|$, *with equality holding if and only if* $\mathscr{C} = \binom{Z}{h}$ *for some* $Z \in \binom{Y}{2h-l}$.

Proof of Proposition 1.4. Define $\mathscr{C} = \mathscr{B}^c = \{Y - B \colon B \in \mathscr{B}\}$. Since for B, $B' \in \mathscr{B}$ $|B \cap B'| \geqslant t$ implies $|B \cup B'| \leqslant 2a + t$, $\mathscr{C}$ is $(m - 2a - t)$-intersecting.

Applying Theorem 5.1 with $h = m - a - t$, $l = m - 2a - t$ gives $|\partial_a(\mathscr{C})| \geqslant |\mathscr{C}|$.

Since $\mathscr{A}$ and $\mathscr{B}$ are cross-intersecting, $\mathscr{A} \cap \partial_a(\mathscr{C}) = \emptyset$. Consequently,

$$|\mathscr{A}| + |\mathscr{B}| \leqslant |\mathscr{A}| + |\partial_a(\mathscr{C})| \leqslant \binom{m}{a} \quad \text{follows.}$$

In case of equality either $\mathscr{B} = \emptyset$ and thus $\mathscr{A} = \binom{Y}{a}$, or equality holds for $\mathscr{C}$ in Theorem 5.1. Set $T = Y - Z$. Then $|T| = m - (2h - l) = t$. Moreover,

$$\mathscr{B} = \mathscr{C}^c = \left\{B \in \binom{Y}{a+t} \colon T \subset B\right\}.$$

Consequently, $\mathscr{A} \subset \{A \in \binom{Y}{a} \colon A \cap T \neq \emptyset\}$. and the statement follows. ∎

Proof of Proposition 1.5. If $|\mathscr{G}| \leqslant \binom{v-1}{r-1}$ then we have nothing to prove. Let us apply Theorem 1.2 to the cross-intersecting families $\mathscr{G}$ and $\mathscr{H}$. Set $\mathscr{L}_0 = \mathscr{L}(|\mathscr{G}|, r)$, $\mathscr{L}_1 = \mathscr{L}(|\mathscr{H}|, r)$. Set also $\mathscr{L}_2 = \mathscr{L}_0 - \mathscr{L}(\binom{v-1}{r-1}, r) \subset \binom{[2, v]}{r}$. Then $0 < |\mathscr{L}_2| \leqslant \binom{v-2}{r-1}$ implies that every member of $\mathscr{L}_2$ contains 2. Hence $\mathscr{L}_2$ is 1-intersecting. Since $\mathscr{L}_0$ contains all r-subsets of Y through 1, every member of $\mathscr{L}_1$ contains 1, that is $|\mathscr{L}_1(1)| = |\mathscr{L}_1|$. Applying Proposition 1.4 to the cross-intersecting families $\mathscr{L}_1(1)$ and $\mathscr{L}_2$ with $a = r - 1$, $t = 1$, $\mathscr{A} = \mathscr{L}_1(1)$, $\mathscr{B} = \mathscr{L}_2$, $m = v - 1$ it follows that

$$|\mathscr{G}| + |\mathscr{H}| - \binom{v-1}{r-1} = |\mathscr{L}_1(1)| + |\mathscr{L}_2| \leqslant \binom{v-1}{r-1},$$

which is the desired inequality.

In case of equality, equality must hold in Proposition 1.4 also. Thus either

$$|\mathscr{L}_1| = \binom{v-1}{r-1}, \qquad \mathscr{L}_2 = \emptyset \qquad \text{or} \qquad |\mathscr{L}_1(1)| = \binom{v-2}{r-2}, \qquad |\mathscr{L}_2| = \binom{v-2}{r-1},$$

concluding the proof. ∎

6. The Case of Equality in Theorem 1.1

For the proof of uniqueness of the optimal families we need the following characterization of equality in Theorem 1.2.

Proposition 6.1. *Suppose that* t, a, b, m *are positive integers,* $a+b<m$, $t\leqslant b$. *Further,* $\mathscr{A}\subset\binom{[1,m]}{a}$, $\mathscr{B}\subset\binom{[1,m]}{b}$ *are cross-intersecting with* $|\mathscr{A}|=\binom{m-1}{a-1}+\cdots+\binom{m-t}{a-1}$, $|\mathscr{B}|=\binom{m-t}{b-t}$. *Then there exists* $T\in\binom{[1,m]}{t}$ *such that* $\mathscr{A}=\{A\in\binom{[m]}{a}: A\cap T\neq\emptyset\}$, $\mathscr{B}=\{B\in\binom{[1,m]}{b}: T\subset B\}$.

Proof. Define $\mathscr{C}=\mathscr{B}^c=\{[1,m]-B: B\in\mathscr{B}\}$. Then $|\mathscr{C}|=|\mathscr{B}|=\binom{m-t}{m-b}$. By the Weak Kruskal–Katona Theorem one has $|\partial_a(\mathscr{C})|\geqslant\binom{m-t}{a}$, with equality holding only if $\mathscr{C}=\binom{S}{m-b}$ for some $S\in\binom{[1,m]}{m-t}$. That is, setting

$$T=[1,m]-S, \qquad \text{if} \quad \mathscr{B}=\left\{B\in\binom{[1,m]}{b}: T\subset B\right\}.$$

Now the statement follows from $\mathscr{A}\cap\partial_a(\mathscr{C})=\emptyset$. ∎

Checking through the proof of (1.6) in Section 2, we see that for $d(\mathscr{F})\geqslant d(\mathscr{F}_3)$ equality can hold only if $\mathscr{F}$ is isomorphic to $\mathscr{F}_i$, $3\leqslant i\leqslant k+1$. In Sections 3 and 4 we might actually suppose that $d(\mathscr{F})\leqslant d(\mathscr{F}_3)-1$. Checking through the proofs we see that this leads to $|\mathscr{F}|\leqslant|\mathscr{F}_3|-1$. ∎

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