JOURNAL OF COMBINATORIAL THEORY, Series A 24, 146–161 (1978)

On Intersecting Families of Finite Sets

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Communicated by the Managing Editors

Received October 20, 1976

Let $X = [1, n]$ be a finite set of cardinality n and let $\mathscr{F}$ be a family of k-subsets of X. Suppose that any two members of $\mathscr{F}$ intersect in at least t elements and for some given positive constant c, every element of X is contained in less than $c \mid \mathscr{F} \mid$ members of $\mathscr{F}$. How large $\mid \mathscr{F} \mid$ can be and which are the extremal families were problems posed by Erdös, Rothschild, and Szemerédi. In this paper we answer some of these questions for $n > n_0(k, c)$. One of the results is the following: let $t = 1$, $3/7 < c < 1/2$. Then whenever $\mathscr{F}$ is an extremal family we can find a 7-3 Steiner system $\mathscr{B}$ such that $\mathscr{F}$ consists exactly of those k-subsets of X which contain some member of $\mathscr{B}$.

1. INTRODUCTION

Let n, t be positive integers. Let $X = [1, n]$ be the set of the first n positive integers. A family of subsets of X is called t-intersecting if any two members of it intersect in at least t elements. Erdös, Ko, and Rado [2] proved that if $\mathscr{F}$ is a t-intersecting family of k-subsets of X and $n > n_0(k, t)$ then $\mid \mathscr{F} \mid \leqslant \binom{n-t}{k-t}$ with equality holding if and only if for some t-element subset Y of X we have $\mathscr{F} = \{F \subseteq X \mid \mid F \mid = k, Y \subseteq F\}$. Hilton and Milner [3] proved that if we exclude this family, i.e., if we make the additional assumption $\mid \bigcap_{F \in \mathscr{F}} F \mid < t$, then we have for $t = 1$

$$\mid \mathscr{F} \mid \leqslant \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1. \tag{1}$$

Equality holds in (1) if and only if for some $x \in X$, $D \subset X$, $\mid D \mid = k$, $x \notin D$, and $\mathscr{F} = \{F \subset X \mid \mid F \mid = k, x \in F, F \cap D \neq \varnothing\} \cup \{D\}$.

Let c be a real number, $0 < c < 1$. Erdös, Rothschild, and Szemerédi (unpublished) have posed the following question. How large a 1-intersecting family of k-subsets of X can be if no element of X is contained in more than $c \mid \mathscr{F} \mid$ members of $\mathscr{F}$. For the case $c = 2/3$, $n > n_0(k)$ they proved

$$\mid \mathscr{F} \mid \leqslant \mid \mathscr{F}_{3,2} = \{F \subset X \mid \mid F \mid = k, \mid F \cap [1, 3] \mid \geqslant 2\} \mid. \tag{2}$$

0097-3165/78/0242–0146$02.00/0

They conjectured that for $c = 3/5$, $n > n_0(k)$

$$|\mathscr{F}| \leqslant |\mathscr{F}_{5,3} = \{F \subset X \mid |F| = k, |F \cap [1, 5]| \geqslant 3\}|, \tag{3}$$

and if $\mathscr{P}$ denotes the set of lines of a projective plane on $[1, 7]$, then for $c = 3/7$ they suggested $(n > n_0(k))$

$$|\mathscr{F}| \leqslant |\mathscr{F}_{\mathscr{P}} = \{F \subset X \mid |F| = k, \exists P \in \mathscr{P}, P \subset F\}|. \tag{4}$$

In this paper we prove (2) and (4) in a stronger form, and obtain some analogous results for the case $t \geqslant 2$. The exact statement of the results is as follows.

THEOREM 1. *Let $\mathscr{F}$ be a t-intersecting family of k-element subsets of $X = [1, n]$. Suppose that $|\bigcap_{F \in \mathscr{F}} F| < t$, and that $|\mathscr{F}|$ is maximal subject to these constraints. Then for $n > n_0(k)$:*

(a) $k > 2t + 1$ *or* $k = 3$, $t = 1$. *There exist* $D_1, D_2 \subset X$, $D_1 \cap D_2 = \varnothing$, $|D_1| = t$, $|D_2| = k - t + 1$ *such that*

$$\mathscr{F} = \mathscr{F}_1 = \{F \subset X \mid |F| = k, F \cap D_2 \neq \varnothing, D_1 \subset F\} \cup \{F \subset X \mid |F| = k, F \supseteq D_2, |F \cap D_1| \geqslant t - 1\}.$$

(b) $k \leqslant 2t + 1$. *There exists a $(t + 2)$-element subset D of X such that*

$$\mathscr{F} = \mathscr{F}_2 = \{F \subset X \mid |F| = k, |F \cap D| \geqslant t + 1\}.$$

THEOREM 2. *Let $\mathscr{F}$ be a t-intersecting family consisting of k-subsets of $X = [1, n]$. Suppose that for some ϵ, $0 < \epsilon < 1/(t + 2)$ and for every j, $1 \leqslant j \leqslant n$ we have $d(j) \leqslant (1 - \epsilon)|\mathscr{F}|$. Then for $n > n_0(k, 2)$*

$$|\mathscr{F}| \leqslant (t + 2)\binom{n - t - 2}{k - t - 1} + \binom{n - t - 2}{k - t - 2},$$

with equality holding if and only if for some $D \subseteq X$, $|D| = t + 2$

$$\mathscr{F} = \{F \subset X \mid |F| = k, |F \cap D| \geqslant t + 1\}.$$

THEOREM 3. *Let $\mathscr{F}$ be a 1-intersecting family consisting of k-element subsets of $X = [1, n]$. Suppose that for some, $0 < \epsilon < 1/14$, and for every j, $1 \leqslant j \leqslant n$ $d(j) \leqslant (\frac{1}{2} - \epsilon)|\mathscr{F}|$ holds. Suppose further that $n > n_0(k, \epsilon)$ and that the cardinality of $\mathscr{F}$ is maximal with respect to these conditions. Then there exists a 7-element subset C of X, and 7 3-element subsets of it $/B_1, B_2, \ldots, B_7/$ which form a projective plane such that*

$$\mathscr{F} = \{F \subset X \mid |F| = k, \exists i, 1 \leqslant i \leqslant 7, F \supseteq B_i\}.$$

THEOREM 4. *Let $\mathscr{F}$ be a family of* 1*-intersecting k-subsets of* $X = [1, n]$. *Suppose that* c, ϵ *are positive real constants,* $c > \epsilon$. *Suppose further that* $|\mathscr{F}| > c\binom{n-3}{k-3}$, $n > n_0(k, \epsilon)$. *Then there exists an* $x \in X$ *such that*

$$d(x) \geqslant \left(\frac{3}{7} - \epsilon\right) |\mathscr{F}|.$$

THEOREM 5. *Let $\mathscr{F}$ be a family of* 1*-intersecting k-subsets of* $X = [1, n]$. *Suppose that* $|\mathscr{F}| > (10 + \epsilon)\binom{n-3}{k-3}$ *where* $\epsilon < 1$ *is a positive constant. Then for* $n > n_0(k, \epsilon)$ *there exists an* $x \in X$ *such that*

$$d(x) > \left(\frac{3}{5} + 0.01\epsilon\right) |\mathscr{F}|.$$

THEOREM 6. *Let $\mathscr{F}$ be a family of* 1*-intersecting k-subsets of* $X = [1, n]$. *Suppose that for every* $j \in X$ *and for some constant* ϵ, $0 < \epsilon < 0.1$ $d(j) < (\frac{3}{5} - \epsilon) |\mathscr{F}|$ *holds. Let the cardinality of $\mathscr{F}$ be maximal and suppose that* $n > n_0(k, \epsilon)$. *Then there exists a* 6*-element subset Y of X and a collection* $\mathscr{C} = \{C_1, ..., C_{10}\}$ *of* 3*-subsets of Y such that the C_i's form a regular,* 1*-intersecting family and* $\mathscr{F} = \mathscr{F}_{Y,\mathscr{C}} = \{F \subset X \mid |F| = k, \exists C \in \mathscr{C}, C \subset F\}$,

THEOREM 7. *Let $\mathscr{F}$ be a family of t-intersecting k-subsets of* $X = [1, n]$. *Let s be a natural integer and ϵ a positive constant such that* $t > 2s(s-1)$ *and* $\epsilon < \epsilon(t, s) < 1/t(t + 2s)$. *Suppose that for every* $1 \leqslant j \leqslant n$ $d(j) < ([(t + s)/(t + 2s)] + \epsilon) |\mathscr{F}|$ *holds. Suppose further that the cardinality of $\mathscr{F}$ is maximal and* $n > n_0(k, s, t)$. *Then for some* $(t + 2s)$*-element subset Z of X we have*

$$\mathscr{F} = \{F \subset X \mid |F| = k, |F \cap Z| \geqslant t + s\}.$$

2. SOME DEFINITIONS AND LEMMAS

A family of sets $\mathscr{B} = \{B_1, ..., B_d\}$ is called a Δ-system of cardinality d if for $D = B_1 \cap B_2 \cap \cdots \cap B_d$ the sets $B_1 - D, ..., B_d - D$ are pairwise disjoint. Erdös and Rado [2] proved the existence of a function $\Phi(k, d)$ such that any family consisting of $\Phi(k, d)$ or more k-element sets contains d members forming a Δ-system of cardinality d.

Let $\mathscr{F}$ be a t-intersecting family of k-subsets of $X = [1, n]$. Let us set: $\mathscr{F}^{(t)} = \{G \subseteq X \mid \exists F \in \mathscr{F}, G \subseteq F, \forall F' \in \mathscr{F}, |G \cap F'| \geqslant t\}$. Obviously we have $\mathscr{F} \subseteq \mathscr{F}^{(t)}$.

Let us define the base $\mathscr{B}$ of $\mathscr{F}$ in the following way.

$$\mathscr{B} = \{B \in \mathscr{F}^{(t)} \mid \nexists F \in \mathscr{F}^{(t)}, F \subset B\}.$$

Let us decompose $\mathscr{B}$ according to the cardinality of its members, i.e., let us set $\mathscr{B} = \mathscr{B}_{l_1} \cup \mathscr{B}_{l_2} \cup \cdots \cup \mathscr{B}_{l_r}$, $l_1 < l_2 < \cdots < l_r$ where for $1 \leqslant j \leqslant r$ $\phi \neq \mathscr{B}_{l_j}$ consits of l_j-element sets. It follows from the definitions $t \leqslant l_1$, $l_r \leqslant k$.

LEMMA 1. *For $1 \leqslant j \leqslant r$ $\mathscr{B}_{l_j}$ does not contain $k - t + 2$ different members $B_1, \ldots, B_{k-t+2}$ forming a Δ-system of cardinality $k - t + 2$.*

Proof. Suppose that the assertion is not true; i.e., we can find different sets $B_i \in \mathscr{B}_{l_j}$, $i = 1, \ldots, k - t + 2$ such that setting $B_1 \cap B_2 = D$ we have $B_{i_1} \cap B_{i_2} = D$ for $1 \leqslant i_1 < i_2 \leqslant k - t + 2$. As $D \subset B_1 \in \mathscr{B}$, it follows $D \notin \mathscr{F}^{(t)}$. We obtain the desired contradiction if we show $|F \cap D| \geqslant t$ for every $F \in \mathscr{F}$. Hence we may suppose that for some $F \in \mathscr{F}$ $|F \cap D| < t$ holds. Let us set $|F \cap D| = t'$. As for $1 \leqslant i \leqslant k - t + 2$ $|F \cap B_i| \geqslant t$, $|F \cap (B_i - D)| \geqslant t - t'$. But the sets $B_i - D$ are pairwise disjoint, so we obtain $|F| \geqslant t' + (k - t + 2)(t - t') > k$ for $t' < t$, a contradiction which proves the lemma.

Let us define

$$\mathscr{B}'_{l_1} = \left\{ B \in \mathscr{B}_{l_1} \mid |\{F \in \mathscr{F} \mid B \in F\}| > k \binom{n - l_1 - 1}{k - l_1 - 1} \right\}.$$

LEMMA 2. *If $B \in \mathscr{B}'_{l_1}$ and $F \in \mathscr{F}^*$ then $|B \cap F| \geqslant t$.*

Proof. Let us suppose that on the contrary $|B \cap F| \leqslant t - 1$ holds for some $B \in \mathscr{B}'_{l_1}$, $F \in \mathscr{F}^*$. By the definition of $\mathscr{F}^*$ it follows $(G - B) \cap F \neq \varnothing$ for every $G \in \mathscr{F}$, $B \subseteq G$; i.e., $\mathscr{G} = \{G - B \mid B \subseteq G, G \in \mathscr{F}\}$ is a family of $(k - l_1)$-sets of the $(n - l_1)$-element set $X - B$, each of them intersecting F, $|F| \leqslant k$. Hence $|\{F \in \mathscr{F} \mid B \subseteq F\}| = |\mathscr{G}| \leqslant k\binom{n-l_1-1}{k-l_1-1}$, a contradiction which proves the lemma.

By Lemma 1 $|\mathscr{B}_{l_j}| < \phi_{k-t+2}(l_j) \leqslant \phi_{k+1}(k)$.

By the definition of $\mathscr{B}$ for every $F \in \mathscr{F}$ there exists $B \in \mathscr{B}$ such that $B \subseteq F$. So the following holds:

LEMMA 3. *Let $\mathscr{F}$ be a t-intersecting family consisting of k-subsets of $X = [1, n]$. Let $\mathscr{B} = \mathscr{B}_{l_1} \cup \cdots \cup \mathscr{B}_{l_r}$ be the decomposition of the base of $\mathscr{F}/l_1 < \cdots < l_r$, and $\mathscr{B}_{l_i}$ consists merely of l_i-element sets. Then*

$$|\mathscr{F}| \leqslant |\mathscr{B}'_{l_1}| \binom{n - l_1}{k - l_1} + c_k \binom{n - l_2}{k - l_2}, \tag{5}$$

or in particular

$$|\mathscr{F}| \leqslant c_k' \binom{n - l_1}{k - l_1}, \tag{6}$$

where c_k, c_k' are constants depending only on k.

We need one more lemma.

LEMMA 4. *Suppose that for some positive integer b the base $\mathscr{B}$ of the t-intersecting family $\mathscr{F}$ has a member B of cardinality b. Then for some $x \in B$ the degree of x in $\mathscr{F}$*, i.e., *the number of members of $\mathscr{F}$ containing x is at least $(t/b) \mid \mathscr{F} \mid$.*

Proof. For $y \in X$ let $d(y)$ denote the degree of y in $\mathscr{F}$. As for $F \in \mathscr{F}$ $\mid F \cap B \mid \geqslant t$, we have

$$\sum_{x \in B} d(x) \geqslant \mid \mathscr{F} \mid t.$$

Hence for at least one $x \in B$ $d(x) \geqslant t \mid \mathscr{F} \mid / \mid \mathscr{B} \mid = (t/b) \mid \mathscr{F} \mid$ holds. Q.E.D.

3. THE PROOF OF THEOREMS 1 AND 2

We start with the proof of Theorem 1.

Let $\mathscr{B} = \mathscr{B}_{l_1} \cup \cdots \cup \mathscr{B}_{l_r}$ be the base of $\mathscr{F}$. We assert $l_1 = t + 1$. If $l_1 = t$ holded then for $B \in \mathscr{B}_{l_1}$ $B \subseteq F$ would follow for every $F \in \mathscr{F}$, yielding $\mid \bigcap_{F \in \mathscr{F}} F \mid \geqslant \mid B \mid = t$, a contradiction.

As both $\mathscr{F}_1$ and $\mathscr{F}_2$ satisfy the conditions, the maximality of $\mid \mathscr{F} \mid$ and (5) imply $l_1 \leqslant t + 1$. Hence $l_1 = t + 1$.

Now $\mathscr{B}'_{l_1} = \mathscr{B}'_{t+1}$ is a t-intersecting family. Again the maximality of $\mid \mathscr{F} \mid$ and (5) imply $\mid \mathscr{B}'_{t+1} \mid \geqslant t + 2 \geqslant 3$.

Let B_1, B_2, B_3 be three different elements of $\mathscr{B}'_{t+1}$. We distinguish between two cases.

a. $B_1 \cap B_2 = B_2 \cap B_3$

Let us set $D_1 = B_1 \cap B_2$. Then by the definition of $\mathscr{B}$ $\mid D_1 \mid \geqslant t$. As $B_1 \neq B_2$, $\mid B_1 \mid = \mid B_2 \mid = t + 1$, it follows $\mid D_1 \mid = t$. Now we assert $D_1 \subset B$ for every $B \in \mathscr{B}'_{t+1}$. If it is not true then we can find an $x \in D_1$ and a $B \in \mathscr{B}'_{t+1}$ such that $x \notin B$. But $\mid B_i \cap B \mid \geqslant t$ implies $B_i \cap B = B_i - x$ for $i = 1, 2, 3$. Hence we obtain $B \supseteq ((B_1 \cup B_2 \cup B_3) - x)$ i.e., $\mid B \mid \geqslant t + 2$, a contradiction.

Now if $\mathscr{B}'_{t+1} = \{B_1, ..., B_s\}$, then we can find s different elements $x_1, ..., x_s$ of $X - D_1$ such that $B_i = D_1 \cup \{x_i\}$ for $i = 1, ..., s$. As $\mid \mathscr{F} \mid \geqslant \mid \mathscr{F}_1 \mid$, Lemma 3 yields $s \geqslant k - t + 1$. On the other hand $s \geqslant k - t + 2$ would contradict Lemma 1, whence $s = k - t + 1$.

Let F be a k-element subset of A, not-containing D_1 but intersecting each of the B_i's in at least t elements. Let x be an element of $D_1 - F$. Then $\mid B_i \mid = t + 1$ and $\mid B_i \cap F \geqslant t$ imply $F \cap B_i = B_i - x$ for $i = 1, ..., s = t-k + 1$. As $\mid F \mid = k$, it follows $F = (D_1 - x) \cup \{x_1, ..., x_{k-t+1}\}$.

Let us set $D_2 = \{x_1, ..., x_{k-t+1}\}$. Then the maximality of $\mathscr{F}$ implies $\mathscr{F} = \mathscr{F}_1$.

b. $B_1 \cap B_2 \neq B_2 \cap B_3$

Let us set $D = B_1 \cup B_2$. $|B_1| = |B_2| = t + 1$ and $|B_1 \cap B_2| = t$ imply $|D| = t + 2$. Let us define $C = B_1 \cap B_2$, $y_1 = B_1 - C$, $y_2 = B_2 - C$. As $|B_2 \cap B_3| = t$, the condition $B_1 \cap B_2 \neq B_2 \cap B_3$ implies that $\mathscr{C} \not\subset B_3$, i.e., $|B_3 \cap C| \leqslant t - 1$. Using $|B_i \cap B_3| \geqslant t$ for $i = 1, 2$ we obtain $|B_3 \cap C| = t - 1$, $\{y_1, y_2\} \subseteq B_3$. Now setting $y_3 = C - B_3$ it follows $B_3 = D - y_3$. $\mathscr{B}'_{t+1} = \{B_1, ..., B_s\}$. $|\mathscr{F}| \geqslant |\mathscr{F}_2|$ implies by (5) $s \geqslant t + 2$. Let $4 \leqslant i \leqslant s$. As $|B_i \cap B_j| \geqslant t$ for $j = 1, 2, 3$, $B_1 \cap B_2 \neq B_2 \cap B_3$ implies $|B_i \cap D| \geqslant t + 1$ i.e., $B_i \subset D$. As $s \geqslant t + 2$ it follows $B'_{t+1} = \{B \subset D \mid |B| = t + 1$. Now the maximality of $\mathscr{F}$ implies $\mathscr{F} = \mathscr{F}_2$.

For $k = t + 1$ $\mathscr{F}_1 = \mathscr{F}_2$. A simple counting shows that for $k > 2t + 1$ $|\mathscr{F}_1| > \mathscr{F}_2|$, while for $k \leqslant 2t + 1$ $|\mathscr{F}_1| \leqslant |\mathscr{F}_2|$ with equality holding if and only if $k = 3$, $t = 1$. Q.E.D.

Now we prove Theorem 2. We proceed as in the proof of Theorem 1. Let $\mathscr{B} = \mathscr{B}_{l_1} \cup \cdots \cup \mathscr{B}_{l_r}$ be the base of $\mathscr{F}$. Then we can see as in the case of Theorem 1 that for an $\mathscr{F}$ of maximal cardinality $l_1 = t + 1$, $|\mathscr{B}_{t+1}| \geqslant t + 2$. We choose again three djfferent elements B_1, B_2, B_3 of $\mathscr{B}_{t+1}$ and we distinguish between the same two cases a and b. In the case a, we have $B_1 \cap B_2 = B_2 \cap B_3 = D_1 \subset B$ for every $B \in \mathscr{B}'_{t+1}$. In view of Lemma 3 $|\{F \in \mathscr{F} \mid \nexists B \in \mathscr{B}'_{t+1}, B \subseteq F\}| \leqslant c_k\binom{n-l_2}{k-l_2}$. Hence for $i \in D_1$ we have

$$d(i) \geqslant |\mathscr{F}| - c_k \binom{n - t - 2}{k - t - 2} > (1 - \epsilon) |\mathscr{F}|$$

for $n > n_0(k, \epsilon)$, a contradiction.

In the case b, i.e., $B_1 \cap B_2 \neq B_2 \cap B_3$, we prove, as in the case of Theorem 1, $B \subset (B_1 \cup B_2) = D$ for every $B \in \mathscr{B}'_{t+1}$. Then $|\mathscr{B}'_{t+1}| \geqslant t + 2$ implies $\mathscr{B}'_{t+1} = \{B \subset D \mid |B| = t + 1$. Now for any set G such that $|G \cap D| \leqslant t$ we can find $B \in \mathscr{B}_{t+1}$ satisfying $|G \cap B| \leqslant t - 1$, yielding $|F \cap D| \geqslant t + 1$ for any $F \in \mathscr{F}$. Hence $\mathscr{F} \subseteq \mathscr{F}_2$. Q.E.D.

4. The Proof of Theorem 3

Let $\mathscr{B} = \mathscr{B}_{l_1} \cup \cdots \cup \mathscr{B}_{l_r}$ be the base of $\mathscr{F}$. By Lemma 3 the maximality of $|\mathscr{F}|$ implies $l_1 \leqslant 3$. On the other hand by Lemma 4 $l_1 \geqslant 3$, whence $l_1 = 3$. Let us set $B_3' = \{B_1, ..., B_s\}$. In view of (5) $s \geqslant 7$.

Let us define for $i = 1, ..., s$

$$c_i = \frac{|\{F \in \mathscr{F} \mid F \supseteq B_i\}|}{\binom{n - 3}{k - 3}}.$$

Then the cardinality of $\mathscr{F}$ can be expressed as follows

$$|\mathscr{F}| = \sum_{i=1}^{s} c_i \binom{n-3}{k-3} + O\binom{n-4}{k-4}. \tag{7}$$

Now the maximality of $|\mathscr{F}|$ implies for $n > n_0(k)$ for example

$$\sum_{i=1}^{s} c_i > 6,9. \tag{8}$$

On the other hand the definition of c_i implies $c_i \leqslant 1$. Now we need a lemma.

LEMMA 5. *Let $B_1, \ldots, B_s$ be a 1-intersecting family of 3-sets. Let us suppose that to each of the sets a real number c_i is associated in such a way that*

$$0 < c_i \leqslant 1, \qquad \sum_{i=1}^{s} c_i > 6,9. \tag{9}$$

Then we can find an element x of some of the sets in such a way that either

$$\sum_{i \mid x \in B_i} c_i \geqslant \frac{1}{2} \sum_{i=1}^{s} c_i, \tag{10}$$

or $s = 7$, and the B_i's are the lines of a 7–3 projective plane.

Proof. Let us suppose that c_1 is the maximal (one of the maximals) among the c_i's. Let us consider two cases separately.

a. *For $2 \leqslant i \leqslant s$ $|B_1 \cap B_i| = 1$*

We may suppose $B_1 = \{1, 2, 3\}$. Let $C_1, \ldots, C_u$; $D_1, \ldots, D_v$; $E_1, \ldots, E_w$ be the collection of the B_i's $i = 2, \ldots, s$ which intersect B_1 in 1, 2, 3, respectively. By symmetry reasons we may suppose $u \geqslant v \geqslant w$. By (9) $u + v + w = s - 1 \geqslant 6$. Let us first suppose $u = v = w = 2$. If $|C_1 \cap C_2| = |D_1 \cap D_2| = |E_1 \cap E_2| = 1$, then we may suppose $C_1 = \{1, 4, 5\}$, $C_2 = \{1, 6, 7\}$. As the B_i's form a 1-intersecting family we may assume $D_1 = \{2, 4, 6\}$, $D_2 = \{2, 5, 7\}$. Then it follows $\{E_1, E_2\} = \{\{3, 4, 7\}, \{3, 5, 6\}\}$ i.e., the B_i's form a 7–3 projective plane.

Now we may assume that for example $|C_1 \cap C_2| = 2$, or more precisely $C_1 = \{1, 4, 5\}$, $C_2 = \{1, 4, 6\}$. If D_1 does not contain 4, then by the intersection property we obtain $D_1 = \{2, 5, 6\}$, and consequently D_2 being different to D_1 has to contain 4. The same argument yields that at least one of the sets E_1, E_2 contains 4. Hence $d(4) \geqslant 4$. Now using (8) and $s = 7$ we conclude

$$\sum_{4 \in B_i} c_i > 3,9 > \frac{1}{2} \sum_{i=1}^{7} c_i.$$

Hence we may assume $u \geqslant 3$. If $v + w \leqslant 3$, then we conclude

$$\sum_{1 \in B_i} c_i \geqslant \sum_{i=1}^{s} c_i - 3 > \frac{1}{2} \sum_{i=1}^{s} c_i .$$

By way of contradiction we obtain $v + w \geqslant 4$. If $w \leqslant 1$, then the maximality of c_1 implies

$$\sum_{1 \in B_i} c_i + \sum_{2 \in B_i} c_i = c_1 + \sum_{i=1}^{s} c_i - \sum_{\substack{B_i \neq \{1,2,3\} \\ 3 \in B_i}} c_i \geqslant \sum_{i=1}^{s} c_i .$$

Consequently, either for $x = 1$ or for $x = 2$ (10) holds. Hence $w \geqslant 2$. We assert $| E_1 \cap E_2 | = 2$. If it is not true then we may assume $E_1 = \{3, 4, 5\}$, $E_2 = \{3, 6, 7\}$. As C_1, C_2, C_3 each intersect both E_1 and E_2, by symmetry reasons we may suppose $C_1 = \{1, 4, 6\}$, $C_2 = \{1, 5, 7\}$, $C_3 = \{1, 4, 7\}$. Now D_1 and D_2 both intersect E_1, E_2, C_1, C_3, but the only such 3-set is $\{2, 4, 7\}$, a contradiction. We may assume $E_1 = \{3, 4, 5\}$, $E_2 = \{3, 4, 6\}$. Then at most one of the sets C_i, namely $\{1, 5, 6\}$, and at most one of the sets D_i, namely $\{2, 5, 6\}$ does not contain 4 which yields that (10) holds for $x = 4$.

b. *There exist* $2 \leqslant j \leqslant s$, *such that* $| B_1 \cap B_j | = 2$

By symmetry reasons we may assume $j = 2$, $B_1 = \{1, 2, 3\}$, $B_2 = \{1, 2, 4\}$. If there is one more set, say B_3, among the B_i's which contains $\{1, 2\}$, then at most one of them, namely $\{3, 4\} \cup (B_3 - \{1, 2\})$ is disjoint to $\{1, 2\}$. Hence we obtain, using the maximality of c_1

$$\sum_{1 \in B_i} c_i + \sum_{2 \in B_i} c_i \geqslant c_1 + c_2 + c_3 + \sum_{i=1}^{s} c_i - \sum_{\{1,2\} \cap B_j = \varnothing} c_j \geqslant \sum_{i=1}^{s} c_i ,$$

showing that for either $x = 1$ or $x = 2$ (10) holds. The same argument yields that there are at least 2 sets among the B_i's which are disjoint to $\{1, 2\}$. Moreover if $B_3, \ldots, B_u$ are these sets then

$$\sum_{j=3}^{u} c_j > c_1 + c_2 . \tag{11}$$

By symmetry reasons, we may suppose that for $3 \leqslant j \leqslant u$ $B_j = \{3, 4, j + 2\}$ If $u \geqslant 5$ then the 1-intersecting property yield that the only B_i which can eventually be disjoint to $\{3, 4\}$ is $\{5, 6, 7\}$. Now the maximality of c_1 and (11) show that (10) holds for either $x = 3$ or $x = 4$. Now we may suppose $u = 4$. Let $B_5, \ldots, B_v$ be the sets among the B_i's which are disjoint to $\{3, 4\}$. Then we have

$$\sum_{3 \in B_i} c_i + \sum_{4 \in B_i} c_i = c_3 + c_4 + \sum_{i=1}^{s} c_i - \sum_{j=5}^{v} c_j . \tag{12}$$

Supposing that (10) does not hold neither for $x = 3$ nor for $x = 4$ from (12) we obtain

$$\sum_{j=5}^{v} c_j > c_3 + c_4 = \sum_{j=3}^{u} c_j > c_1 + c_2 . \tag{13}$$

In particular by the maximality of c_1 $v \geqslant 6$. For B_5 and B_6 there are no other possibilities than $\{B_5, B_6\} = \{\{5, 6, 1\}, \{5, 6, 2\}\}$. Hence $v = 6$ and the only B_i's disjoint to $\{5, 6\}$ are B_1 and B_2. Using (13) we obtain

$$\sum_{5\in B_i} c_i + \sum_{6\in B_i} c_i = c_5 + c_6 + \sum_{j=1}^{s} c_j - c_1 - c_2 > \sum_{j=1}^{s} c_j . \tag{14}$$

Equation (14) shows that for either $x = 5$ or $x = 6$ (10) holds which finishes the proof of the lemma.

Now we apply the lemma to the proof of Theorem 3.

By the assumption $d(i) \leqslant (\frac{1}{2} - \epsilon) \mid \mathscr{F} \mid$ for $1 \leqslant i \leqslant n$ we obtain that the first alternative in Lemma 5 cannot hold. Hence $s = 7$ and $B_1, \ldots, B_7$ form a projective plane. By the definition of the base each element F of $\mathscr{F}$ intersects each of the sets B_i $i = 1, \ldots, 7$. As the 7–3 projective plane is 3-chromatic for each $F \in \mathscr{F}$ there is a j, $1 \leqslant j \leqslant 7$, such that $B_j \subseteq F$. Now the maximality of $\mathscr{F}$ yields

$$\mathscr{F} = \{F \subset X \mid \mid F \mid = k, \exists j, 1 \leqslant j \leqslant 7, B_j \subseteq F\}. \qquad \text{Q.E.D.}$$

5. The Proof of Theorem 4 and Some Remarks

Let $\mathscr{B} = \mathscr{B}_{l_1} \cup \cdots \cup \mathscr{B}_{l_r}$ be the base of $\mathscr{F}$. As in the proof of Theorem 3 we can prove $l_1 = 3$. Let $\mathscr{B}_3' = \{B_1, \ldots, B_s\}$. Let us define again

$$c_i = \frac{|\{F \in \mathscr{F}, B_i \subseteq F\}|}{\binom{n-3}{k-3}} .$$

Then for $n > n_0(k)$ we have $\sum_{i=1}^{s} c_i > c/2$.

Let c_1 be the maximal one among the c_i's. Let us suppose first that $s \leqslant 7$. We may suppose $B_1 = \{1, 2, 3\}$, and that

$$d'(1) = \sum_{1\in B_i} c_i \geqslant \sum_{2\in B_i} c_i \geqslant \sum_{3\in B_i} c_i . \tag{15}$$

From (15) we obtain

$$d'(1) \geqslant c_1 + \frac{1}{3}\sum_{i=2}^{s} c_i = \frac{1}{3}\sum_{i=1}^{s} c_i + \frac{2}{3} c_1 \geqslant \left(\frac{1}{3} + \frac{2}{21}\right) \sum_{i=1}^{s} c_i = \frac{3}{7} \sum_{i=1}^{s} c_i . \tag{16}$$

From (16) it follows

$$d(1) \geqslant \frac{3}{7} \sum_{i=1}^{s} c_i \binom{n-3}{k-1} + O\binom{n-4}{k-4} > \left(\frac{3}{7} - \epsilon\right) |\mathscr{F}|,$$

whenever $n > n_0(k, \epsilon)$. Now let us suppose $s \geqslant 8$. Then by a result of Deza [1] we cannot have $|B_i \cap B_j| = 1$ for all the pairs $1 \leqslant i < j \leqslant s$. Hence there are two sets, say B_{i_1}, B_{i_2}, such that $|B_{i_1} \cap B_{i_2}| = 2$ and whenever for a pair $1 \leqslant i < j \leqslant s$ $|B_i \cap B_j| = 2$, then $c_{i_1} + c_{i_2} \geqslant c_i + c_j$. We may assume $B_{i_1} = \{1, 2, 3\}$, $B_{i_2} = \{1, 2, 4\}$. Let $B_{i_3}, \ldots, B_{i_u}$ be the collection of the B_i's which are disjoint to $\{1, 2\}$. By the 1-intersection property each of these sets contains 3 and 4. Hence we may assume $B_{i_v} = \{3, 4, v + 2\}$ for $v = 3, \ldots, u$. If $u \geqslant 5$, then there is no 3-element set which is disjoint to $\{3, 4\}$ and intersects each of the sets B_{i_v} $v = 1, \ldots, 5$. Indeed, it should contain 5, 6, and 7 and it should meet $\{1, 2\}$ as well. Hence in this case for $n > n_0(k, \epsilon)$ either 3 or 4 is contained in at least $\frac{1}{2} |F| - O(\binom{n-4}{k-4}) > (\frac{3}{7} - \epsilon) |F|$ members of $\mathscr{F}$, a contradiction. In the case $u = 4$ the special choice of i_1, i_2 implies

$$d(1) + d(2) \geqslant \left(c_{i_1} + c_{i_2} - c_{i_3} - c_{i_4} + \sum_{i=1}^{s} c_i\right)\binom{n-3}{k-3} + O\left(\binom{n-4}{k-4}\right)$$

$$\geqslant \sum_{i=1}^{s} c_i \binom{n-3}{k-3} + O\left(\binom{n-4}{k-4}\right) > 2 \cdot \left(\frac{3}{7} - \epsilon\right) |\mathscr{F}|$$

$$\text{for} \quad n > n_0(k, \epsilon),$$

a contradiction. Using the same argument we may assume $u = 3$, $c_{i_3} > c_{i_1} + c_{i_2}$. Hence by the definition of i_1, i_2 for $1 \leqslant j \leqslant s$, $j \neq i_3$ $|B_{i_3} \cap B_j| = 1$.

If for every $1 \leqslant j \leqslant s$ $B_j \cap \{1, 5\} \neq \varnothing$ holds then we deduce $d(1) + d(5) \geqslant \sum_{i=1}^{s} c_i \binom{n-3}{k-3} = |\mathscr{F}| + O(\binom{n-4}{k-4}) > 2(\frac{3}{7} - \epsilon)$ $|\mathscr{F}|$, a contradiction. Hence there exists a set, say B_{i_4} (and using the same argument an other one B_{i_5}) which is disjoint to $\{1, 5\}$ ($\{2, 5\}$, respectively). Using the 1-intersection property of $\mathscr{B}_3$ we may assume $B_{i_4} = \{2, 3, 6\}$. Let us set $Y = B_1 \cup \cdots \cup B_s$. If $|Y| \leqslant 7$ then we obtain $\sum_{y \in Y} d(y) \geqslant \sum_{i=1}^{s} |B_i| \, C_i \binom{n-4}{k-4} = 3$ $|\mathscr{F}| + O(\binom{n-3}{k-3})$, yielding that for $n > n_0(k, \epsilon)$ for at least one $y \in Y$ $d(y) > (\frac{3}{7} - \epsilon)$ $|\mathscr{F}|$ holds. So we may assume $|Y| \geqslant 8$.

For B_{i_5} there are 3 essentially different possibilities: $\{1, 3, 7\}$, $\{1, 4, 6\}$, $\{1, 3, 6\}$. However in the latter cases $|Y| \geqslant 8$ implies that there is a set, say B_{i_6} which is not contained in [1, 6]. So we may assume $7 \in B_{i_6}$. We know $|B_{i_6} \cap \{3, 4, 5\}| = 1$. If $5 \in B_{i_6}$ then B_{i_6} does not contain either 3 or 4, but it intersects $\{2, 3, 6\}$ and $\{1, 2, 3\}$ nontrivially. Hence $B_{i_6} = \{2, 5, 7\}$, but this set is disjoint to both $\{1, 4, 6\}$ and $\{1, 3, 6\}$, a contradiction. If $B_{i_5} = \{1, 3, 6\}$ then $4 \in B_{i_6}$ yields essentially the same contradiction. Hence in this case $3 \in B_{i_6}$, and by symmetry reasons we may assume $B_{i_6} = \{1, 3, 7\}$. If $B_{i_6} =$

$\{1, 4, 6\}$, then we have symmetry in 3 and 4. Hence we may assume $3 \in B_{i_6}$. Then $B_{i_6} \cap B_{i_5} \neq \varnothing$ and $B_{i_4} \cap \{1, 2, 4\} \neq \varnothing$ imply $B_{i_6} = \{1, 3, 7\}$. So we have proved that we may suppose $B_{i_5} = \{1, 3, 7\}$. As $|Y| \geqslant 8$, there is a member, say B_{i_6} of $\mathscr{B}_3$ which is not contained in [1, 7]. We may suppose $8 \in B_{i_6}$. From $|B_{i_6} \cap \{3, 4, 5\}| = 1$, $B_{i_6} \cap B_{i_j} \neq \varnothing$ for $j = 4, 5$ it follows $3 \in B_{i_6}$. As $\{1, 2, 4\} \cap B_{i_6} \neq \varnothing$ it follows that the third element of B_{i_6} is either 1 or 2. By symmetry reasons we may assume $B_{i_6} = \{1, 3, 8\}$. If a set $B \in \mathscr{B}_3$ is disjoint to $\{1, 3\}$ then it should be $\{2, 7, 8\}$ as it has to intersect $\{1, 3, 2\}, \{1, 3, 7\}$, and $\{1, 3, 8\}$. But $\{2, 7, 8) \cap \{3, 4, 5\} \neq \varnothing$, a contradiction. Hence for every $B \in \mathscr{B}_3$ either 1 or 3 is contained in B, yielding

$$d(1) + d(3) \geqslant \sum_{i=1}^{s} c_i \binom{n-3}{k-3} > 2\left(\frac{3}{7} - \epsilon\right) |\mathscr{F}|$$

for $n > n_0(k, \epsilon)$ i.e., either 1 or 3 has degree greater than $(\frac{3}{7} - \epsilon)\ |\mathscr{F}|$.
Q.E.D.

Remark 1. The proof of Theorem 4 seems to foreshadow how complicated it will be to solve the same problem for the case $|\mathscr{F}| > c\binom{n-v}{k-v}$, $v \geqslant 4$ is given. The case $v = 3$ suggests that the bound given by the projective plane is optimal, i.e., there is always a point of degree $\geqslant ([v/(v^2 - v + 1)] - \epsilon)\ |\mathscr{F}|$ for any positive ϵ and $n > n_0(k, \epsilon)$.

In the case $v = 2$ an easy modification of the argument of the proof of Theorem 2 yields that the optimal bound is 2/3.

Remark 2. Erdös conjectured recently that if there exists a regular intersecting v-graph on m points then $m \leqslant v^2 - v + 1$. If this conjecture is not true then there exists a family $\mathscr{D} = \{D_1, \ldots, D_s\}$ of 1-intersecting v-sets, which form a regular v-graph on $m \geqslant v^2 - v + 2$. Let us define for $k \geqslant v$ $\mathscr{F}_{\mathscr{D}} = \{F \subset X \mid |F| = k, \exists D \in \mathscr{D}, D \subseteq F\}$. Then $|\mathscr{F}| \geqslant \binom{n-v}{k-v}$ but for any $i \in X$ $d(i) \leqslant ([v/(v^2 - v + 2)] + o(1)\ |\mathscr{F}|$. Thus if the bound given using the projective plane is optimal then the conjecture of Erdös is true. So Theorem 4 establishes it for $v = 3$.

6. The Proof of Theorems 5 and 6

First we prove a lemma.

Lemma 6. *Let $\mathscr{B} = \{B_1, \ldots, B_s\}$ be a 1-intersecting family of 3-subsets of $[1, n]$. Suppose that for $1 \leqslant i \leqslant s$ there is a constant c_i, $0 \leqslant c_i \leqslant 1$ associated with the set B_i. Suppose further that for some $0 < \delta < 1$*

$$\sum_{i=1}^{s} c_i > 10 + \delta. \tag{17}$$

Then there exists a j, $1 \leqslant j \leqslant n$ *such that*

$$\sum_{j \in B_i} c_i \geqslant \frac{6+\delta}{10+\delta} \sum_{i=1}^{s} c_i . \tag{18}$$

Proof. It follows from (17) that $s \geqslant 11$.

If for $1 \leqslant i_1 \leqslant i_2 \leqslant s \mid B_{i_1} \cap B_{i_2} \mid = 1$ then in view of a result of Deza [1] $s \leqslant 7$. Hence there exist two sets, say B, $B' \in \mathscr{B}$ such that $| B \cap B' | = 2$. That is to say there exist 2-element subsets of X which are contained in more than one of the B_i's. Let C be a 2-element set which is contained in a maximal number of the B_i's. We may assume $C = [1, 2]$, and that $\mathscr{C} = \{\{1, 2, 3\}, \{1, 2, 4\},..., \{1, 2, u\}\}$ are the B_i's containing C. If $B \in \mathscr{B}$ and $B \cap C = \varnothing$ then the intersection property of $\mathscr{B}$ implies $[3, u] \subseteq B$. Let $\mathscr{D} = \{D_1 ,..., D_v\}$ be the collection of the B_i's disjoint to C.

Let us suppose first $| \mathscr{D} | \leqslant 1$.

Let us divide the members of $\mathscr{B} - (\mathscr{C} \cup \mathscr{D})$ into two families $\mathscr{E}_1$, $\mathscr{E}_2$ according to whether they intersect C in $\{1\}$ or in $\{2\}$. By symmetry reasons we may assume $| \mathscr{E}_1 | \geqslant | \mathscr{E}_2 |$. Suppose $| \mathscr{E}_2 | \geqslant 4$. Let us consider first the case when there are two sets, say E, $E' \in \mathscr{E}_2$ such that $| E \cap E' | = 1$. Let $E = \{2, e_1, e_2\}$, $E' = \{2, e_1', e_2'\}$ where $e_1, e_2, e_1'\ e_2'$ are four different elements of $[3, n]$. The 1-intersection property and $| \mathscr{E}_1 | \geqslant | \mathscr{E}_2 | \geqslant 4$ imply $\mathscr{E}_1 = \{\{1, e_1, e_1'\}, \{1, e_1, e_2'\}, \{1, e_2, e_1'\}, \{1, e_2, e_2'\}\}$. But now we cannot find any 3-element set different from E, E', containing 2, and nontrivially intersecting each member of $\mathscr{E}_1$. However this contradicts $| \mathscr{E}_2 | \geqslant 4$. Now we may assume $| E \cap E' | \geqslant 2$ for E, $E' \in \mathscr{E}_2$. Then the sets $E - 2$, $E \in \mathscr{E}_2$ form a 1-intersecting family of 2-element sets. Hence $| \mathscr{E}_2 | \geqslant 4$ implies that there exists an element r which is common to each of the sets $E - 2$, $E \in \mathscr{E}_2$. Now $| \mathscr{E}_1 | \geqslant 4$ and the 1-intersection property entail that r belongs to every member of $\mathscr{E}_1$ as well. Hence we have proved that every member of $\mathscr{B} - \mathscr{D}$ intersects $\{1, 2, r\}$ in at least two points. If $D \in \mathscr{D}$ then the 1-intersection property implies $r \in D$ as otherwise D should contain all the different elements $E - \{2, r\}$, $E \in \mathscr{E}_2$, but $| \mathscr{E}_2 | > | D |$. So we obtain

$$\sum_{1 \in B_i} c_i + \sum_{2 \in B_i} c_i + \sum_{r \in B_i} c_i \geqslant 2 \sum_{i=1}^{s} c_i - 1.$$

Consequently for either $j = 1$ or $j = 2$ or $j = r$

$$\sum_{j \in B_i} c_i \geqslant \frac{2 \sum_{i=1}^{s} c_i - 1}{3} \geqslant \frac{(2 - 1/(10 + \delta)) \sum_{i=1}^{s} c_i}{3} \geqslant \frac{6+\delta}{10+\delta} \sum_{i=1}^{s} c_i .$$

Suppose now $| \mathscr{E}_2 | \leqslant 3$. Then every member of $\mathscr{B} - (\mathscr{E}_2 \cup \mathscr{D})$ contains 1.

As $|\mathscr{E}_2 \cup \mathscr{D}| \leqslant 4$, we obtain

$$\sum_{1 \in B_i} c_i \geqslant \sum_{i=1}^{s} c_i - 4 \geqslant \left(1 - \frac{4}{10+\delta}\right) \sum_{i=1}^{s} c_i = \frac{6+\delta}{10+\delta} \sum_{i=1}^{s} c_i,$$

i.e., (18) holds for $j = 1$.

Now we must consider the case $|\mathscr{D}| \geqslant 2$. As we proved $[3, u] \subseteq D$ for every $D \in \mathscr{D}$, this case is possible only for $u = 4$. Then the special choice of C implies $|\mathscr{D}| = 2$. We may assume $\mathscr{D} = \{\{3, 4, 5\}, \{3, 4, 6\}\}$. Let $\mathscr{E} = \{E_1, ..., E_w\}$ be the collection of B_i's disjoint to $[3, 4]$. As each of the E_j's contains 5 and 6 it follows from the maximal choice of C that $w \leqslant 2$. If $w \leqslant 1$ then replacing $[1, 2]$ by $[3, 4]$ we come back to the preceeding case $|\mathscr{D}| \leqslant 1$. If $w = 2$ then the 1-intersection property yields $\{E_1, E_2\} = \{\{5, 6, 1\}, \{5, 6, 2\}\}$. Now each of the remaining members of $\mathscr{B}$ has to intersect $\{1, 2\}$, $\{3, 4\}$, and $\{5, 6\}$.

Thus being a 3-element set it is contained in $[1, 6]$. But then the Erdös–Ko–Rado theorem yields $|\mathscr{B}| \leqslant \binom{6-1}{3-1} = 10 < 11$, a contradiction. Q.E.D.

Now we apply the lemma to the proof of the theorem. Let $\mathscr{B}' = \mathscr{B}_{l_1} \cup \cdots \cup \mathscr{B}_{l_q}$ be the base of $\mathscr{F}$. Then as in the preceeding sections, $|\mathscr{F}| \geqslant (10 + \epsilon)\binom{n-3}{k-3}$ implies for $n > n_0(k, \epsilon)$ $l_1 = 3$. We apply the lemma for $\mathscr{B} = \mathscr{B}_3$ and

$$c_i = \frac{|F \in \mathscr{F} \mid B_i \subseteq F|}{\binom{n-3}{k-3}} \qquad \text{for} \quad B_i \in \mathscr{B}_3'.$$

Setting $\delta = \epsilon/2$ the validity of (17) follows for $n > n_0(k, \epsilon)$. Now Lemma 6 yields that there exists a $j \in [1, n]$ such that

$$d(j) > \frac{6+\delta}{10+\delta} \sum_{i=1}^{s} c_i \binom{n-3}{k-3} + O\left(\binom{n-4}{k-4}\right) > \left(\frac{3}{5} + 0.01\epsilon\right) |\mathscr{F}|$$

for $n > n_0(k, \epsilon)$. Q.E.D.

Now we turn to the proof of Theorem 6.

Let us recall the proof of Lemma 6. Let us suppose that instead of $\delta > 0$ we assume only $\delta > -1$. It still ensures us of $s \geqslant 10$ but not of $s \geqslant 11$. However the fact $s \geqslant 11$ was used only at the very end of the proof. If we assume only $s \geqslant 10$ then we have to deal yet with the case $\mathscr{B}$ consists of 10 subsets of $[1, 6]$. If there is an $i \in [1, 6]$ which is contained in at least 6, i.e., not contained in at most 4 members of $\mathscr{B}$ then for this i we have

$$\sum_{i \in B_i} c_j \geqslant \sum_{j=1}^{s} c_j - 4 \geqslant \frac{6+\delta}{10+\delta} \sum_{j=1}^{s} c_j .$$

Otherwise every element of [1, 6] has degree 5, i.e., $\mathscr{B}$ is a regular 1-intersecting family. Hence the proof of Lemma 6 yields:

LEMMA 7. *Let $\mathscr{B} = \{B_1, ..., B_s\}$ be a 1-intersecting family of 3-subsets of $[1, n]$. Suppose that for $1 \leqslant i \leqslant s$ there is a constant associated with the set B_i Suppose further that for some δ, $-1 < \delta < +1$,*

$$\sum_{i=1}^{s} c_i > 10 + \delta. \tag{19}$$

Then either there exists a $j \in [1, n]$ for which (18) holds or the B_i's form a regular, 1-intersecting family of cardinality 10 on some 6-element subset Y of X.

Now we use Lemma 7 to prove Theorem 6.

From the maximality of $|\mathscr{F}|$ it follows again that if $\mathscr{B}' = \mathscr{B}_{l_1} \cup \cdots \cup \mathscr{B}_{l_r}$ is the usual decomposition of the base of $\mathscr{F}$ then $l_1 = 3$. Moreover if we define

$$c_i = \frac{|\{F \in \mathscr{F} \mid B_i \subseteq F\}|}{\binom{n-3}{k-3}} \quad \text{for} \quad B_i \in \mathscr{B}_3',$$

then it follows $\sum_{i=1}^{s} c_i > 10 - \epsilon$. Setting $\delta = -\epsilon$ it follows from Lemma 7 that either we have for some $j \in X$

$$d(j) \geqslant \frac{6-\epsilon}{10-\epsilon} |\mathscr{F}| + O\left(\binom{n-4}{k-4}\right) > \left(\frac{3}{5} - \epsilon\right) |\mathscr{F}| \quad \text{for} \quad n > n_0(k, \epsilon),$$

a contradiction or $|\mathscr{B}_3| = 10$ and for some 6-subset Y of X the members of $\mathscr{B}_3$ form a 1-intersecting, regular 3-graph on it.

Now we prove that every subset of X intersecting nontrivially each member of $\mathscr{B}_3'$ contains a member of $\mathscr{B}_3'$. Obviously it suffices to prove that every 4-element subset, G of Y contains a member of $\mathscr{B}_3'$. As $\binom{6}{3} = 2 \cdot 10$. $\mathscr{B}_3'$ contains exactly one of each 3-subset of Y and its complement. So if G does not contain any member of $\mathscr{B}_3'$, then each one of the four 3-subsets of Y containing Y–G belongs to $\mathscr{B}_3'$. From the 1-intersection property it follows that the remaining members of $\mathscr{B}_3'$ intersect Y–G nontrivially. Hence at least one of the two elements of Y–G is contained in at least 7 members of $\mathscr{B}_3'$ contradicting the regularity of it. Setting $\mathscr{C} = \mathscr{B}_3'$ it follows now $\mathscr{F} \subseteq \mathscr{F}_{y,\mathscr{C}}$.

Q.E.D.

7. The Proof of Theorem 7

Let $\mathscr{B} = \mathscr{B}_{l_1} \cup \cdots \cup \mathscr{B}_{l_r}$ be the usual decomposition of the base on $\mathscr{F}$. From the maximality of $|\mathscr{F}|$ it follows $l_1 \leqslant t + s$. On the other hand Lemma 4 yields $l_1 \geqslant t + s$, i.e., $l_1 = t + s$. Let $\mathscr{B}'_{t+s} = \{B_1, ..., B_q\}$. The maximality of $|\mathscr{F}|$ implies $q \geqslant \binom{t+2s}{t+s}$. Let us define

$$c_i = \frac{|\{F \in \mathscr{F} \mid B_i \subseteq F\}|}{\binom{n - t - s}{k - t - s}} \quad \text{for} \quad i = 1, ..., s.$$

We need a lemma.

Lemma 8. *Let $B_1, ..., B_q$ be a t-intersecting family of $(t + s)$-element subsets of $X = [1, n]$. Suppose $q \geqslant \binom{t+2s}{t+s}$. Then either there exists a $(t + i)$-element subset Y of X for some $0 \leqslant i < s$, satisfying $|B_j \cap Y| \geqslant t + i - s + 1$ for $j = 1, ..., q$, or there exists a $(t + 2s)$-set Z such that $\{B_1, ..., B_q\} = \{B \subset Z \mid B = t + s\}$.*

Proof. Let us consider the intersections $B_1 \cap B_j$ $j = 1, ..., q$. From the binomal identity

$$\binom{t + 2s}{t + s} = \sum_{i=0}^{s} \binom{t + s}{t + i}\binom{s}{s - i}$$

it follows that either for some $(t + i)$-subset Y of B_1, $0 \leqslant i < s$

$$|\mathscr{D}_Y = \{B_j \mid 1 \leqslant j \leqslant q, B_j \cap B_1 = Y\}| > \binom{s}{s - i}, \tag{20}$$

or $q = \binom{t+2s}{t+s}$ and for every i, $0 \leqslant i < s$, and every $(t + i)$-subset of B_1 we have equality in (20). Let us consider the first possibility. We assert $|B_j \cap Y| \geqslant t + i - s + 1$ for $1 \leqslant j \leqslant q$. Suppose that it is not true, i.e., for some $1 \leqslant j \leqslant q$ $|B_j \cap Y| \leqslant t + i - s$. The t-intersection property implies $B_j \cap Y = t + i - s$, and $B_j \supseteq (B_r - Y)$ for $r = 1$ and for the values of r satisfying $B_r \cap B_1 = Y$. Now (20) implies that for these values of r $|\bigcup_r (B_r - Y)| > s$ yielding $|B_j| > t + i - s + (s - i) + s = t + s$, a contradiction.

From this argument follows that if the second possibility holds then not only we have equality in (20) for every $0 \leqslant i < s$ and every $(t + i)$-subset Y of B_1 but there exists an s-element subset Z_Y of $X - B_1$ such that $\mathscr{D}_Y = \{B \subset X \mid B \cap B_1 = Y, (B - Y) \subset Z_Y, |B| = t + s\}$. The statement of the lemma would follow if we proved Z_Y does not depend on Y, i.e., for Y, $Y' \subseteq B_1$, $|Y| = t + i$, $|Y'| = t + i'$ $Z_Y = Z_{Y'}$. If it is not true then we may assume that it does not hold for a pair Y, Y' satisfying the additional requirements $Y \cup Y' = B_1$, $s > i + i'$. Now let us choose $B \in \mathscr{D}_Y$ and $B' \in \mathscr{D}_{Y'}$ in such a way that $|(B - Y) \cap (B' - Y')| < s - i - i'$—it is possible as $Z_Y \neq Z_{Y'}$ and $s > i + i'$. But then we have $|B \cap B'| < (t + i + i' - s) + (s - i - i') = t$, a contradiction proving the lemma.

Now we apply the lemma to the proof of the theorem. If the first possibility holds then it follows that for some element $y \in Y$

$$d(y) \geqslant \frac{t + i - s + 1}{t + i} |\mathscr{F}| + O\left(\binom{n - t - s - 1}{k - t - s - 1}\right)$$

$$\geqslant \frac{t - s + 1}{t} |\mathscr{F}| + O(|\mathscr{F}|) > \left(\frac{t + s}{t + 2s} + \epsilon(t, s)\right) |\mathscr{F}|$$

as $t > 2s(s - 1)$, and $n > n_0(k, s, t)$, a contradiction proving the theorem for this case.

In the second case we have for some $(t + 2s)$-element subset Z of X $\mathscr{B}_{t+s} = \{B \subset Z \mid |B| = t + s\}$, and consequently $|F \cap Z| \geqslant t + s$ for every $F \in \mathscr{F}$ follows from the t-intersection property.

Now the maximality of $|\mathscr{F}|$ yields $\mathscr{F} = \{F \subset X \mid |F| = k, |F \cap Z| \geqslant t + s\}$. Q.E.D.

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