JOURNAL OF COMBINATORIAL THEORY, Series A 42, 200–206 (1986)

All Rationals Occur as Exponents

PETER FRANKL

C.N.R.S., Paris, 15 Quai Anatole France, 75007, Paris, France

Communicated by the Managing Editors

Received July 3, 1985

For integers $n \geqslant k \geqslant 1$ and $L \subset \{0, 1,..., k-1\}$, $m(n, k, L)$ denotes the maximum number of k-subsets of an n-set so that the size of the intersection of any two among them is in L. It is proven that for every rational number $r \geqslant 1$ there is a choice of k and L so that $cn^r < m(n, k, L) < dn^r$, where c, d depend on k and L but not on n.

1. INTRODUCTION

Suppose $n \geqslant k \geqslant 1$, $L \subseteq \{0, 1,..., k-1\}$. Let X be a finite set, $|X| = n$. A family $\mathscr{F}$ of subsets of X is called an L-system if for any two distinct members F, F' of $\mathscr{F}$ one has $|F \cap F'| \in L$. Define

$$m(n, L) = \{\max |\mathscr{F}| : \mathscr{F} \text{ is an } L\text{-system}\};$$

$$m(n, k, L) = \{\max |\mathscr{F}| : \mathscr{F} \text{ is an } L\text{-system and } |F| = k \text{ for all } F \in \mathscr{F}\}.$$

There is a wide variety of problems related to $m(n, L)$ and $m(n, k, L)$. For example, $m(n, k\{0, 1,..., t-1\}) \leqslant \binom{n}{t}/\binom{k}{t}$ with equality holding if and only if a (n, k, t)-Steiner-system exists. This already shows that the determination of these functions is hopeless in general. Let us mention three general upper bounds:

(1) $m(n, L) \leqslant \sum_{0 \leqslant i \leqslant |L|} \binom{n}{i}$ [13]

(2) $m(n, k, L) \leqslant \binom{n}{|L|}$ [12]

(3) $m(n, k, L) \leqslant \prod_{l \in L} (n-l)/(k-l)$ for $n > n_0(k)$ [1].

Let us mention some of the recent papers concerning $m(n, L)$ and $m(n, k, L)$: [5, 6, 7, 8, 9, 10, 11, 14].

0097-3165/86 $3.00

Let us use the notation $m(n, k, L) = \Theta(n^\alpha)$ to denote that there exist constants c, d depending on k and L but not on n so that $cn^\alpha < m(n, k, L) < dn^\alpha$. It is not known whether such an α exists for all choices of k and L. However, if $\alpha = \alpha(k, L)$ exists then obviously $\alpha \geqslant 1$.

THEOREM 1.1. *For every rational number r, $r \geqslant 1$ there exists k and L so that $m(n, k, L) = \Theta(n^r)$.*

The author has examined all cases with $k \leqslant 10$ and proved the existence of $\alpha(k, L)$. Actually $\alpha(k, L)$ is an integer for all cases with $k \leqslant 9$ and all but two cases with $k = 10$. Its value in the exceptional cases is $\frac{5}{2}$. In fact Theorem 1.1 follows from the following result.

THEOREM 1.2. *Suppose that $s, d, a_0, a_1, \ldots, a_d$ are non-negative integers with $s \geqslant d \geqslant 1$, $a_d \geqslant 1$, and $a_1 > \sum_{i \neq 1} a_i \binom{s-1}{i}$, define $p(x) = \sum_{i=0}^{d} a_i \binom{x}{i}$. Then*

$$m(n, p(s), \{p(0), \ldots, p(s-1)\}) = \Theta(n^{s/d}).$$

2. SOME PREPARATIONS

A family $\mathscr{A}$ of sets is called *closed under intersection* (or shortly *closed*) if $A, A' \in \mathscr{A}$ implies $A \cap A' \in \mathscr{A}$. Clearly, to every family $\mathscr{B}$ there is a smallest closed family $\bar{\mathscr{B}}$ with $\mathscr{B} \subseteq \bar{\mathscr{B}}$, $\bar{\mathscr{B}}$ is called the closure of $\mathscr{B}$.

For an arbitrary set D, the family $\bar{\mathscr{B}}_{|D} = \{B \cap D: B \in \bar{\mathscr{B}}\}$ is closed again.

By a simple averaging argument (cf, [3]) every $\mathscr{F} \subset \binom{X}{k}$ contains a subfamily $\mathscr{F}'$, $|\mathscr{F}'|/|\mathscr{F}| \geqslant k!/k^k$ and $\mathscr{F}'$ being k-partite, i.e., there exist disjoint sets $X_1, \ldots, X_k$ satisfying $|F \cap X_i| = 1$ for all $F \in \mathscr{F}'$ and $i = 1, \ldots, k$.

For a set G satisfying $|G \cap X_i| \leqslant 1$ define the *canonical projection* $\pi(G)$ of G by $\pi(G) = \{i: |G \cap X_i| = 1\}$. Note that $|G| = |\pi(G)|$. Also, if $\mathscr{A}$ is an arbitrary family and G as above, then the families $\mathscr{A}_{|G}$ and $\pi(\mathscr{A}_{|G}) = \{\pi(A): A \in \mathscr{A}_{|G}\}$ are isomorphic.

THEOREM 2.1 ([8]). *Suppose $\mathscr{F}$ is an (n, k, L)-system. Then there exists a positive constant $c(k, L)$, independent of n, a closed L-system $\mathscr{A} \subset 2^{\{1,2,\ldots,k\}}$, and $\mathscr{F}^* \subset \mathscr{F}$ so that*

(i) *$\mathscr{F}^*$ is k-partite,*

(ii) $|\mathscr{F}^*| \geqslant c(k, L)\,|\mathscr{F}|$,

(iii) *For every $F \in \mathscr{F}^*$ one has $\pi(\bar{\mathscr{F}}^*_{|F}) = \mathscr{A}$.*

Note that (iii) implies that $\bar{\mathscr{F}}^*$ is an L-system, i.e., the size of the intersection of any number of members of $\mathscr{F}^*$ is in L.

Since we are only interested in the order of magnitude of $m(n, k, L)$, we may suppose $\mathscr{F} = \mathscr{F}^*$. To express this fact we say that $\mathscr{F}$ is *canonical*, we call $\mathscr{A}$ the *intersection pattern* of $\mathscr{F}$.

Let us mention without proof the following easy fact.

PROPOSITION 2.2. *Suppose $\{\mathscr{A}_1,..., \mathscr{A}_m\}$ and $\{\mathscr{B}_1,..., \mathscr{B}_m\}$ are two families of sets satisfying for all j and all $1 \leqslant i_1 < \cdots < i_j \leqslant m$,*

$$|A_i \cap \cdots \cap A_{i_j}| = |B_i \cap \cdots \cap B_{i_j}|.$$

Then they are isomorphic.

Suppose $\mathscr{F}$ is a canonical family with intersection pattern $\mathscr{A}$. For $A, B \in \mathscr{A}$ satisyfing $A \subset B$ and $G \in \bar{\bar{\mathscr{F}}}$ with $\pi(G) = A$ define

$$\mathscr{I}_G(A, B) = \{H \in \bar{\bar{\mathscr{F}}}: G \subset H, \pi(H) = B\}.$$

We say that B *covers* A if $A, B \in \mathscr{A}$, $A \subset B$ but there is no $C \in \mathscr{A}$ with $A \subset C \subset B$.

LEMMA 2.3 (Monotonity lemma). *Suppose A, B, C, $D \in \mathscr{A}$ satisfy $A \subset B \subset D$, with D covering B, $A \subset C \subset D$ and $C \nsubseteq B$. Then for all G, $H \in \bar{\bar{\mathscr{F}}}$ satisfying $\pi(G) = A$, $\pi(H) = B$, and $G \subset H$ one has*

$$|\mathscr{I}_G(A, C)| \geqslant |\mathscr{I}_H(B, D)|.$$

Proof. Suppose $\mathscr{I}_H(B, D) = \{K_1,..., K_s\}$. Let L_i be the unique subset of K_i satisfying $\pi(L_i) = C$—such L_i exists because $C \subset D = \pi(K_i)$. In view of Theorem 2.1 (iii) $L_i \in \bar{\bar{\mathscr{F}}}$ holds.

Since $A \subset C$ and $G \subset H$, $G \subset L_i$ holds. To conclude the proof we must show that the L_i's are distinct.

Consider $\pi(K_i \cap K_j)$ for $i \neq j$. Since $K_i \neq K_j$, it is a proper subset of D, containing B. As D covers B, $\pi(K_i \cap K_j) = B$ follows. Thus $K_i \cap K_j = H$. Consequently $L_i \cap L_j \subseteq H$. But $\pi(L_i) = \pi(L_j) = C$ and $C \not\subset \pi(H) = B$ proving $L_i \neq L_j$. ∎

3. The Lower Bound in Theorem 1.2

The construction we use here was given in [4]. Since we need it in the proof of the upper bound, we repeat it shortly.

Let b be an integer and Z a set of cardinality $a_0 + a_1 b + a_2\binom{b}{2} + \cdots + a_d\binom{b}{d}$ which we consider as the disjoint union of a_i copies of $\binom{[1, b]}{i}$, $i = 0,..., d$. For $A \subseteq [1, b]$, let $\varphi(A)$ be the corresponding subset of Z

with $|\varphi(A)|=\sum_{i=0}^{d} a_i\binom{|A|}{i}$. It is very easy to check that for $A, B\subseteq[1,b]$, $\varphi(A\cap B)=\varphi(A)\cap\varphi(B)$ holds. Thus for s arbitrary the family $\{\varphi(A): A\subset[1,b],\ |A|\leqslant s\}$ is a closed $\{p(0),..., p(s-1)\}$-system showing $m(p(b), p(s), \{p(0),..., p(s-1)\})\geqslant\binom{b}{s}$. By choosing $b=\Omega(n^{1/a})$ the desired lower bound follows.

4. Proof of the Upper Bound Part of Theorem 1.2

W.l.o.g. let $\mathscr{F}$ be a canonical closed L-system, $L=\{p(0),..., p(s-1)\}$, let $\mathscr{A}$ be the sample family on $[1, p(s)]$. Also, let $\mathscr{P}=\mathscr{P}^{(s)}$ be the sample family from our construction (the point set Y of $\mathscr{P}^{(s)}$ is the disjoint union of a_0 copies of the singleton $\binom{[1,s]}{0}$, a_1 copies of $\binom{[1,s]}{1}$,...., a_d copies of $\binom{[1,s]}{d}$), say

$$Y=\bigcup_{i=0}^{d}\ \bigcup_{1\leqslant j\leqslant a_i} Y^i_{a_i}.$$

For a subset $B\subset[1,s]$ denote by $\rho(B)$ the subset of size $p(|B|)$ of Y which is the union of the corresponding subsets of $Y^i_{a_i}$. Then

$$\mathscr{P}=\{\rho(B) : B\subseteq[1,s]\}.$$

Note that as a lattice $\mathscr{P}$ is isomorphic to $A^{[1,s]}$, in particular, all maximal chains have the same size s.

We are going to show that $\mathscr{A}$ can be embedded into $\mathscr{P}$, that is, there exists a 1–1 map φ: $[1, p(s)]\to Y$ so that $\varphi(A)\in\mathscr{P}$ holds for all $A\in\mathscr{A}$.

Call a subset $C\subset[1,p(s)]$ an *atom* if $C\cap A\neq\varnothing$ implies $C\subseteq A$ for all $A\in\mathscr{A}$. An element $x\in A\in\mathscr{A}$ is a *generic point* for A if for all $B\in\mathscr{A}$, $x\in B$ implies $A\subseteq B$.

Note that if C is an atom, $|C|=a_1$, then $\mathscr{A}\cup\{C\}$ will be a closed family. Adding atoms of size a_1 successively one obtains finally a family $\mathscr{A}'$ to which one cannot add atoms of size a_1. When proving the imbeddability we may assume $\mathscr{A}=\mathscr{A}'$.

Call a set $A\in\mathscr{A}$ with $|A|=p(i)$ *filled* if it contains i atoms of size a_1. For a filled set let $D(A)$ be the union of its atoms, $|D(A)|=ia_1$.

Claim 4.1. *All $A\in\mathscr{A}$ are filled.*

Proof of Claim 4.1. The claim clearly holds if $|A|=a_1$. Let A be a counterexample of minimal size $|A|=p(i)$. Set $\mathscr{B}=\{B\in\mathscr{A},\ B\subsetneqq A\}$.

Define $M=M(A)=\bigcup_{B\in\mathscr{B}} B$. Since $A-M$ is an atom,

$$|A-M|<a_1 \text{ holds.}$$

Define also,

$$K = K(A) = \bigcup_{B \in \mathscr{B}} D(B).$$

Then $K \subset M$, K is the union of atoms of size a_1, thus

$$|K| = ja_1 \qquad \text{holds with some } j < i.$$

For definiteness let $C_1, \ldots, C_j$ be these atoms. For $B \in \mathscr{B}$ define $T(B) = \{v: C_v \subset B\}$. Since B is filled, $|T(B)| = \lfloor |B|/a_1 \rfloor$ holds. If for $B, B' \in \mathscr{B}$, $|T(B) \cap T(B') = t$ then $ta_1 \leqslant |B \cap B'| \leqslant ta_1 + |B - D(B)| < (t+1)a_1$ holds. Therefore $|B \cap B'| = p(t)$.

Consequently, the map $B \to \rho(T(B))$ defines an embedding of $\mathscr{B}$ into $\mathscr{P}^{(j)}$ (here we used Proposition 2.2). In particular

$$|M| = \left| \bigcup_{B \in \mathscr{B}} B \right| \leqslant \left| \bigcup_{P \in \mathscr{P}^{(j)}} P \right| = p(j). \tag{1}$$

Thus $p(i) = |A| < p(j) + a_1 < p(i)$, a contradiction. ∎

Applying the claim to $[1, p(s)] \in \mathscr{A}$, we see that there are s atoms $C_1, \ldots, C_s$ of size a_1 in it. Define for all $A \in \mathscr{A}$, $T(A) = \{v: C_v \subset A\}$. Then $A \to \rho(T(A))$ gives the desired embedding of $\mathscr{A}$ into $\mathscr{P}^{(s)}$.

Note that this implies that every $A \in \mathscr{A}$ with $|A| = p(d)$ has a generic point (no $B \in \mathscr{A}$ with $B \subsetneq A$ can contain elements which are mapped on a copy of $\binom{[1,s]}{d}$). In particular, if $s = d$, then $|\mathscr{F}| \leqslant n$ follows and this will be the starting case of the induction.

Also, we can add to $\mathscr{F}$ all subsets of members of $\mathscr{F}$ which have projection in $\mathscr{P}$, i.e., the family

$$\mathscr{H} = \{H: \pi(H) \in \mathscr{P}, \exists F \in \mathscr{F}, H \subseteq F\}$$

is still closed.

Suppose $s > d$ and the upper bound is proved for $s - 1$. Define

$$\mathscr{H}_1 = \{H \in \mathscr{H}: |H| = p(s-1)\}.$$

By induction

(4) $|\mathscr{H}_1| \leqslant \Omega(n^{(s-1)/d})$ holds.

Set $\mathscr{H}_0^{(0)} = \mathscr{F}$, $\mathscr{H}_1^{(0)} = \mathscr{H}_1$. If $\mathscr{H}_0^{(i)}$ and $\mathscr{H}_1^{(i)}$ are defined and some member $G \in \mathscr{H}_1^{(i)}$ is contained in less than $n^{1/d}$ members of $\mathscr{H}_0^{(i)}$ then define $\mathscr{H}_1^{(i+1)} = \mathscr{H}_1^{(i)} - \{G\}$, $\mathscr{H}_0^{(i+1)} - \{H \in \mathscr{H}_0^{(i)}: G \subset H\}$ and continue. In view of (4) altogether less than $n^{1/d} O(n^{(s-1)/d}) = O(n^{s/d})$ sets are thrown away. Thus it will be sufficient to prove the upper bound for the remaining family, which we denote, by abuse of notation, by $\mathscr{F}$. Define

$$\mathscr{H}_i = \{H \in \mathscr{H} : |H| = p(s-i)\}, \qquad 0 \leqslant i \leqslant s.$$

CLAIM 4.2. *Suppose* $G \in \mathscr{H}_i$, $i > 0$, $A, C \in \mathscr{P}$, $\pi(G) = A \subset C$, $|C| = p(s-i+1)$. *Then* $|\mathscr{I}_G(A, C)| \geqslant n^{1/d}$.

Proof. Apply induction on i. The case $i = 1$ is fine by the construction. Let $A_0(C_0)$ be the subset of $[1, s]$ satisfying $\varphi(A_0) = A$ ($\varphi(C_0) = C$), respectively. Of course, $|A_0| = |C_0| - 1 = s - i$. Let j be an arbitrary element of $[1, s] - C_0$. Define $B = \varphi(A_0 \cup \{j\})$, $D = \varphi(C_0 \cup \{j\})$. Take $G, H \in \mathscr{H}$ with $G \subset H$, $\pi(G) = A$, $\pi(H) = B$. By the induction hypothesis and by Lemma 2.3 we have

$$|\mathscr{I}_G(A, C)| \geqslant |\mathscr{I}_H(B, D)| \geqslant n^{1/d}. \quad \blacksquare$$

CLAIM 4.3. *For* $1 \leqslant i \leqslant s-1$ *one has* $|\mathscr{H}_i| \leqslant (s/\binom{s}{i})\, |\mathscr{H}_1|\, n^{-(i-1)/d}$.

Proof. The statement is trivial for $i = 1$. Suppose it has been proved for $i - 1$. Consider the bipartite graph with vertex set $\mathscr{H}_i$, $\mathscr{H}_{i-1}$ with (G, H) forming an edge if $G \in \mathscr{H}_i$, $H \in \mathscr{H}_{i-1}$, and $G \subset H$. Now the degree of H is $s - i + 1$ while the degree of G is at least $in^{1/d}$. This implies

$$|\mathscr{H}_i| \leqslant \frac{s-i+1}{i} n^{-1/d} |\mathscr{H}_{i-1}| \leqslant |\mathscr{H}_1| \frac{\binom{s}{i-1}}{s} \frac{s-i+1}{i} n^{-(i-1)/d}$$

$$= |\mathscr{H}_1| \frac{\binom{s}{i}}{s} n^{-(i-1)/d}.$$

Now the upper bound is immediate: for an arbitrary $F \in \mathscr{F} = \mathscr{H}_0$ let $A_1(F), ..., A_s(F)$ be the s atoms in F, i.e., $\pi(A_i(F) = \varphi(\{i\})$. Then no other member F' of $\mathscr{F}$ contains $A_1(F), ..., A_s(F)$ because otherwise $|F \cap F'| \geqslant sa_1 > p(s-1)$, a contradiction. Consequently,

$$|\mathscr{F}| \leqslant \binom{|\mathscr{H}_{s-1}|}{s} = O(n^{s/d}). \quad \blacksquare$$

5. CONCLUDING REMARKS

First of all let us mention an old conjecture of Erdös and Simonovits which has an apparent similarity with Theorem 1.1.

For a class $\mathscr{C}$ of graphs let $\operatorname{ex}(n, \mathscr{C})$ denote the maximum number of edges in a graph with no subgraphs isomorphic to a member of $\mathscr{C}$.

Conjecture 5.1 [2]. For every rational number r, $1 < r < 2$, there exists a finite class $\mathscr{C}$ of bipartite graphs so that $\operatorname{ex}(n, \mathscr{C}) = \Theta(n^r)$ holds.

Suppose $p(x) = \sum_{i=0}^{d} a_i \binom{x}{i}$, where $a_d \geqslant 1$, a_i is integer for $i = 0, \ldots, d-1$. Then there exists a smallest non-negative integer $t = t(p)$ so that substituting $y = x - t$ into $p(x)$ will give a polynomial $q(y) = p(y+t) = \sum_{i=0}^{t} b_i \binom{y}{i}$ with $b_d = a_d$ and $b_i \geqslant 0$, integer.

Conjecture 5.2. Suppose $p(x) = \sum_{i=0}^{d} a_i \binom{x}{i}$ and $t = t(p)$ are as above. Then for $s \geqslant s_0(p)$ one has

$$m(n, p(s), \{p(j): 0 < j < s\}) = \Theta(n^{(s-t)/d}).$$

We can prove the above conjecture in several special cases not covered by Theorem 1.2 and can obtain as well a general upper bound of the form $O(n^{a(p) + s/d})$, where $a(p)$ is a constant depending only on the polynomial p.

References

1. M. Deza, P. Erdös, and P. Frankl, Intersection theorems for systems of finite sets, *Proc. London Math. Soc.* (3) **36** (1978), 369–384.
2. P. Erdös, On the combinatorial problems which I would most like to see solved, *Combinatorica* **1** (1981), 25–42.
3. P. Erdös and D. J. Kleitman, Coloring graphs to maximize the proportion of multicolored k-edges, *J. Combin. Theory* **5** (1968), 164–169.
4. P. Frankl, Constructing finite sets with given intersection, *Ann. Discrete Math.* **17** (1983), 289–291.
5. P. Frankl, Families of finite sets with three intersections, *Combinatorica* **4** (1984), 141–148.
6. P. Frankl and Z. Füredi, On hypergraphs without two edges intersecting in a given number of vertices, *J. Combin. Theory Ser. A* **36** (1984), 230–236.
7. P. Frankl and Z. Füredi, Forbidding just one intersection, *J. Combin. Theory Ser. A* **39** (1985), 160–176.
8. Z. Füredi, On finite set-systems whose every intersection is a kernel of a star, *Discrete Math.* **47** (1983), 129–132.
9. Z. Füredi, Families of finite sets with 3 intersections, *Combinatorica*, in press.
10. L. Pyber, An extension of a Frankl–Füredi theorem, *Discrete Math.* **52** (1984), 253–268.
11. V. Rödl, A packing and covering problem, *European J. Combin.* **6** (1985), 69–78.
12. D. K. Ray-Chaudhuri and R. M. Wilson, On t-designs, *Osaka J. Math.* **12** (1975), 737–744.
13. P. Frankl and R. M. Wilson, Intersection theorems with geometric consequences, *Combinatorica* **1** (1981), 357–368.
14. P. Frankl and V. Rödl, Forbidden intersections, *Transactions AMS*, in press.