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Some Intersection Theorems for Ordered Sets and Graphs

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A classical topic in combinatorics is the study of problems of the following type: What are the *maximum* families **F** of subsets of a finite set with the property that the intersection of any two sets in the family satisfies some specified condition?

Typical restrictions on the intersections $F \cap F'$ of any F and F' in **F** are:

(i) $F \cap F' \neq \emptyset$, where all $F \in \mathbf{F}$ have k elements (Erdös, Ko, and Rado (1961)).

(ii) $|F \cap F'| \geqslant j$ (Katona (1964)).

In this paper, we consider the following general question: For a given family **B** of subsets of $[n] = \{1, 2, ..., n\}$, what is the largest family **F** of subsets of $[n]$ satsifying

$$F, F' \in \mathbf{F} \Rightarrow F \cap F' \supseteq B \qquad \text{for some } B \in \mathbf{B}.$$

Of particular interest are those **B** for which the maximum families consist of so-called "kernel systems," i.e., the family of all *supersets* of some fixed set in **B**. For example, we show that the set of all (cyclic) translates of a block of consecutive integers in $[n]$ is such a family. It turns out rather unexpectedly that many of the results we obtain here depend strongly on properties of the well-known *entropy* function (from information theory).

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I. Introduction

A classical topic in combinatorics is the study of questions of the following type: What are the *maximum* families $\mathbf{F}$ of subsets of a finite set with the property that the intersection of any two sets in the family satisfies some specified condition?

Typical restrictions on the intersections based on F and F' in $\mathbf{F}$ are:

(i) $\bar{F} \cap F' \neq \varnothing$, where $\bar{F}$ denotes the complement of F [16];

(ii) $F \cap F' \neq \varnothing$, where all $F \in \mathbf{F}$ have k elements [3];

(iii) $|F \cap F'| \geqslant j$ [8].

Good surveys of our current state of knowledge in this area can be found in [6, 7, 9, 17], in addition to the results in [5, 12, 13, 14, 18].

In this note we investigate the following question: For a given family $\mathbf{B}$ of subsets of $[n] := \{1, 2, \ldots, n\}$, what is the largest family $\mathbf{F}$ of subsets of $[n]$ satisfying:

$$F, F' \in \mathbf{F} \Rightarrow F \cap F' \supseteq B \qquad \text{for some } B \in \mathbf{B}. \tag{1}$$

In particular, let $v(\mathbf{B})$ denote the cardinality of the largest family $\mathbf{F}$ satisfying (1).

An Easy Example

As a prelude to the general results, we first consider a simple special case. For $\mathbf{B} = \mathbf{B}_2$ we take the set of all pairs $\{i, i+1\}$, $1 \leqslant i < n$. For the family $\mathbf{B}_2$ we prove

$$v(\mathbf{B}_2) = 2^{n-2}. \tag{2}$$

Proof of (2): Define S_i, $i = 1, 2$, by

$$S_i := \{j \in [n] : j \equiv i \pmod 2\}. \tag{3}$$

Observe that for all i and all $B \in \mathbf{B}$

$$S_i \cap B \neq \varnothing. \tag{4}$$

Suppose $\mathbf{F} \subseteq 2^{[n]}$ satisfies (1). Define the induced families $\mathbf{F}(S_i)$ by

$$\mathbf{F}(S_i) := \{F \cap S_i : F \in \mathbf{F}\}, \qquad i = 1, 2. \tag{5}$$

Note that if $G, G' \in \mathbf{F}(S_i)$ then

$$\begin{aligned} G \cap G' &= (F \cap S_i) \cap (F' \cap S_i) \qquad \text{for some } F, F' \in \mathbf{F} \\ &= F \cap F' \cap S_i \neq \varnothing \end{aligned} \tag{6}$$

since $F \cap F' \supseteq B'$ for some $B' \in \mathbf{B}$ and by construction $S_i \cap B \neq \emptyset$ for *every* $B \in \mathbf{B}$. Thus, for $i = 1, 2$, $\mathbf{F}(S_i)$ is a family of subsets of S_i with the property that no two sets in $\mathbf{F}(S_i)$ are disjoint. This implies that

$$|\mathbf{F}(S_i)| \leqslant \tfrac{1}{2} \cdot 2^{|S_i|} \tag{7}$$

since we cannot have a set X and its complement $S_i - X$ both in $\mathbf{F}(S_i)$. Since any set $F \in \mathbf{F}$ is determined by its intersections $F \cap S_i$, $i = 1, 2$, then by (7)

$$|\mathbf{F}| \leqslant \tfrac{1}{2} \cdot 2^{|S_1|} \cdot \tfrac{1}{2} \cdot 2^{|S_2|} = \tfrac{1}{4} \cdot 2^{|S_1| + |S_2|} = 2^{n-2}. \tag{8}$$

On the other hand, for the family $\mathbf{F}'$ given by $\mathbf{F}' = \{X \subseteq [n]\colon \{1, 2\} \subseteq X\}$, we have

$$F \cap F' \subseteq \{1, 2\} \in \mathbf{B} \qquad \text{for all } F, F' \in \mathbf{F}'$$

and

$$|\mathbf{F}'| = 2^{n-2}.$$

This proves (2). ∎

Note that the content of (2) is just that no family satisfying (1) for B_2 can have more sets than can be achieved in a trivial way, i.e., by taking all subsets of $[n]$ containing a fixed $B_0 \in \mathbf{B}$. In general, we call such a family a *kernel system* with kernel B_0. Of course, (2) does not imply that every maximum family $\mathbf{F}$ is a kernel system.

In what follows, we will be especially interested in those families $\mathbf{B}$ for which $v(\mathbf{B})$ is attained by kernel systems. This seems to be true, for example, for any family $\mathbf{B}$ formed by taking the (cyclic) translates of a fixed set in $[n]$ (although we do not prove this).

II. Partitions of $[n]$

Although we study set *intersections* here, it is sometimes useful to consider the following variation of set intersection, namely, the complement of the symmetric difference of two sets, defined for $X, Y \subseteq [n]$ by

$$X \nabla Y := (X \cap Y) \cup (\bar{X} \cap \bar{Y}) = \overline{X \Delta Y}$$

where $\bar{X} = [n] - X$. For a given family $\mathbf{B}$ of subsets of $[n]$, let $\bar{v}(\mathbf{B})$ denote the cardinality of the largest family $\mathbf{F}$ satisfying

$$F, F' \in \mathbf{F} \Rightarrow F \nabla F' \supseteq B \qquad \text{for some } B \in \mathbf{B}.$$

Obviously $v(\mathbf{B}) \leqslant \bar{v}(\mathbf{B})$.

Slightly less obvious is the following.

FACT 1.

$$v(\mathbf{B}) = \bar{v}(\mathbf{B}) \text{ for all } \mathbf{B}. \tag{9}$$

Sketch of proof. Assume $\mathbf{F}$ is a maximum ∇-family for $\mathbf{B}$, i.e., $|\mathbf{F}| = \bar{v}(\mathbf{B})$. Select, if possible, some element $t \in [n]$ so that for some $F \in \mathbf{F}$, $F \cup \{t\} \notin \mathbf{F}$. Replace *all* such $F \in \mathbf{F}$ (simultaneously) by $F \cup \{t\}$, forming a new family $\mathbf{F}'$. It is easy to check that $\mathbf{F}'$ is also a ∇-family for $\mathbf{B}$, and $|\mathbf{F}'| = |\mathbf{F}|$. Continue this process as long as possible, finally forming the family $\mathbf{F}^*$, which has the property that for any $F \in \mathbf{F}'$, if $t \notin F$ then $F \cup \{t\} \in \mathbf{F}^*$. Thus, $\mathbf{F}^*$ is an upper ideal in the lattice of subsets $2^{[n]}$, i.e., $[n] \supseteq G \supset F \in \mathbf{F}^*$ implies $G \in \mathbf{F}^*$. It now follows easily that $\mathbf{F}^*$ is in fact an $\cap$-family for $\mathbf{B}$, i.e., $F, F' \in \mathbf{F}^*$ implies $F \cap F' \supseteq B$ for some $B \in \mathbf{B}$. Since $|\mathbf{F}^*| = |\mathbf{F}| = \bar{v}(\mathbf{B})$ then we have $v(\mathbf{B}) \geqslant (\mathbf{B})$ which implies (9). ∎

THEOREM 1. *Suppose $[n] = S_1 \cup \cdots \cup S_k$ is a partition of $[n]$ into k non-empty subsets. For $X \subseteq [n]$, define $f(X) = \{i : S_i \cap X \neq \emptyset\}$. Let $\mathbf{B}$ be a family of subsets of $[n]$ and define $\mathbf{B}^* = \{f(X) : X \in \mathbf{B}\} \subseteq 2^{[k]}$. Then we have*

$$v(\mathbf{B}) \leqslant v(\mathbf{B}^*)\, 2^{n-k}. \tag{10}$$

Proof. By Fact 1, it is enough to prove

$$\bar{v}(\mathbf{B}) \leqslant \bar{v}(\mathbf{B}^*)\, 2^{n-k}. \tag{10$'$}$$

Let $\mathbf{F}$ be a ∇-family for $\mathbf{B}$, i.e., $F, F' \in \mathbf{F}$ implies $F \nabla F' \supseteq B$ for some $B \in \mathbf{B}$. Also, let $\mathbf{W}$ denote the subspace of $2^{[n]}$ (considered as an n-dimensional vector space under the operation Δ) generated by the S_i. Partition $2^{[n]}$ into cosets $C_i \Delta \mathbf{W}$, $1 \leqslant i \leqslant 2^{n-k}$. It will suffice to show that each coset $C \Delta \mathbf{W}$ contains at most $\bar{v}(\mathbf{B}^*)$ elements of $\mathbf{F}$. Since $(X \Delta C) \nabla (Y \Delta C) = X \nabla Y$, it suffices to prove that $\mathbf{W}$ contains at most $\bar{v}(\mathbf{B}^*)$ elements of $\mathbf{F}$. Note that f is a one-to-one map of $\mathbf{W}$ to $2^{[k]}$ and it is easily checked that $f(X \nabla Y) = f(X) \nabla f(Y)$. Hence, for $F, F' \in F \cap \mathbf{W}$, we have

$$f(F) \nabla f(F') = f(F \nabla F') \supseteq f(B) \in \mathbf{B}^*$$

for some $B \in \mathbf{B}$. Therefore, $\mathbf{W}$ contains at most $\bar{v}(\mathbf{B}^*)$ elements of $\mathbf{F}$ and Theorem 1 is proved. ∎

As an immediate consequence of Theorem 1, we have the following result, which has also been obtained independently by Faudree, Schelp, and Sós [4].

THEOREM 2. *Suppose* $[n] = S_1 \cup \cdots \cup S_k$ *is a partition of* $[n]$ *into* k *non-empty sets, and* $\mathbf{B} \subseteq 2^{[n]}$ *is a family with the property that for some* j, $1 \leqslant j \leqslant k$, *each* $B \in \mathbf{B}$ *intersects at least* j *of the* S_i, $1 \leqslant i \leqslant k$. *Then*

$$v(\mathbf{B}) \leqslant 2^{n-k} g(k, j) \tag{11}$$

where

$$g(k, j) = \begin{cases} \sum_{t \geqslant v} \binom{k}{t} & \text{if } k + j = 2v, \\ \sum_{t \geqslant v} \binom{k}{t} + \binom{k-1}{v-1} & \text{if } k + j = 2v - 1. \end{cases}$$

Proof. By Fact 1 and Theorem 1 we have

$$v(\mathbf{B}) \leqslant v(\mathbf{B}^*)\, 2^{n-k}.$$

Since $\mathbf{B}^*$ is a family of subsets of $[k]$ each containing at least j elements, then a result of Kleitman [10] (also see Ahlswede and Katona [1]) implies $v(\mathbf{B}^*) \leqslant g(k, j)$. This proves Theorem 2. ∎

In order to apply Theorem 2 to a particular family $\mathbf{B}$, we need to choose a suitable partition $[n] = \bigcup_{i=1}^{k} S_i$ (which determines some maximal value of j associated with it). It is always possible to use trivial partitions and indeed, these are sometimes optimal. For example, for $[n] = S_1$ we have $k = 1$, $j = 1$, $g(k, j) = 1$, and so,

$$v(\mathbf{B}) \leqslant 2^{n-1}$$

for any family $\mathbf{B}$ (which does not contain $\varnothing$). Of course, for $\mathbf{B} = \{\{1\}\}$, for example, the family $\mathbf{F} = \{X \subseteq [n]: 1 \in X\}$ shows that this bound can be achieved.

On the other hand, suppose we take for $\mathbf{B}$ the family of all j-element subsets of $[n]$. For the (maximum) partition $[n] = \bigcup_{i=1}^{n} S_i$ with $S_i = \{i\}$, the condition that $F \cap F' \supseteq B$ for some $B \in \mathbf{B}$ is equivalent to $|F \cap F'| \geqslant j$, i.e., $F \cap F'$ intersects at least j of the S_i. In this case, it follows that

$$v(\mathbf{B}) \leqslant g(n, j). \tag{12}$$

In fact, a theorem of Katona in [6] shows that we actually have equality in this case as well.

For any family $\mathbf{B}$, if $\mu(\mathbf{B})$ denotes the cardinality of a minimum set B_0 in $\mathbf{B}$ then by forming a maximum kernel system with kernel B_0, we have

$$v(\mathbf{B}) \geqslant 2^{n-\mu(\mathbf{B})}. \tag{13}$$

In order to obtain the exact value of $v(\mathbf{B})$ using (11) and (13) it is necessary that

$$g(k, j) = 2^{k-\mu(\mathbf{B})}. \tag{14}$$

As an illustration of (14) let $\mathbf{B}(t)$ denote the family of n t-sets of $[n]$ formed by choosing (cyclically) t *consecutive* elements of Z_n. We claim that if $n \geqslant t^2 - t$ then it is always possible to partition $[n]$ into $t+1$ subsets S_i, $1 \leqslant i \leqslant t+1$, so that the distance (in the corresponding n-cycle C_n) between any s and $s' \in S_i$ is at least t. (An easy way to do this is to write $n = ut + v$, $0 \leqslant v < t$, write down the string $1, 2, ..., t, 1, 2, ..., t, ..., 1, 2, ..., t$ of u copies of $1, 2, ..., t$, and then "insert" v copies of $t+1$ which are all at distance at least t from one another; this now defines a partition of $[n]$ into $t+1$ subsets with the desired property.) Since any $B \in \mathbf{B}(t)$ intersects at least t of the $t+1$ S_i's then the appropriate values of k and j to use in (11) are $k = t+1$, $j = t$. However, since $\mu(\mathbf{B}(t)) = t$ then

$$g(t+1, t) = 2 = 2^{t+1-t}$$

i.e., (14) holds, and consequently

$$v(\mathbf{B}(t)) = 2^{n-t}$$

when $n \geqslant t^2 - t$. In the next section we will extend this to all values of $n > t$.

III. On Translates of a Block

In this section we will show that for any $t < n$, the collection $\mathbf{B}(t) \subseteq 2^{[n]}$ consisting of a kernel system is the largest intersection family for $\mathbf{B}(t)$ which consists of all cyclic translates of t consecutive numbers. First we will make some easy observations.

Fact 2. *Let $r \leqslant n/2$. Let X be a subset of the n-cycle C_n such that for $u, v \in X$, the distance between u and v in C_n is no more than $r-1$. Then $|X| \leqslant r$.*

Proof. Note that each vertex v in X excludes an interval, denoted by $I(v)$, of length $n+1-2r \geqslant 1$. We will encounter the $I(v)$, $v \in X$, in the following order. Choose a fixed vertex $v = v_1$. In general, v_i is defined to be the vertex in $X - \{v_i, ..., v_{i-1}\}$ closest to $\{v_1, ..., v_{i-1}\}$ (in case of a tie, choose arbitrarily). Now $I(v_1)$ eliminates $n+1-2r$ vertices from C_n. Each additional $I(v_i)$ eliminates at least one more vertex from C_n. Hence the total number of excluded vertices is at least $n+1-2r+|X|-1$. These together with the $|X|$ points in X, total at most n. Therefore, $n+2|X|-2r \leqslant n$, i.e., $|X| \leqslant r$. ∎

THEOREM 3. *Suppose $t<n \leqslant 2t$. Let $\mathbf{B}'(t)$ consist of the cyclic translates of both $\{1, 2,..., t\}$ (mod n) together with $\{1, 2,..., t\}$ (mod$(n-1)$). Let $\mathbf{F}$ be a family of subsets of $[n]$ with the property that $F, F' \in \mathbf{F} \Rightarrow F \nabla F' \supseteq B$ for some $B \in \mathbf{B}'(t)$. Then we have $|\mathbf{F}| \leqslant 2^{n-t}$.*

Proof. Since $(X \Delta Z) \nabla (Y \Delta Z) = X \nabla Y$, we may consider $\mathbf{F}' = \{F \Delta F_0: F' \in \mathbf{F}\}$ for a fixed subset F_0 in $\mathbf{F}$. Thus, $\mathbf{F}'$ contains the empty set and $F, F' \in \mathbf{F}' \Rightarrow F \nabla F' \supseteq B$ for some $B \in \mathbf{B}'(t)$. Furthermore $|\mathbf{F}| = |\mathbf{F}'|$ since $F' \neq F''$ if and only if $F_0 \Delta F' \neq F_0 \Delta F''$. It suffices to show $|\mathbf{F}'| \leqslant 2^{n-t}$. Let U denote the set $\bigcup_{F \in \mathbf{F}'} F = \{x: x \in F \in \mathbf{F}'\}$. Suppose $i, j \in U$, $i, j \neq n$, and $[n]$ (mod n) is viewed as an n-cycle. Then we claim the distance between i and j is at most $n-t-1$. Assume the contrary. First, suppose i and j both are in $F \in \mathbf{F}'$. Then $F \nabla \varnothing = \bar{F}$ does not contain i and j and cannot contain a cyclic translate of $\{1,..., t\}$ (mod n) or (mod$(n-1)$), which is a contradiction. Suppose i and j are in different subsets F, F' in $\mathbf{F}$. Then again we have $i, j \notin F \nabla F'$ and $F \nabla F'$ cannot contain a cyclic translate of $\{1,..., t\}$ (mod n) or (mod $(n-1)$). Hence, by Fact 2, U contains at most $n-t$ elements of $[n-1]$ or $n-t+1$ elements of $[n]$. Clearly, $\mathbf{F}' \subseteq 2^U$. Hence if $|U| \leqslant n-t$, then $|\mathbf{F}'| \leqslant 2^{n-t}$. Suppose $|U| = n-t+1$. Let X be a subset of U and $X' = U - X$. Since $(X \nabla X') \cap U = \varnothing$, then $|X \nabla X'| \leqslant n - |U| \leqslant t-1$. Therefore $X \nabla X'$ cannot contain a translate of $\{1,..., t\}$ and X, X' cannot both be in U. Hence $\mathbf{F}'$ contains at most half of the subsets in 2^U, i.e., $|\mathbf{F}'| \leqslant \frac{1}{2} \cdot 2^{n-t+1} = 2^{n-t}$, which completes the proof of Theorem 3. ∎

THEOREM 4. *Let $\mathbf{F}$ be a family of subsets of $[n]$ such that $F, F' \in \mathbf{F} \Rightarrow F \nabla F'$ contains some cyclic translate of $\{1,..., t\}$. Then $|\mathbf{F}| \leqslant 2^{n-t}$.*

Proof. By Theorem 3 we only have to consider the case that $n > 2t$. We can write any n as $im + j(m-1)$ for some m, $t < m \leqslant 2t$, where i, j are non-negative and i is nonzero. Partition $[n]$ into m subsets S_i, $1 \leqslant i \leqslant m$, so that the distance between any s and $s' \in S_i$ is at least $m-1$. Using Theorem 1 we have $v(\mathbf{B}(t)) \leqslant v(\mathbf{B}^*(t)) 2^{n-m}$. Theorem 3 then implies $v(\mathbf{B}^*(t)) \leqslant 2^{m-t}$. Therefore we have $v(\mathbf{B}(t)) \leqslant 2^{n-t}$ as desired. ∎

As an immediate consequence we have the following:

THEOREM 5. *Let $\mathbf{F}$ be a family of subsets of $[n]$ such that $F, F' \in \mathbf{F} \Rightarrow F \cap F'$ contains some cyclic translate of $\{1,..., t\}$. Then $|\mathbf{F}| \leqslant 2^{n-t}$.*

We remark that the kernel system formed by all supersets of $\{1,\dots, t\}$ has 2^{n-t} subsets and hence is a largest possible family.

IV. On Translates of a Fixed Set

We have shown that kernel systems form the best intersection families when **B** consists of all the (cyclic) translates of $\{1, 2\dots, t\}$. It appears that this may hold much more generally.

Conjecture 1. If $\mathbf{B}(X)$ consists of the set of all the cyclic translates of a fixed set $X \subseteq [n]$ then

$$v(\mathbf{B}(X)) = 2^{n-|X|}. \tag{15}$$

Of course a kernel system with kernel X shows that $v(\mathbf{B}(X))$ is at least as large as $2^{n-|X|}$. Although we could not prove this conjecture, the following results provide some evidence in support of the conjecture.

Let $\mathbf{B}_n(X)$ denote the set of all n cyclic translates of X in $[n]$ and let $\mathbf{B}_n^*(X)$ denote the subset of all translates of X. It follows immediately that

$$2^{n-|X|} \leqslant v(\mathbf{B}_n^*(X)) \leqslant v(\mathbf{B}_n(X)). \tag{16}$$

Since $v(\mathbf{B}_{n+1}^*(X)) \geqslant 2v(\mathbf{B}_n^*(X))$, $v(\mathbf{B}_n^*(X))/2^n$ is non-decreasing in n. Consequently,

$$r^*(X) := \lim_{n\to\infty} \frac{v(\mathbf{B}_n^*(X))}{2^n} \qquad \text{exists.} \tag{17}$$

If X is a block of t consecutive integers, then $r^*(X) = 2^{-t}$. We will prove the following:

Theorem 6.

$$r(X) := \lim_{n\to\infty} \frac{v(\mathbf{B}_n(X))}{2^n} \quad \text{exists}$$

and

$$r(X) = r^*(X).$$

Proof. From (16) and (17) we have $v(\mathbf{B}_n(X))/2^n \geqslant v(\mathbf{B}_n^*(X))/2^n$ and $\lim_{n\to\infty} v(\mathbf{B}_n(X))/2^n = r^*(X)$. Hence, it clearly suffices to show that for any $\varepsilon > 0$ there exists n_0 so that for all $n > n_0$ we have

$$\frac{v(\mathbf{B}_n(X))}{2^n} \leqslant r^*(X) + \varepsilon.$$

To prove this, it is enough to show for an intersection family $\mathbf{F}$, we can find a set H of h consecutive integers, where $X \subseteq [h]$, such that

$$|\{F \in \mathbf{F}: F \cap H = \emptyset\}| \geq |\mathbf{F}|/2^h(1+\varepsilon). \tag{18}$$

To see this, note that $|\{F \in \mathbf{F}: F \cap H = \emptyset\}| \leq v(\mathbf{B}^*_{n-h}(X))$. Combining this with (18) we get

$$\frac{|\mathbf{F}|}{2^n} \leq (1+\varepsilon)\frac{v(\mathbf{B}^*_{n-h}(X))}{2^{n-h}} \leq r^*(X) + \varepsilon'. \tag{19}$$

We only have to consider $\mathbf{F}$ with

$$|\mathbf{F}| \geq 2^{n-h}. \tag{20}$$

Now we partition $[n]$ into $m = \lceil n/h \rceil$ blocks, i.e., $[n] = S_1 \cup S_2 \cup \cdots \cup S_m$, where $|S_m| \leq h$ and S_i, $i \neq m$, is a set of h consecutive numbers. We consider a random variable X assuming values in $\mathbf{F}$ so that each element of $\mathbf{F}$ is equally likely. For $1 \leq i \leq m$, let $X_i = X \cap S_i$ be the associated random variable taking values in $F_i = \{F \cap S_i : F \in \mathbf{F}\}$. We consider the entropy (see [11])

$$H(X) = \sum_F -p_F \log_2 p_F = \log_2 |\mathbf{F}|,$$

where $p_F := \text{Prob}(X = F)$ and the sum is taken over all $F \in \mathbf{F}$. Since $X_1, ..., X_m$ determine X, we have

$$H(X) \leq \sum_{i=1}^{m} H(X_i)$$

which with (20) implies

$$n - h \leq \sum_{i=1}^{m} H(X_i)$$

i.e.,

$$\sum_{i=1}^{m} (|S_i| - H(X_i)) \leq h.$$

Therefore there exists an i, say $i = 1$, such that

$$|S_1| - H(X_1) \leq \frac{h}{m-1} < \frac{h^2}{n-2h}. \tag{21}$$

Suppose $\text{Prob}(X_1=\varnothing)<1/(1+\varepsilon)2^h$. Then there exists $\delta=\delta(\varepsilon)>0$, such that

$$H(X_1)<|S_1|-\delta.$$

Therefore we have $h^2/(n-2h)\geqslant\delta$ which contradicts the fact that $n>2h+h^2/\delta$ for n sufficiently large. Thus, $\text{Prob}(X_1=\varnothing)\geqslant 1/(1+\varepsilon)2^h$ and (19) holds. This completes the proof of Theorem 6. ∎

Let $X+i$ denote the set $\{x+i(\text{mod}\, n)\colon x\in X\}$. We have the following.

THEOREM 7. *Suppose* $X\subseteq[n]$ *satisfies* $|X\cup(X+i)|>|X|+\log_2\binom{n}{2}$ *for all* $1\leqslant i<n$. *Then* $v(\mathbf{B}(X))=2^{n-|X|}$ *where equality holds only for kernel systems with kernel* $X+j$, *for some* j.

Proof. Let $\mathbf{F}\subseteq 2^{[n]}$ be a family of sets such that for any $F, F'\in\mathbf{F}$, $F\cap F'$ contains $X+i$ for some i. We distinguish two cases:

(i) There exists $F\in\mathbf{F}$ such that F contains only one translated copy, say $X+i$, of X. Then $X+i\subseteq F\cap F'$ holds for all $F'\in\mathbf{F}$, i.e., $\mathbf{F}$ is contained in the kernel system $\{F\subseteq[n]\colon X+i\subseteq F\}$, which has size $2^{n-|X|}$.

(ii) For every $F\in\mathbf{F}$ there are at least two different numbers i,j, $1\leqslant i<j\leqslant n$ such that $(X+i)\subset F$, $(X+j)\subset F$ hold. Since there are only $\binom{n}{2}$ choices for (i,j) there is a particular choice, say k,l, such that $(X+k)\cup(X+l)\subseteq F$ holds for at least $|\mathbf{F}|/\binom{n}{2}$ sets $F\in\mathbf{F}$.

However, $|((X+k)\cup(X+l))|=|X\cup(X+(l-k))|>|X|+\log_2\binom{n}{2}$, which means that

$$|\{F\subseteq[n]\colon((X+k)\cup(X+l))\subseteq F\}|<2^{n-|X|-\log_2\binom{n}{2}}=2^{n-|X|}/\binom{n}{2}.$$

Consequently $|\mathbf{F}|<\binom{n}{2}2^{n-|X|}/\binom{n}{2}=2^{n-|X|}$ holds and Theorem 7 is proved. ∎

If $c>2$ is a constant and $c\log_2 n<t<n-c\log_2 n$ then for almost all t-element subsets X of $[n]$, the assumption of Theorem 7 can be verified. Thus we have:

COROLLARY. *Given* c *and* t *satisfying* $c>2$, $c\log_2 n<t<n-c\log_2 n$, *then for almost all* t*-subsets* X *of* $[n]$ *we have*

$$v(\mathbf{B}(X))=2^{n-|X|}.$$

V. A Product Theorem

The following result, which seems to be a very useful tool in many extremal problems in combinatorics, was first proved by one of us (JBS) in

1978 (unpublished). A simpler related result was used by Bombieri [2] in connection with a question of J.-P. Serre.

THE PRODUCT THEOREM. *Let S be a finite set and let $A_1,\ldots, A_m$ be subsets of S such that every element of S is contained in at least k of $A_1,\ldots, A_m$. Let $\mathbf{F}$ be a collection of subsets of S and let $\mathbf{F}_i=\{F\cap A_i: F\in\mathbf{F}\}$ for $1\leqslant i\leqslant m$. Then we have*

$$|\mathbf{F}|^k\leqslant\prod_{i=1}^{m}|\mathbf{F}_i|.$$

Proof. Let X be a random variable assuming values in $\mathbf{F}$ so that each element of $\mathbf{F}$ is equally likely. For $1\leqslant i\leqslant m$, let $X_i=X\cap A_i$ be the associated random variable taking on values in $\mathbf{F}_i$. We will prove

$$kH(X)\leqslant\sum_{i=1}^{m}H(X_i). \tag{22}$$

If $k=1$, then $S=A_1\cup\cdots\cup A_m$. Thus, $X_1,\ldots, X_m$ determine X and consequently, $H(X)\leqslant\sum_{i=1}^{m}H(X_i)$ as desired. Now assume $k>1$. Let j denote the minimum number of A_i's whose union is S. Clearly $1\leqslant j\leqslant m$. We will prove (22) by induction on k and j. If $j=1$, say $A_1=S$, we have (by induction on k)

$$(k-1)H(X)\leqslant\sum_{i\neq 1}H(X_i)$$

and consequently

$$kH(X)\leqslant\sum_{i=1}^{m}H(X_i).$$

Suppose $j>1$. We may assume without loss of generality that $A_1\cup A_2\cup\cdots\cup A_j=S$. Let $A_1'=A_1\cup A_2$, $A_2'=A_1\cap A_2$. Clearly every element of S is in at least k of $A_1', A_2', A_3,\ldots, A_m$. By induction on j we have

$$kH(X)\leqslant\sum_{i\neq 1,2}H(X_i)+H(X')+H(X'')$$

where $X'=X\cap A_1'$ and $X''=X\cap A_2'$. Since it can be shown (by the convexity of H) that

$$H(X')+H(X'')\leqslant H(X_1)+H(X_2)$$

then we have $kH(X)\leqslant\sum_{i=1}^{m}H(X_i)$.

Now, $H(X)=\log_2|\mathbf{F}|$ and $H(X_i)\leqslant\log_2|\mathbf{F}_i|$. Thus we have

$$|\mathbf{F}|^k \leqslant \prod_{i=1}^{m} |\mathbf{F}_i|$$

and the proof is complete. ∎

The following inequalities of interest in information theory can be proved in a similar way. We will state these inequalities but omit the proofs.

$$\begin{aligned} H(X, Y, Z) &\leqslant \tfrac{1}{2}(H(X, Y)+H(Y, Z)+H(X, Z)) \\ &\leqslant H(X)+H(Y)+H(Z). \end{aligned}$$

More generally,

$$H(X_1,..., X_t) \leqslant \binom{t-1}{j-1}^{-1} \sum_{\{i_1,...,i_j\}\subseteq 2^{[t]}} H(X_{i_1},..., X_{i_j}).$$

We will now use the Product Theorem to prove two theorems on intersection families of graphs.

THEOREM 8. *Suppose* $\mathbf{F}$ *is a family of (labelled) subgraphs of the complete graph* K_n *such that for all* $F, F' \in \mathbf{F}$, $F\cap F'$ *does not contain any isolated vertices. Then*

$$|\mathbf{F}| \leqslant 2^{\binom{n}{2}-\frac{n}{2}}.$$

Proof. Choose A_i to be the (spanning) star at vertex v_i and let $E(A_i)$ denote the set of edges of A_i. Clearly every edge is in exactly two of $A_1,..., A_n$. Now $\mathbf{F}_i=\{F\cap A_i : F\in\mathbf{F}\}$ has the intersection property (i), i.e.,

$$(F\cap A_i)\cap(F'\cap A_i)=(F\cap F')\cap A_i\neq\varnothing.$$

Therefore $|\mathbf{F}_i|\leqslant 2^{|E(A_i)|-1}=2^{n-2}$ since for any $T\subset A_i$, T and A_i-T cannot both be in $\mathbf{F}_i$. Using the Product Theorem, we have

$$|\mathbf{F}|^2 \leqslant \prod_{i=1}^{n} |\mathbf{F}_i| \leqslant 2^{n(n-2)}.$$

Therefore

$$|\mathbf{F}| \leqslant 2^{n(n-2)/2} = 2^{\binom{n}{2}-\frac{n}{2}}$$

which proves Theorem 8. ∎

We note that the bound in Theorem 8 is best possible for the case of n even since one such family is a kernel system consisting of all subgraphs of K_n containing a fixed matching.

THEOREM 9. *Suppose* $\mathbf{F}$ *is a family of (labelled) subgraphs of* K_n *such that* $F \cap F'$ *contains a triangle for all* $F, F' \in \mathbf{F}$. *Then*

$$|\mathbf{F}| \leqslant 2^{\binom{n}{2} - 2}.$$

Proof. First, suppose n is even. We choose A_i, $1 \leqslant i \leqslant \frac{1}{2}\binom{n}{n/2}$, to be all possible disjoint unions of two complete (labelled) graphs of $n/2$ vertices each. Then $\mathbf{F}_i = \{F \cap A_i \mid F \in \mathbf{F}\}$ has the intersection property (i) since no triangle can be contained in a bipartite graph. Therefore

$$|\mathbf{F}_i| \leqslant 2^{|E(A_i)| - 1}.$$

Each edge of K_n is in exactly $\binom{n-2}{n/2}$ A_i's. Therefore by the Product Theorem we have

$$|\mathbf{F}|^{\binom{n-2}{n/2}} \leqslant 2^{1/2\left(2\binom{n/2}{2} - 1\right)\binom{n}{n/2}}$$

i.e.,

$$|\mathbf{F}| \leqslant 2^{\binom{n}{2} - n(n-1)/n(n/2-1)}$$
$$\leqslant 2^{\binom{n}{2} - 2}.$$

For the case of n odd, the proof is quite similar and will be omitted. We remark that the largest such family we can find so far is the kernel system of all $2^{\binom{n}{2} - 3}$ graphs which contain a fixed triangle. The above result supplies evidence in favor of the old conjecture of Simonovits and Sós [15].

Conjecture 2. If $\mathbf{F}$ is a family of (labelled) subgraphs of K_n such that for any $F, F' \in \mathbf{F}$, $F \cap F'$ contains a triangle then $|\mathbf{F}| \leqslant 2^{\binom{n}{2} - 3}$.

Let $G = K(r_1, r_2, r_3)$ denote the complete tripartite graph on the vertex sets R_i of size r_i, $1 \leqslant i \leqslant 3$. Suppose $\mathbf{F}$ is a family of (labelled) subgraphs of G such that $F \cap F'$ contains a triangle for all $F, F' \in \mathbf{F}$. One such family is a kernel system of G containing some fixed triangle. Clearly such a family has $2^{r_1 r_2 + r_2 r_3 + r_3 r_1 - 3}$ graphs in it. We will show that no family $\mathbf{F}$ satisfying the hypothesis can have more than this many graphs. To see this, partition the edge set E of G into three classes E_i, $1 \leqslant i \leqslant 3$, where E_i denotes the sets of edges which are not incident to a vertex in R_i. It follows from the structure of G that $F \cap F'$ must intersect every R_i since all triangles do. Thus, by Theorem 1 we have

$$|\mathbf{F}| \leqslant 2^{|E|-3} g(3, 3)$$
$$= 2^{r_1 r_2 + r_2 r_3 + r_3 r_1 - 3}$$

as claimed.

Here is another tantalizing conjecture:

Conjecture 3. Suppose $\mathbf{F}$ is a family of (labelled) subgraphs of K_n such that for any $F, F' \in \mathbf{F}$, $F \cap F'$ contains a path of three edges. Then

$$|\mathbf{F}| \leqslant 2^{\binom{n}{2} - 3}$$

i.e., kernel systems give the largest possible families.

At present all that is known is that

$$2^{\binom{n}{2} - 3} \leqslant \max_{\mathbf{F}} |\mathbf{F}| \leqslant 2^{\binom{n}{2} - 1},$$

the upper bound resulting from the observation that $\mathbf{F}$ cannot contain a graph and its complement. We remark that if we only consider paths of length 2, then it is not difficult to show that $\max_{\mathbf{F}} |\mathbf{F}| = 2^{\binom{n}{2} - 1 + o(1)}$.

Finally, we mention one more (related) conjecture of Simonovits and Sós [15]:

Conjecture 4. If $\mathbf{F}$ is a family of subsets of $[n]$ such that $F, F' \in \mathbf{F} \Rightarrow F \cap F'$ contains a 3-term arithmetic progression, then $|\mathbf{F}| \leqslant 2^{n-3}$.

Note that this bound, if true, would be best possible, since in this case the kernel system formed by all sets containing a fixed 3-term arithmetic progression has 2^{n-3} sets in it.

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