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On Sperner Families Satisfying an Additional Condition

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Let $\mathscr{F}$ be a Sperner family consisting of subsets of a finite set X of cardinality n such that the union of any three sets belonging to $\mathscr{F}$ is different from X.

In this paper we prove that for large enough n (e.g. for $n > 1000$)

$$|\mathscr{F}| \leqslant \binom{n-1}{\left[\frac{n-1}{2}\right]} + \epsilon,$$

where $\epsilon = 0$ for odd and $\epsilon = 1$ for even values of n. The extremal families are determined also.

We prove further a generalization of the Erdös–Chao–Ko–Rado theorem which we apply to the proof of the main theorem.

1. INTRODUCTION AND NOTATION

Let X be a finite set. A family $\mathscr{F}$ of subsets of X is called a Sperner family if $F, G \in \mathscr{F}$ implies $F \not\subset G$ unless $F = G$.

$|\mathscr{F}|$ denotes the cardinality of $\mathscr{F}$ while $|X|$ denotes the cardinality of X.

Sperner [6] proved that if $|X| = n$, then for any Sperner family $\mathscr{F}$, consisting of subsets of X we have

$$|\mathscr{F}| \leqslant \binom{n}{\left[\frac{n}{2}\right]}.$$

E.C. Milner proved the following generalization of Sperner's theorem (it is contained in Milner [5] though it is stated there for intersections instead of unions):

Let $\mathscr{F}$ be a Sperner family of the subsets of X, $|X| = n$, and suppose that for any two sets A, B belonging to $\mathscr{F}$ we have

$$|A \cup B| \leqslant k \leqslant n,$$

then

$$|\mathscr{F}| \leqslant \binom{n}{\left[\frac{k}{2}\right]}.$$

After this it is natural to ask what happens if we consider triple (quadruple,...) unions. In this paper we shall answer this question for $k = n - 1$, n sufficiently large, proving the following:

THEOREM. *Let X be a finite set of cardinality n. Let $\mathscr{F}$ be a Sperner family consisting of subsets of X, satisfying the additional condition that for any F, G, $H \in \mathscr{F}$ we have*

$$F \cup G \cup H \neq X.$$

Then for n sufficiently large we have:

$$|\mathscr{F}| \leqslant \binom{n-1}{\left[\frac{n-1}{2}\right]} + \epsilon$$

where $\epsilon = 0$ for odd and $\epsilon = 1$ for even values of n.

If $\mathscr{F}$ is a family of sets then $\mathscr{F}_i$ denotes the family of sets belonging to $\mathscr{F}$ and having cardinality i, while

$$\mathscr{F}' = \{F \mid \exists G \in \mathscr{F}, F \subset G, |G \backslash F| = 1\}.$$

2. Some Reductions

The Erdös–Chao–Ko–Rado theorem states if we are given a Sperner family of subsets of cardinality not exceeding i of a set of cardinality n, $2i \leqslant n$, in such a way that no 2 sets belonging to this family are disjoint, then the cardinality of the family does not exceed

$$\binom{n-1}{i-1}.$$

Katona [1] gave a simple proof of this theorem. We shall generalize his method to obtain the following:

LEMMA. *Let X be a finite set, $|X| = n$. Let $\mathscr{F}$ be a family of subsets of cardinality i of X such that for $F_1, ..., F_k \in \mathscr{F}$ $\bigcap_{j=1}^{k} F_j \neq \varnothing$.*

If $ki/(k-1) \leqslant n$, then $|\mathscr{F}| \leqslant \binom{n-1}{i-1}$.

Proof. Let $\mathscr{F}^c = \{K \mid X \backslash K \in \mathscr{F}\}$. Then every member of $\mathscr{F}^c$ has cardinality $n - i \geqslant n/k$, and for $K_1, ..., K_k \in \mathscr{F}^c$, we have $\bigcup_{j=1}^{k} K_j \neq X$. Let $x_1, x_2, ..., x_n, x_1$ be a cyclic ordering of the elements of X. We shall estimate the number of K's consisting of $n - i$ consecutive elements relative to this cyclic ordering. If there exists at least one such K then we may suppose that its last element is x_n (here and henceforward last means that its neighbour to the right is not contained in K). To every K consisting of consecutive elements relative to this ordering we associate the index of its last element, to the set ending with n we associate every integer s, $n \leqslant s \leqslant k(n - i)$. If there are r sets consisting of consecutive elements relative to this cyclic ordering then we have associated with them $r + k(n - i) - n$ indices from the interval $[1, k(n - i)]$, the elements of which can be divided into k disjoint classes such that the elements in each of the classes have the same residue modulo k. If we can pick out k sets consisting of consecutive elements relative to this cyclic ordering such that the integers associated with them completely cover one of the classes, then one can easily see that the union of these sets is X. Hence by our assumption there exists an element in each of the classes to which we have not associated any of the sets. As we have associated with different sets different indices we get: $-n + r + k(n - i) + (n - i) \leqslant k(n - i)$ or, what is equivalent,

$$r \leqslant i.$$

If we count the number of pairs consisting of a cyclic ordering and a set consisting of consecutive elements with respect to this ordering we get:

$$| \mathscr{F}^c | \, i!(n - i)! \leqslant (n - 1)! \, i$$

whence $| \mathscr{F}^c | \leqslant \binom{n-1}{i-1}$ Q.E.D.

We shall now use the following estimate which follows from the Kruskal–Katona theorem (cf. Katona [2] and Kruskal [3]).

Let $\mathscr{A}$ be a family of k-element subsets of the finite set X. If

$$| \mathscr{A} | \leqslant \binom{s}{k}$$

then

$$| \mathscr{A}' | \geqslant | \mathscr{A} | \binom{s}{k-1} \Big/ \binom{s}{k} \qquad (*)$$

with equality holding if and only if $\mathscr{A}$ consists of all the k-element subsets of an s-element subset of X. Now we are able to prove the following generalization of the Erdös–Chao–Ko–Rado theorem:

THEOREM 1. *Let X be a finite set of cardinality n, let $\mathscr{A}$ be a Sperner family consisting of subsets of X of cardinality not exceeding i. Let the sets belonging to $\mathscr{A}$ be k-wise nondisjoint, and $ki \leqslant (k-1)n$. Then the following inequality holds*:

$$\sum_{j=1}^{i} \frac{|\mathscr{A}_j|}{\binom{n-1}{j-1}} \leqslant 1. \tag{1}$$

Remark. This is really a generalization of the Erdös–Chao–Ko–Rado theorem as for $k = 2$, $2i \leqslant n$, and beyond $n/2$ $\binom{n-1}{i-1}$ increases monotonically, so we have

$$1 \geqslant \sum_{j=1}^{i} \frac{|\mathscr{A}_j|}{\binom{n-1}{j-1}} \geqslant \frac{\sum_{j=1}^{i} |\mathscr{A}_j|}{\binom{n-1}{i-1}} = \frac{|\mathscr{A}|}{\binom{n-1}{i-1}}.$$

Proof. For $i = 1$ the theorem is trivial. We apply induction on the number of nonzero $|\mathscr{A}_j|$'s.

If this number is one then the theorem follows from the above lemma. If it is not the case then let p be the smallest and r the second-smallest index for which $|\mathscr{A}_j| \neq 0$. Then using the estimate (*) $(r-p)$-times we get

$$|B_r = \{B \mid \exists A \in \mathscr{A}_p, B \supset A, |B| = r\}| \geqslant |\mathscr{A}_p| \frac{\binom{n-1}{r-1}}{\binom{n-1}{p-1}}.$$

The family $\mathscr{B} = (\mathscr{A} \backslash \mathscr{A}_p) \cup \mathscr{B}_r$ also satisfies the assumptions of the theorem, and the number of nonzero $|\mathscr{A}_j|$'s is one less, whence by the induction hypothesis we have:

$$\sum_{j=1}^{i} \frac{|\mathscr{A}_j|}{\binom{n-1}{j-1}} \leqslant \sum_{j=1}^{i} \frac{|\mathscr{B}_j|}{\binom{n-1}{j-1}} \leqslant 1 \qquad \text{Q.E.D.}$$

Let now $\mathscr{F}$ be a Sperner family for which the union of no k members belonging to this family is X, and $|\mathscr{F}|$ is maximal with respect to these properties.

Let $\mathscr{G}$ be the family consisting of the sets belonging to $\mathscr{F}$ and having cardinality less than n/k. $\mathscr{G}$ is a Sperner family whence satisfies Lubell's inequality

$$\sum_{j=1}^{n} \frac{|G_j|}{\binom{n}{j}} = \sum_{j=1}^{\left[\frac{n-1}{k}\right]} \frac{|G_j|}{\binom{n}{j}} \leqslant 1.$$

Hence

$$|G| \leqslant \binom{n}{\left[\frac{n-1}{k}\right]}. \tag{2}$$

From (1) and (2) it follows that

$$|\mathscr{F}| \leqslant \binom{n-1}{\left[\frac{n-1}{2}\right]} + \binom{n}{\left[\frac{n-1}{k}\right]} \tag{3}$$

or more exactly denoting $|\mathscr{F}_j|/\binom{n-1}{j-1}$ by r_j:

$$|\mathscr{F}| \leqslant \binom{n}{\left[\frac{n-1}{k}\right]} + \sum_{j=\left[\frac{n+k-1}{k}\right]}^{n} r_j \binom{n-1}{j-1} \tag{4}$$

where by (1)

$$\sum_{j=\left[\frac{n+k-1}{k}\right]}^{n} r_j \leqslant 1. \tag{5}$$

Inequality (3) does not yield the desired result, but it is true for every value of n.

We shall prove now that for $2t \geqslant n$ $\mathscr{F}_t$ is empty.

Let T be the greatest index such that $\mathscr{F}_T$ is nonempty. If $2T > n$ or if $2T = n$ but $|\mathscr{F}_T| \neq \binom{n-1}{T}$ then it follows from (*) that the family $(\mathscr{F}\backslash\mathscr{F}_T) \cup \mathscr{F}_T'$ has cardinality greater than $|\mathscr{F}|$ and satisfies the assumptions of the theorem, contradicting the maximality of $\mathscr{F}$. In the only remaining case $\mathscr{F}_{n/2}$ consists exactly of the $n/2$-element subsets of X not containing a fixed element x of X, otherwise we would not have equality in (*). In this case $\mathscr{F}$ cannot have any other members, for if some $F \in \mathscr{F}$, $x \in F$, then we could find two $n/2$-element subsets G, H not containing x for which

$$G \cup H = X\backslash x.$$

Hence $G \cup H \cup F = X$, contradicting our assumptions. If $x \notin F$ then F

contains or is contained in an $n/2$-element subset of $X\backslash x$, whence by the Sperner-property we have indeed $\mathscr{F} = \mathscr{F}_{n/2}$, which implies

$$|\mathscr{F}| = \binom{n-1}{\frac{n}{2}}$$

while the family consisting of the $(\frac{n}{2} - 1)$-element subsets of $X\backslash x$ and the set $\{x\}$ satisfies the assumptions of the theorem, too, and has cardinality greater than $|\mathscr{F}|$.

Now we can assume that $\mathscr{F}_t$ is empty for $2t \geqslant n$.

3. The Proof of the Main Theorem

We restate the theorem in the following stronger version:

Theorem 2. *Let X be a finite set of cardinality $2t + \epsilon = n$, where ϵ equals 0 or 1. Let $\mathscr{F}$ be a Sperner family of subsets of X. Let us further suppose that for any $F, G, H \in \mathscr{F}$ we have*

$$F \cup G \cup H \neq X$$

and $|\mathscr{F}|$ is maximal. Then if $\epsilon = 1$ and n is sufficiently large, there exists an element x of X such that $\mathscr{F}$ consists exactly of the t element subsets of $X\backslash x$. If $\epsilon = 0$ and n is sufficiently large, there exists an element x of X such that $\mathscr{F}$ consists exactly of the $t - 1$-element subsets of $X\backslash x$, and the set $\{x\}$.

Proof. By (4) we have:

$$|\mathscr{F}| \leqslant r_{\lceil\frac{n-1}{2}\rceil}\binom{n-1}{\left\lceil\frac{n-1}{2}\right\rceil} + (1 - r_{\lceil\frac{n-1}{2}\rceil})\binom{n-1}{\left\lceil\frac{n-1}{2}\right\rceil - 1} + \binom{n}{\left\lceil\frac{n-1}{3}\right\rceil}$$

$$= \binom{n-1}{\left\lceil\frac{n-1}{2}\right\rceil} + \binom{n}{\left\lceil\frac{n-1}{3}\right\rceil} - (1 - r_{\lceil\frac{n-1}{2}\rceil})\left(\binom{n-1}{\left\lceil\frac{n-1}{2}\right\rceil} - \binom{n-1}{\left\lceil\frac{n-1}{2}\right\rceil - 1}\right)$$

$$\leqslant \binom{n-1}{\left\lceil\frac{n-1}{2}\right\rceil} + \binom{n}{\left\lceil\frac{n-1}{3}\right\rceil} - (1 - r_{\lceil\frac{n-1}{2}\rceil})\frac{2}{n+1}\binom{n-1}{\left\lceil\frac{n-1}{2}\right\rceil}. \tag{6}$$

Now we shall estimate the ratio $\binom{n-1}{[n-1]/2}/\binom{n}{[n-1]/3}$

$$\frac{\binom{n-1}{\left[\frac{n-1}{2}\right]}}{\binom{n}{\left[\frac{n-1}{3}\right]}} = \frac{(n-1)!\left[\frac{n-1}{3}\right]!\left(n-\left[\frac{n-1}{3}\right]\right)!}{n!\left[\frac{n-1}{2}\right]!\left[\frac{n}{2}\right]!}$$

$$= \left(\prod_{i=0}^{n-3-\left[\frac{n-1}{3}\right]-\left[\frac{n-1}{2}\right]} \frac{n-\left[\frac{n-1}{3}\right]-i}{\left[\frac{n-1}{2}\right]-1-i}\right)$$

$$\cdot\left(\frac{\left[\frac{n}{2}\right]+1}{\left[\frac{n-1}{2}\right]}\right)\left(\frac{\left[\frac{n}{2}\right]+2}{n}\right) > \frac{1}{2}\prod_{i=0}^{n-3-\left[\frac{n-1}{3}\right]-\left[\frac{n-1}{2}\right]}$$

$$\times \frac{\frac{2n}{3}-i-1}{\frac{n}{2}-i-\frac{3}{2}} > \frac{1}{2}\prod_{i=0}^{\frac{n}{6}-3}\frac{4}{3} \geqslant \frac{1}{5}\left(\frac{4}{3}\right)^{\frac{n}{6}}. \tag{7}$$

Now we shall examine separately the cases n odd and n even.

(a) n is odd.

If we knew that there are two disjoint t-element subsets of X belonging to $\mathscr{F}$, then it would immediately follow that any set belonging to $\mathscr{F}$ is contained in their union, which is a $2t$-element subset of X. Hence $\mathscr{F}$ is a Sperner family consisting of subsets of a $2t$-element set, and the statement of the theorem follows.

So we may assume that any two sets belonging to $\mathscr{F}_t$ intersect nontrivially whence by the lemma we have

$$|\mathscr{F}_t| \leqslant \binom{n-1}{\frac{n-1}{2}-1} = \binom{n-1}{\frac{n-1}{2}} - \frac{2}{n+1}\binom{n-1}{\frac{n-1}{2}}.$$

Using (6) we get

$$|\mathscr{F}| \leqslant \binom{2t}{t} + \binom{n}{\left[\frac{n+1}{3}\right]} - \frac{4}{(n+1)(n+1)}\binom{n-1}{\frac{n-1}{2}}$$

which is by (7)(for example for $n \geqslant 300$) less than $\binom{2t}{t}$ Q.E.D.

(b) n is even.

Let us return to the cyclic orderings. Let us suppose that for a cyclic ordering $x_1, x_2, \ldots, x_n, x_1$ there are exactly $t+1$ members of $\mathscr{F}_{t-1}$ consisting of consecutive elements with respect to this cyclic ordering. Let us form for any i the pairs of $(t-1)$-tuples:

$$A_i = \{x_{i+1}, x_{i+2}, \ldots, x_{i+t-1}\},\ B_i = \{x_{i+t}, \ldots, x_{i+2t-2}\}$$

$i = 1, \ldots, n$ for $n < j \leqslant 2n$ x_j stands for x_{j-n}.

Every set consisting of $t-1$ consecutive elements relative to this ordering is contained in exactly 2 pairs. There are $2t$ such pairs all together and we have $t+1$ sets belonging to $\mathscr{F}_{t-1}$. Hence there is a pair each member of which is contained in $\mathscr{F}_{t-1}$. We may suppose that this pair is:

$$A_n = \{x_1, x_2, \ldots, x_{t-1}\},\ B_n = \{x_t, x_{t+1}, \ldots, x_{2t-2}\}.$$

None of the sets A_i can belong to $\mathscr{F}_{t-1}$ for $i = t+1, \ldots, 2t-2$ otherwise A_n, B_n and this set would have X for their union. These are $t-2$ sets. If both A_t and A_{2t-1} belong to $\mathscr{F}_{t-1}$ then it implies as above that none of the sets $A_1, \ldots, A_{t-2}$ belongs to $\mathscr{F}_{t-1}$. But this impossible for $t \geqslant 4$, as there are at most $t-1$ A_i's which do not belong to $\mathscr{F}_{t-1}$. As we are not interested in the case $n < 8$, we may assume that the only remaining failing set is either A_t or A_{2t-1}. In either cases we get that there is an element x_r of X such that A_j belongs to $\mathscr{F}_{t-1}$ if and only if it does not contain x_r. If $r_{t-1} \leqslant 1 - 1/n(t+1)$, then using (6) and (7) we get, as for the case of odd values of n, that this is impossible (for example for $n > 1000$).

Hence we may assume

$$r_{t-1} > 1 - \frac{1}{n(t+1)}. \tag{8}$$

Let c denote the number of cyclic orderings with respect to which there are exactly $t+1$, $t-1$-tuples belonging to $\mathscr{F}_{t-1}$, and consisting of consecutive elements with respect to this ordering. Then we have:

$$|\mathscr{F}_{t-1}| \leqslant \frac{c(t+1) + ((n-1)! - c)t}{(t-1)!(t+1)!}$$

$$= \binom{n-1}{t-1}\left(1 - \frac{1}{t+1}\,\frac{(n-1)! - c}{(n-1)!}\right).$$

Using (8) we get:

$$c > (n-1)!\,\frac{n-1}{n} > (n-2)(n-2)! \tag{9}$$

If for a cyclic ordering there is an element y of X such that any set consisting of $t-1$ consecutive elements with respect to this ordering belongs to $\mathscr{F}_{t-1}$ iff it does not contain y, then we say that y corresponds to this cyclic ordering. By (9) we may suppose that there is a cyclic ordering $x_1, x_2, \dots, x_n, x_1$ to which corresponds some element y of X. By symmetry we may suppose $y = x_1$.

If we are given a cyclic ordering $y_1, y_2, \dots, y_{n-1}, y_1$ of the set $X \backslash x$ then placing x in all the possible $n-1$ places we can construct $n-1$ different cyclic orderings of X. In such a manner we can divide the cyclic orderings of X into $(n-2)!$ disjoint $(n-1)$-blocks. By (9) we can find a block such that we have associated with every cyclic ordering in this block an element y. Let $y_1, y_2, \dots, y_{n-1}, y_1$ be the cyclic ordering from which we have constructed the orderings in this block. Let us first suppose that $u \neq x_1$ corresponds to a cyclic ordering in this block. Let $u = y_j$. In this ordering either y_j and y_{j+1} or y_{j-1} and y_j are consecutive elements. By symmetry we may suppose that y_j and y_{j+1} are consecutive elements, i.e., x_1 does not lie between them. Then the union of the $(t-1)$-set ending with the neighbour to the left of y_j and the $(t-1)$-set beginning with the neighbour to the right of y_{j+1} is $X \backslash \{y_j, y_{j+1}\}$. As both these sets belong to $\mathscr{F}_{t-1}$, no set belonging to $\mathscr{F}$ contains both y_j and y_{j+1}. It follows that to any ordering of this block in which x_1 does not lie between y_j and y_{j+1} corresponds either y_j or y_{j+1}. Let us choose one of the elements x_2, x_{t+1}, x_{2t} which is different from both y_j and y_{j+1}, and denote this element by z. One of the pairs $(\{x_3, x_4, \dots, x_{t+1}\}, \{x_{t+2}, x_{t+3}, \dots, x_{2t}\})$ $(\{x_2, \dots, x_t\}$ $\{x_{t+2}, x_{t+3}, \dots, x_{2t}\})(\{x_2, x_3, \dots, x_t\}, \{x_{t+1}, \dots, x_{2t-1}\})$ has $X \backslash \{x_1, z\}$ for the union of its members which belong to $\mathscr{F}_{t-1}$. Now let us consider a cyclic ordering from the chosen block in which x_1 and z are neighbours. As neither x_1 nor z corresponds to this ordering, we can find a set belonging to $\mathscr{F}_{t-1}$ which contains both of them. In this way we have found 3 sets belonging to $\mathscr{F}$ and having X for their union, which is a contradiction.

Hence we may assume that to every cyclic ordering in this block x_1 is the corresponding element. Let us suppose that there is a set belonging to $\mathscr{F}$ and containing x_1 and another element z, too. Let $z = y_j$. As x_1 corresponds to the cyclic ordering $y_1, y_2, \dots, y_j, x_1, y_{j+1}, \dots, y_{n+1}, y_1$, the sets $\{y_{j+1}, \dots, y_{j+t-1}\}$ and $\{y_{j+t}, \dots, y_{j+2t-2}\}$, where the indices have to be taken modulo $n-1 = 2t-1$, belong to $\mathscr{F}_{t-1}$. The union of these two sets and the set containing both z and x_1 is X, which is a contradiction.

So far we have proven that any set belonging to $\mathscr{F}$ is either a subset of $X \backslash x_1$, or $\{x_1\}$. We have proven that for $s \geqslant t$ $\mathscr{F}_s$ is empty. The sets different from $\{x_1\}$ and belonging to $\mathscr{F}$ form a Sperner family of subsets of $X \backslash x_1$.

Hence by Lubell's inequality we have [4]:

$$\sum_{\mathscr{A}\neq\{x_1\},\mathscr{A}\in\mathscr{F}} \frac{1}{\binom{n-1}{|\mathscr{A}|}} \leqslant 1.$$

Hence

$$|\mathscr{F}| \leqslant 1 + \binom{n-1}{\left[\frac{n-1}{2}\right]}$$

and equality can hold only if $\mathscr{F}\backslash\{x_1\}$ consists exactly of the $(t-1)$-element subsets of $X\backslash x_1$ Q.E.D.

To conclude this paper we mention two problems:

Problem 1. Let $\mathscr{F}$ be a family consisting of s-element subsets of a finite set X, $|X| = n$. Let us suppose that for any $F, G, H \in \mathscr{F}$ we have:

$$|F \cap G \cap H| \geqslant 2.$$

Does there exist a positive ϵ such that for

$$s \leqslant (\tfrac{1}{2} + \epsilon)n$$

we have $|\mathscr{F}| \leqslant \binom{n-2}{s-2}$?

Problem 2. Let $\mathscr{F}$ be a Sperner family consisting of subsets of a finite set X, $|X| = n$. Let us suppose that the union of any three sets belonging to $\mathscr{F}$ has cardinality less or equal $n - 2$. Let ϵ be an arbitrary positive real number. Is it true that for $n > n_0(\epsilon)$ we have

$$|\mathscr{F}| \leqslant (1 + \epsilon)\binom{n-2}{\left[\frac{n-2}{2}\right]} ?$$

Remark. If the answer for Problem 1 is affimative then the answer to the Problem 2 is affirmative also.

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