Discrete Mathematics 50 (1984) 323–328
North-Holland 

COMMUNICATION

# AN EXACT RESULT FOR 3-GRAPHS

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Communicated by A. Hajnal
Received 24 January 1984

The aim of this paper is to prove Theorem 1 which gives a full description of families of 3-subsets in which any 4 points contain 0 or 2 members of the family.

## 1. Introduction

A hypergraph $H=(V, \mathscr{E})$ consists of a vertex set $V$ and an edge set $\mathscr{E}$, which is a family of subsets of $V$ satisfying $\bigcup \mathscr{E} = V$. It is called an $r$-graph if all edges have $r$ elements. For $W \subset V$ we set $\mathscr{E}_W = \{E \in \mathscr{E} : E \subset W\}$, these are the edges *spanned* by $W$.

Given integers $n > k > r$, $0 < s \leqslant \binom{k}{r}$ let us denote by $m(n, r, k, s)$ the maximum number of edges in an $r$-graph on $n$ vertices in which any $k$ vertices span less than $s$ edges. The determination of $m(n, r, k, s)$ is a hopelessly difficult problem in general even for $r=2$. The well-known Turàn's theorem is the case $s=\binom{k}{2}$.

For $r \geqslant 3$ and $s=\binom{k}{r}$ we come to Turàn's problem: What is the maximum number of edges in an $r$-graph without a complete subgraph on $k$ vertices. This is a challenging open problem for all $k > r$, Erdös offers a monotone increasing sum for its solution (at present 3000 dollars).

Turàn (cf. [3, 7]) conjectured that $m(n, 3, 4, 3)$ is asymptotic to $n^3/24$.

Let us consider the following 3-graph on 6 points: $S(6)=\{(123), (124), (345), (346), (561), (562), (135), (146), (236), (245)\}$ (cf. Fig. 1). One can check that any 4 points span 2 edges in $S(6)$.

**Example 1.** Suppose $|V|=n$ and $V$ is partitioned into $V=V_1 \cup \cdots \cup V_6$. Define a 3-graph $H_S=(V, \mathscr{E})$, where

$$\mathscr{E}=\{(v_{i_1}v_{i_2}v_{i_3}): 1 \leqslant i_1 < i_2 < i_3 \leqslant 6,\ v_{i_j} \in V_{i_j},\ (i_1 i_2 i_3) \in S(6)\}.$$

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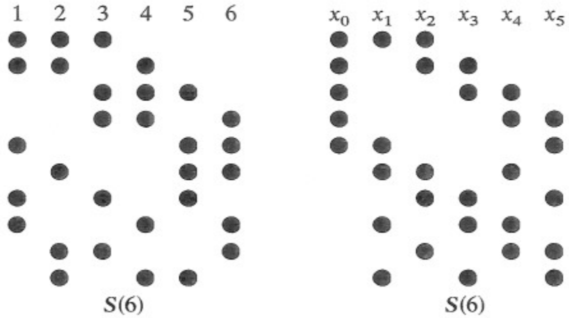


Fig. 1.

If we choose a partition satisfying $|V_i| \geq \lfloor n/6 \rfloor$, then $H_S$ has more than $10\lfloor n/6 \rfloor^3$ edges which is more than $n^3/24$, disproving Turàn's conjecture – we will give a counter-example with more edges in Section 2.

Let us note that in $H_S$ any four points span either zero or two edges.

We shall show that a 3-graph having this property cannot have more edges than $H_S$ (Theorem 2). This result is deduced from Theorem 1, which gives a complete description of 3-graphs with the above properties: apart the 3-graphs given by Example 1 they are of the following form:

**Example 2.** Let $V$ consist of $n$ points on the unit circle, and let the edges be those 3-tuples that the origin is contained in the triangle formed by them (tacitly we assumed that the origin is contained in the convex hull of the points but it is on none of the lines joining two points).

The fact that in this 3-graph any 4 points span 0 or 2 edges can be verified easily.

Now we state our main results formally.

**Theorem 1.** *Suppose $H = (V, \mathscr{E})$ is a 3-graph in which any 4 points span 0 or 2 edges. Then $H$ is isomorphic to one of the 3-graphs in Examples 1 or 2.*

**Theorem 2.** *Suppose that $H = (V, \mathscr{E})$, $|V| = n \geq 5$ and any 4 points of $V$ span 0 or 2 edges. Then $\max |\mathscr{E}|$ is attained exactly for $H$ of the form $H_S$ for some equipartition, i.e., $\lfloor n/6 \rfloor \leq |V_i| \leq \lceil n/6 \rceil$.*

## 2. A remark on $m(n, 3, 4, 3)$

If one only wants to satisfy the condition: no four points span more than 2 edges, then one can add edges to $H_S$ iteratively: first partition each $V_i$ into 6 sets, say $W_{j1}, \ldots, W_{j6}$ and add to $H_S$ all 3-tuples $(w_{ji_1} w_{ji_2} w_{ji_3})$ with $(i_1 i_2 i_3) \in S(6)$. Then repeat this with the $W_{ji}$, etc.

Making the partitions always as equal as possible, finally one obtains $n^3(1+o(1))/21$ edges. This gives the first part of the following:

**Theorem 3.**

$$\frac{2+o(1)}{7}\binom{n}{3} \leqslant m(n,3,4,3) \leqslant \frac{1}{3}\binom{n}{3}\frac{n}{n-2}.$$

The upper bound was proved by Caen [2].

Let us mention that the lower bound of the theorem was proved independently by Giraud [5] also.

We shall discuss other related extremal problems at the end of this paper.

## 3. The proof of Theorem 1

Suppose that $H=(V,\mathscr{E})$ is a 3-graph satisfying the assumptions, i.e., whenever $(xyz) \in \mathscr{E}$ and $v$ is another element of $V$, then there is exactly one more edge spanned by $\{x, y, z, v\}$.

For $v \in V$ let us define the *neighbourhood* of $v$: $N(v)=\{(xy):(xyv)\in\mathscr{E}\}$.

We distinguish two cases.

(a) For some $v \in V$, $N(v)$ contains an odd cycle: $(x_1x_2), (x_2x_3), \ldots, (x_{2t+1}x_1)$.

**Proposition 1.** *$N(v)$ contains a cycle of length 5.*

**Proof.** Suppose $x_1, \ldots, x_{2t+1}$ form an odd cycle of minimal length. Since $t=1$ would yield $\geqslant 3$ edges on 4 points, $t \geqslant 2$. Suppose for contradiction, $t \geqslant 3$.

By the minimal choice of $t$, $(v, x_ix_j) \notin \mathscr{E}$ holds unless $i=j+1$ or $j=i+1$. Looking at 4 vertices $v, x_i, x_{i+1}, x_j$ with $j \neq i-1, i+1, i+2$, we conclude $(x_ix_{i+1}x_j) \in \mathscr{E}$. Consequently the 4 points $x_1, x_2, x_4, x_5$ span 4 edges, a contradiction. □

Actually our argument gave for the five-cycle $x_1, \ldots, x_5$ that $(x_ix_{i+1}x_{i+3}) \in \mathscr{E}$ holds for $i=1, \ldots, 5$ (the subscripts are understood mod 5). That is the 3-graph spanned by $v, x_1, \ldots, x_5$ is isomorphic to $S(6)$. Set $x_0 = v$.

First we show that $w, x_0, x_1, \ldots, x_5$ span a 3-graph of the form $H_S$, given by Example 1.

Looking at the 4-tuple $w, x_0, x_1, x_2$ and using that the automorphism group of $S(6)$ (which is $A_5$) is doubly transitive, we may assume $(wx_1x_2) \in \mathscr{E}$.

Similarly, with $w, x_1, x_2, x_3$ we infer $(wx_2x_3) \in \mathscr{E}$.

In particular $(wx_0x_i) \notin \mathscr{E}$ holds for $i=1,2,3$.

We claim that the same holds for $i=4,5$. Consider $wx_1x_2x_4$, we infer

$(wx_2x_4) \notin \mathscr{E}$. Now look at $wx_0x_2x_4$. They span no edge, especially $(wx_0x_4) \notin \mathscr{E}$. Similarly $(wx_0x_5) \notin \mathscr{E}$ holds. Looking at the sets $wx_0x_ix_{i+1}$ we conclude $(wx_ix_{i+1}) \in \mathscr{E}$, i.e., $w, x_1, x_2, \ldots, x_5$ span $S(6)$.

These show that if we have $S(6)$ plus one point, then they span a 3-graph of the form $H_S$. Now the general case follows easily, e.g. by induction on $n$.

(b) $N(v)$ is bipartite for all $v \in V$.

Let $x$ be a fixed vertex and $(A, B)$ a fixed bipartition of $N(x)$.

**Proposition 2.** *If $(ab) \in N(x)$, $(ab') \notin N(x)$, then $(abb') \in \mathscr{E}$.*

**Proof.** It is sufficient to consider the 4-tuple $(xabb')$. □

Denote by $\deg(c)$ the degree of the vertex $c$ in $N(x)$. Order the elements of $A$ and $B$ according to their degrees: $A = \{a_1, \ldots, a_k\}$; $B = \{b_1, \ldots, b_l\}$; $\deg(a_1) \geqslant \cdots \geqslant \deg(a_k)$, $\deg(b_1) \geqslant \cdots \geqslant \deg(b_l)$.

**Proposition 3.** *Suppose $(a_ib_j) \in N(x)$, $i' \leqslant i$, $j' \leqslant j$. Then $(a_{i'}b_{j'}) \in N(x)$ holds.*

**Proof.** Clearly, it is sufficient to prove the statement in the case $i = i'$. Set $a = a_i$ and suppose for contradiction $(ab_{j'}) \notin N(x)$. Now $\deg(b_{j'}) \geqslant \deg(b_j)$ implies the existence of $a' \in A$ with $(a'b'_j) \in N(x)$, $(a', b_j) \notin N(x)$.

Applying Proposition 2 four times we infer that $a, a', b'_j, b_j$ span 4 edges, a contradiction. □

**Proposition 4.** *Neither $A$ nor $B$ contains an edge of $H$.*

**Proof.** Suppose by symmetry $(a_1a_2a_3)$ is an edge of $H$ contained in $A$. Considering the four points $x, a_1, a_2, a_3$ we arrive at a contradiction with $(A, B)$ being a bipartition of $N(x)$. □

**Proposition 5.** *Suppose $a \in A$, $b, b' \in B$. Then $(abb') \in \mathscr{E}$ if and only if exactly one out of $(ab)$ and $(ab')$ belongs to $N(x)$.*

**Proof.** Since $N(x)$ is bipartite, $(xbb') \notin \mathscr{E}$. Now the statement follows by considering the four point $x, a, b, b'$. □

Let us call $v, w \in V$ *equivalent* if $N(v) = N(w)$. It is easy to see that adding or removing equivalent points does not change the four points property.

**Proposition 6.** *Two points $v, w$ are equivalent if and only if there is no edge in $H$ containing both of them.*

**Proof.** If for some $z$, $(vwz)\in\mathscr{E}$, then $(vz)\notin N(v)$ but $(vz)\in N(w)$, that is $v$ and $w$ are not equivalent. Suppose now that $\{v, w\}$ is not contained in any edge. Choose $y, z\in V-\{v, w\}$. Considering $v, w, y, z$ it follows that either both $(vyz)$ and $(wyz)$ are edges or none of them, proving the proposition. □

Clearly it is sufficient to show that $H$ is isomorphic to Example 2 in the case: there are no two equivalent vertices. In view of Proposition 6 we assume that any pair of vertices is contained in at least one edge. Consequently $N(x)$ has no isolated vertices.

If $a, a'\in A$ have the same degree, then by Proposition 3, they have the same neighbourhood in $N(x)$. Thus, by Propositions 4 and 5, $\{a, a'\}$ is contained in no edge, a contradiction. We infer

$$l\geqslant \deg(a_1)>\cdots>\deg(a_k)\geqslant 1; \qquad k\geqslant \deg(b_1)>\cdots>\deg(b_l)\geqslant 1.$$

This is only possible if $k=l$ and $\deg(a_i)=\deg(b_i)=k-i+1$. More exactly – using Proposition 3 – $(a_i, b_j)\in N(x)$ if and only if $i+j\leqslant k+1$.

Now imagine that we have placed these $2k+1$ points in the vertices of a regular $(2k+1)$-gon on the unit circle in the order $x, a_k, \ldots, a_1, b_1, \ldots, b_k$. Then $(a_ib_j)\in N(x)$ if and only if the triangle $xa_ib_j$ contains the origin. Now Proposition 5 yields that 3 points form an edge in $H$ if and only if the corresponding triangle contains the origin, concluding the proof of Theorem 1. □

**Remark 1.** Our proof showed that in Example 2 one can always move the points to the vertices of some regular $(2k+1)$-gon without altering the 3-graph. In particular, if any two vertices are covered by an edge, then $n$ *is odd.*

To prove Theorem 2 just note that if $H$ can be obtained by putting $d_1, \ldots, d_{2k+1}$ points into the vertices of a regular $(2k+1)$-gon, then the number of edges is maximized if the $d_i$'s are as equal as possible. Thus it is upperbounded by

$$(n/(2k+1))^3(2k+1)\binom{k+1}{2}\Big/3 \ (<n^3/24),$$

which is always less than or equal to the maximal size of a 3-graph coming from Example 1, with equality holding only for $n\leqslant 5$ – then the two examples coincide.

## 4. Concluding remarks and open problems

The value of our Theorem 1 is given partially by the fact that there are very few exact results concerning 3-graphs.

Following a conjecture of Katona [6], Bollobàs [1] proved that if in a 3-graph on $n$ vertices no edge is containing the symmetric difference of two other edges, then it has at most $\lfloor n/3 \rfloor \lfloor (n+1)/3 \rfloor \lfloor (n+2)/3 \rfloor$ edges. In [4] for $n > 800$ we gave a more exact form of this result by showing that if the 3-graph has at least as many as above edges than either it contains 3 edges of the form (123), (124), (345) or it is the complete 3-partite graph, that is $V = V_1 \cup V_2 \cup V_3$, the $V_i$'s are disjoint and the edges are the triples meeting all the $V_i$ (Bollobàs excluded the configuration (123), (124), (134) also, however, his result holds for all $n$).

Let us mention the following:

**Conjecture 1** (Erdös and Sós [3]). *Suppose $H = (V, \mathscr{E})$ is a 3-graph in which $N(x)$ is bipartite for all $x \in V$. Then $|\mathscr{E}| < n^3/24$.*

Example 2 shows that one can have as much as $n^3(1+o(1))/24$ edges. Another, more general example is provided by taking a random tournament on $n$ points and the 3-cycles of it as edges.

**Problem 1.** Suppose $H = (V, \mathscr{E})$ is an $r$-graph on $n$ points in which any $r+1$ points span zero or two edges. Determine $\max |\mathscr{E}|$.

An interesting example of such $r$-graphs is given by

**Example 3.** Put $n$ points on the surface of the unit sphere in $r-1$ dimension. Let $r$ points form an edge if the corresponding simplex contains the origin.

It is easy to see, that choosing the points at random gives $\binom{n}{r}(1+o(1))/2^{r-1}$ edges and one cannot have more edges in this example.

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