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Injection Geometries

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In this paper a new concept, injection geometries, is considered. This provides a common generalization of matroids and permutation geometries. Different systems of axioms and various examples are given. Theorem 5.1 provides an extremal set theoretic characterization of injection designs. © 1984 Academic Press, Inc.

1. INTRODUCTION

Denote by N the set of positive integers i, $1 \leqslant i \leqslant n$, $N^d = \{(a_1,\dots,a_d)\colon 1 \leqslant a_i \leqslant n, 1 \leqslant i \leqslant d\}$ is the set of all sequences of length d in N.

A partial diagonal or *injective* set is a set $A \subset N^d$ such that all elements of A are different in all d coordinates, i.e. there are no $\mathbf{a}, \mathbf{b} \in A$ with $a_i = b_i$ for some $1 \leqslant i \leqslant d$. Of course, each A with $|A| \leqslant 1$ is injective.

Clearly, the definition implies $|A| \leqslant n$. If equality holds A is called *diagonal.*

Note that in the case $d = 1$ each $A \subset N$ is injective, for $d = 2$, A is injective iff it is the graph of a bijective function between subsets of N. In particular A is diagonal iff it is the graph of a bijection $N \to N$, i.e., it corresponds to a permutation. In general, an injective set $A \subset N^d$ can also be considered as a function f with domain $\{a_1\colon (a_1,\dots,a_d) \in A\}$ and $f(a_1) = (a_2, a_3,\dots, a_d)$. The name "injective" comes from the fact that A is injective if and only if each projection $f_i(a_1) \overset{\text{def}}{=} a_i$ is an injective function.

The intersection of injective sets is necessarily injective ($\varnothing$ is injective!) but their union need not be injective. Two injective sets A and B are called *disjoint* if $A \cap B = \varnothing$ and $A \cup B$ is injective.

Now we are ready to define injection geometries. Suppose $X \subset N^d$, $\mathscr{A}$ is a family of injective subsets of X, i.e., $\mathscr{A} \subset 2^X$ and $\mathscr{A}$ is partitioned into $\mathscr{A}_0 \cup \mathscr{A}_1 \cup \cdots \cup \mathscr{A}_r$, $\mathscr{A}_r \neq \varnothing \neq \mathscr{A}_0$. Then $\mathscr{A}$ is called an ***injection geometry*** of rank r if F(i) through F(iii) hold.

F(i) $\mathscr{A}$ is closed under intersection ($A, B \in \mathscr{A}$ implies $A \cap B \in \mathscr{A}$).

F(ii) If $A \in \mathscr{A}_i$, $B \in \mathscr{A}_j$, and $A \subset B$, then $i \leqslant j$ holds.

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F(iii) Given disjoint injective sets A, B with $|B| = 1$ and $A \in \mathscr{A}_i$, $0 \leqslant i < r$, there exists a unique $A' \in \mathscr{A}_{i+1}$ verifying $A \cup B \subset A'$. Moreover, for any $A'' \in \mathscr{A}$ verifying $A \cup B \subset A''$, $A'' \supset A'$ holds.

The family $\mathscr{A}_i$ is called *the system of flats of rank i* of the injection geometry.

In analogy with matroids the members of the family $\mathscr{A}$ along with the ground set X can be called the *closed* sets.

Note that for $d = 1$ an injection geometry is simply the family of flats of a matroid (cf. [13]). For $d = 2$ it is slightly more general than permutation geometries (cf. [3]).

EXAMPLE 1.1 (Free Injection Geometry). For an arbitrary $X \subset N^d$ define $\mathscr{F}_i(X) = \{F \subset X, |F| = i, \quad F \text{ is injective}\}$. If $\mathscr{F}_r \neq \varnothing$, then $\mathscr{F} = \mathscr{F}_0 \cup \cdots \cup \mathscr{F}_r$ is the free injection geometry of rank r on X.

EXAMPLE 1.2. Suppose $\mathscr{S}_1 = \{S_1, ..., S_t\}$ is a *star* with center C, i.e., $S_i \cap S_j = C$ for $1 \leqslant i < j \leqslant t$, $X \overset{\text{def}}{=} \cup \mathscr{S}_1 \subset N^d$. Moreover, each S_i is injective. Define $\mathscr{S}_0 = \{C\}$. Then $\mathscr{S}_0 \cup \mathscr{S}_1$ is an injection geometry of rank 1.

Note that in this example $C = \varnothing$ is permitted. Moreover, $\mathscr{S}' = \mathscr{S}'_0 \cup \mathscr{S}'_1$, where $\mathscr{S}'_0 = \{\varnothing\}$ and $\mathscr{S}'_1 = \{S - C = S \in \mathscr{S}_1\}$, is an injection geometry on $X - C$.

An important special case of Example 1.2 is

EXAMPLE 1.3 (Permutation Cube, cf. [5]). Suppose $\mathscr{P}_1$ is a set of n^{d-1} pairwise disjoint diagonals of N^d, $\mathscr{P}_0 = \{\varnothing\}$. Then $\mathscr{P} = \mathscr{P}_0 \cup \mathscr{P}_1$ is an injection geometry of rank 1.

If each $A \in \mathscr{A}$ is contained in some $A' \in \mathscr{A}_r$, then the injection geometry $\mathscr{A}$ is called *regular*. If $\mathscr{A}$ is regular and each member of $\mathscr{A}_r$ is a diagonal then $\mathscr{A}$ is called *full*.

The injection geometry $\mathscr{S}$ from Example 1.2 is regular. It is full iff $|S_i| = n$ holds for all i.

It is much more complicated to give examples of full injection geometries of higher rank.

Suppose P is a set of permutations on $\{1, 2, ..., n\}$, i.e., P is a subset of S_n. P is called *sharply l-transitive* if for any two ordered l-tuples $((a_1, ..., a_l), (b_1, ..., b_l))$ of distinct elements of N there is a *unique* $\pi \in P$ satisfying $\pi(a_i) = b_i$ for $1 \leqslant i \leqslant l$. Sharply 1-transitive sets are just Latin squares. Each abstract finite group has a representation as sharply 1-transitive permutation group. For $l \geqslant 2$ all sharply l-transitive groups have been classified by Jordan and Zassenhaus (cf. [12]).

EXAMPLE 1.4. Suppose P is a sharply l-transitive set, and $X = N^d$. Define $\mathscr{A}_i = \{A \subset X: A$ is injective of size $i\}, 0 \leqslant i < l$, $\mathscr{A}_l = \{\{(i, \pi_2(i),..., \pi_d(i)), 1 \leqslant i \leqslant n\}: \pi_j \in P, 2 \leqslant j \leqslant d\}$. Then $\mathscr{A} = \mathscr{A}_0 \cup \cdots \cup \mathscr{A}_l$ is an injection geometry of rank l.

Given a permutation group G on N, define a sequence $G_0, G_1, G_2,...,$ of subgroups of G and a sequence $x_1, x_2,...,$ of points of N inductively by the rules:

(i) $G_0 = G$;

(ii) x_{i+1} is a point not fixed by G_i, and G_{i+1} its stabilizer in G_i. This sequence must terminate, since N is finite: that is $G_r = 1$ for some r.

DEFINITION 1.5 (cf. [3]). G is called geometric of rank r if for $0 \leqslant i \leqslant r-1$, G_i is transitive on the set of points which it does not fix and $G_r = 1$.

EXAMPLE 1.6. Suppose G is a geometric group of rank r and $Y^i = \{y^1,..., y^{l_i}\}$ is the set of fixpoints of $G_i, 0 \leqslant i \leqslant r$. Define $\mathscr{A}_i = \{\{(\pi_1(y^j), \pi_2(y^j),..., \pi_d(y^j)): 1 \leqslant j \leqslant l_i\}: \pi_s \in G, 1 \leqslant s \leqslant d\}, 0 \leqslant i \leqslant r$. Then $\mathscr{A} = \mathscr{A}_0 \cup \cdots \cup \mathscr{A}_r$ is an injection geometry of rank r, clearly $\mathscr{A}$ is full and for $A \in \mathscr{A}_i$ we have $|A| = |Y^i| = l_i$.

Note that every l-sharply transitive group is geometric of rank l. Geometric groups with $l_0 = 0, l_1 = 1$; i.e., doubly transitive, have been classified by Kantor [9]. The non sharply transitive ones are $GL(r, 2)$, $n = 2^r - 1$; $AGL(r-1, q)$, $n = q^{r-1}$, and two more groups on 15 and 16 points, respectively.

In general $GL(r, q)$ is geometric of rank r with $l_0 = 0, l_1 = q - 1$, $n = q^r - 1$.

In [4] the procedure of *blow-up* is described which provides new examples from existing ones. As a matter of fact in [4] only the case $d = 2$ is considered but the general cases can be treated similarly. Another class of geometric groups (those with $r = 2, l_0 = 0$) was considered in [1]. They act as sharply edge-transitive group of automorphisms of some digraphs.

2. SOME BASIC PROPERTIES OF INJECTION GEOMETRIES

PROPOSITION 2.1. *Let $\mathscr{A} = \mathscr{A}_0 \cup \cdots \cup \mathscr{A}_r$ be an injection geometry of rank r.*

(i) *If $A, B \in \mathscr{A}_i$ and $A \leqslant B$, then $A = B$;*

(ii) $|\mathscr{A}_0| = 1$;

(iii) *Any element of* $\mathscr{A} - \mathscr{A}_0$ *is the least upper bound (in* $\mathscr{A}$*) of elements in* $\mathscr{A}_1$.

Proof. (i) Suppose $A \neq B$, $i < r$. Then $B - A$ is nonempty, we may take $\mathbf{b} \in B - A$. Thus by the axiom F(iii) there is a unique $C \in \mathscr{A}_{i+1}$ which covers A and $\mathbf{b}$. Again by F(iii), $C \leqslant B$ holds—in contradiction with F(ii). The argument is similar for $i = r$.

(ii) Let A be the intersection of all elements of $\mathscr{A}$. By F(ii) $A \in \mathscr{A}_0$, and by the definition of $A, A \subseteq B$ holds for all $B \in \mathscr{A}_0$. Thus part (i) of the proposition implies $A = B$.

(iii) Choose $A \in \mathscr{A}$ and let $A' \in \mathscr{A}$ be maximal subject to the conditions that $A' \leqslant A$ and A' is the last upper bound of elements of $\mathscr{A}_1$. Suppose for contradicition $A' \neq A$ and let a be an element of $A - A'$. Then $\{a\}$ is disjoint from the unique element of $\mathscr{A}_0$ thus it is covered by a unique element $B \in \mathscr{A}_1$. As $A \cap B \notin \mathscr{A}_0, A \cap B \in \mathscr{A}_1$ and thus $B \leqslant A$ holds. Then A' and B have an upper bound in $\mathscr{A}$ which is strictly larger than A', a contradiction. ∎

For $A, B \in \mathscr{A}$, and $A \subset B$ we define the *interval* $[A, B]$ by $[A, B] = \{C \in \mathscr{A} : A \leqslant C \leqslant B\}$.

The next proposition is obvious.

PROPOSITION 2.2. *For* $A \subset B, A, B \in \mathscr{A}$ *the set-system* $[A, B]$ *is the family of flats of a matroid on B.* ∎

PROPOSITION 2.3. *Suppose* $\mathscr{A}$ *is an injection geometry of rank* r $A \in \mathscr{A}_i$, $B \in \mathscr{A}_j, A \cap B \in \mathscr{A}_t, i + j - t \leqslant r$, *and* $A \cup B$ *is injective. Then there exists* $D \in \mathscr{A}_l$ *for some* $l \leqslant i + j - t$ *such that* $A \leqslant D, B \leqslant D$.

Proof. We apply induction on $i - t$. If $i = t$ then $A \cap B \leqslant A$ implies $A \leqslant B$, i.e., $A \cup B = B$ is an upper bound for A and B. Now suppose $a \in A - B$ and let B' be the unique upper bound of B and a in $\mathscr{A}_{j+1}$. Also let C' be the unique upper bound of $A \cap B$ and a in $\mathscr{A}_{t+1}$. Define t' by $A \cap B' \in \mathscr{A}_{t'+1}$. By F(iii) we have $C' \leqslant A \cap B'$. Thus $t \leqslant t'$. Consequently the least upper bound D of A and B' in $\mathscr{A}$ (which exists by the induction hypothesis) satisfies $A \cup B' \subseteq D$ and $D \in \mathscr{A}_k$, for some $k' \leqslant i + j + 1 - (t' + 1) \leqslant i + j - t$ as desired. ∎

3. RANK FUNCTION IN INJECTION GEOMETRIES

Let us define the rank function $r_{\mathscr{A}}: 2^X \to \{0, ..., r\} \cup \{\infty\}$ for each injection geometry $\mathscr{A}$ of rank r by $r_{\mathscr{A}}(F) = \min\{i : F \subset A \in \mathscr{A}_i\}$ for $F \subset X$ ($r_{\mathscr{A}}(F) = \infty$ is understood if F is not contained in any $A \in \mathscr{A}$). Usually we omit the subscript $\mathscr{A}$ and write simply $r(F)$.

The next proposition collects important properties of the rank function $r(F)$.

PROPOSITION 3.1.

R(i) *$r(F)$ is a nonnegative integer or ∞.*

R(ii) *$r(\varnothing)=0$, $r(x)\leqslant 1$ or $r(x)=\infty$ for all $x\in X$.*

R(iii) *$r(F)$ is monotone ($A\subset B$ implies $r(A)\leqslant r(B)$).*

R(iv) *If F is not injective then $r(F)=\infty$ holds.*

R(v) *Suppose $A\subset B\cap C$, $r(A)=r(B)$, and $r(C)<\infty$. Then $r(B\cup C)<\infty$.*

R(vi) *If $F, G\subset X$ and $r(F\cup G)<\infty$ then $r(F)+r(G)\geqslant r(F\cap G)+r(F\cup G)$ holds (submodularity).*

Proof. Only (v) and (vi) need a proof.

Proof of (v). Set $r(A)=i$ and suppose $A\subset F, B\subset G, F, G\in\mathscr{A}_i$. Now $A\subset B$ implies $A\subset F\cap G$. Using $r(A)=i$ and Proposition 2.1(i), we infer $F=G$. Choose $H\in\mathscr{A}$ satisfying $C\subset H$, this is possible because of $r(C)<\infty$. Then $A\subset B\cap C\subset F\cap H$ implies again $F=F\cap H$, i.e., $F\leqslant H$. Thus $B\subset H$. Hence $B\cup C\subset H$, proving $r(B\cup C)<\infty$. ∎

Proof of (vi). Suppose $r(F\cup G)=k, r(F)=i, r(G)=j, r(F\cap G)=t$, and let D, A, B, C be corresponding minimal sets in $\mathscr{A}$ containing them. As $F\subseteq A, G\subseteq B, F\cup G\subseteq D$, we have $F\subseteq A\cap D, G\subseteq B\cap D$. As $A\cap D$, $B\cap D\in\mathscr{A}, A\subseteq D, B\subseteq D$. By the minimality of D, D is the least upper bound of A and B, thus by Proposition 2.3, we have

$$k=\operatorname{rank} D\leqslant r(A)+r(B)-r(A\cap B)$$
$$=i+j-r(A\cap B)\leqslant i+j-r(C)=i+j-t. \quad ∎$$

Remark 3.2. If $r(F)$ is finite, then $r(G)$ is finite for all $G\subseteq F$ in view of (iv). Thus this proposition shows that $r(G)$ defines a matroid in the usual way, moreover this matroid is the same as $[\varnothing, F]$.

THEOREM 3.3. *Suppose r is a function on X satisfying* R(i–vi) *of Proposition* 3.1, *moreover for some positive integer k we have that $r(F)$ is finite for every injective subset of size k of X. Define $\mathscr{A}_i=\{A\subset X: r(A)=i$, $A\subsetneqq B$ implies $r(B)>i\}, 0\leqslant i\leqslant k$. Then $\mathscr{A}=\mathscr{A}_0\cup\cdots\cup\mathscr{A}_k$ is an injection geometry of rank k whose rank function coincides with r for all $F\subset X$ with $r(F)\leqslant k$.*

Proof of Theorem 3.3. We have to prove that F(i–iii) hold. F(ii) holds trivially. Let us prove F(i). Suppose $A\in\mathscr{A}_i, B\in\mathscr{A}_j$. As $A\cap B<A$,

$r(A \cap B) = t$ is finite. So for some $C \in \mathscr{A}_t$ we have $A \cap B \subseteq C$. Suppose for contradiction that the inequality is strict and say $A \not\supset C$.

Since $r(A \cap B) = r(C)$, R(v) implies that $r(A \cup C)$ is finite. By definition $r(A \cup C) > r(A)$, yielding $r(A \cup C) + r(A \cap C) \geqslant r(A \cup C) + r(A \cap B) > r(A) + r(C)$, contradicting (vi). Now we prove F(iii).

Suppose $r(A) = i < k$. Then the matroid $[\varnothing, A]$ has rank i, consequently it has a basis $B = \{x_1, ..., x_i\}$. Suppose $A \cup \{b\}$ is injective, then $B \cup \{b\}$ is injective of size at most k, thus $r(B \cup \{b\}) < \infty$. R(vi) yields $i \leqslant r(B \cup \{b\}) \leqslant i + 1$. If $r(B \cup \{b\}) = i$, then R(v) yields $r(A \cup \{b\}) < \infty$, and by R(vi) $r(A \cup \{b\}) \leqslant i$, contradicting the maximality of A. Thus $r(B \cup \{b\}) = i + 1$. Let H be a maximal set of rank $i + 1$ containing $B \cup \{b\}$. Then $H \in \mathscr{A}$, consequently $A \cap H \in \mathscr{A}$. We infer from $B \subset A \cap H \subset A$ that $A \cap H = A$, i.e., $A \subset H$. This means $H \in \mathscr{A}_{i+1}, A \cup \{b\} \subset H$. Suppose $A \cup \{b\} \subset H' \in \mathscr{A}$. Then $H \cap H' \in \mathscr{A}, A \cup \{b\} \subseteq H \cap H'$, yielding $i + 1 = r(A \cup \{b\}) \leqslant r(H \cap H') \leqslant r(H) = i + 1$, i.e., $H \cap H' \in \mathscr{A}_{i+1}$. Since $H, H \cap H' \in \mathscr{A}_{i+1}, H \cap H' \subset H$, the definition of $\mathscr{A}_{i+1}$ implies $H = H \cap H'$, i.e., $H \subset H'$. ∎

4. Independence in Injection Geometries

Suppose $X \subset N^d$ and $r: 2^X \to N \cup \{\infty\}$ is a rank function verifying R(i–vi). Call a subset $I \subset X$ *independent* if $r(I) = |I|$ (in particular, $r(I) < \infty$), and *critical* if $r(I) = \infty$ but $r(J) < \infty$ for all proper subsets J of I. Let $\mathscr{I}(\mathscr{C})$ be the collection of all independant (critical) subsets of X, respectively. Set $\mathscr{C}^* = \{D \subset X: \exists C \in \mathscr{C}, C \subset D\}$, i.e., $D \in \mathscr{C}^*$ iff $r(D) = \infty$.

Proposition 4.1.

I(i) *$\mathscr{I}$ is a nonempty complex of injective subsets of X (i.e., $\varnothing \in \mathscr{I}$, and $A \subset B \in \mathscr{I}$ implies $A \in \mathscr{I}$), if $\{x, y\}$ is not injective then $\{x, y\} \in \mathscr{C}^*$.*

I(ii) *If $A \subset C \notin \mathscr{C}^*, A \in \mathscr{I}, C \cup \{x\} \in \mathscr{C}^*$ then $(A \cup \{x\}) \in \mathscr{C}^* \cup \mathscr{I}$.*

I(iii) *If $A, B \in \mathscr{I}, |A| < |B|$, and $r(A \cup B) < \infty$ then there exists $b \in B$ such that $(A \cup \{b\}) \in \mathscr{I}$.*

Proof. (i) is evident by R(iv) and the submodularity of r. To prove (ii) set $B = A \cup \{x\}$ and suppose for contradiction $r(B) = r(A) < \infty$. Now A, B, C verify the assumptions of R(v). Thus $r(B \cup C) = r(C \cup \{x\}) < \infty$, contradicting $(C \cup \{x\}) \in \mathscr{C}^*$. The validity of (iii) follows from the fact that $[\varnothing, A \cup B]$ is a matroid. ∎

Proposition 4.2. *If $B \subsetneq D \in \mathscr{C}$ then $B \in \mathscr{I}$.*

Proof. Suppose for contradiction $r(B) < |B| < \infty$ and let $A \in \mathscr{I}$ satisfy $A \subset B, r(A) = r(B)$. Choose $x \in B - A$ and define $C = D - \{x\}$. The application of I(ii) yields $B \in \mathscr{C}^*$, *a* contradiction. ∎

Given the families of independent and critical subsets one can define a rank function in the following way. Set $r(F) = \infty$ if $C \subset F$ holds for some $C \in \mathscr{C}$. Otherwise set $r(F) = \max\{|I| : I \subset F, I \in \mathscr{I}\}$.

THEOREM 4.3. *Suppose* $\mathscr{I}, \mathscr{C} \subset 2^X$ *verify* I(i–iii), $\mathscr{I} \cap \mathscr{C} = \varnothing$, *and the rank function r is defined as above. Then r fulfills* R(i–vi).

Proof. R(i–iv) hold trivially. To prove R(vi) just note that if $r(F \cup G) < \infty$, then I(i) and I(iii) give that $[\varnothing, F \cup G]$ is a matroid, yielding the submodularity.

Finally we prove R(v). Suppose A, B, C is a counterexample in which $|B \cup C|$ is minimal. We may suppose $A \in \mathscr{I}$. In fact, otherwise just replace A by $A' \subset A, A' \in \mathscr{I}$, and $r(A') = r(A) = r(C)$. Suppose $x \in (B - C)$. The minimality of $|B \cup C|$ implies that $A, B - \{x\}, C$ is not a counterexample, i.e., $(B \cup C - \{x\}) \notin \mathscr{C}^*$. Thus we may apply I(ii) to $A' = A, B' = A \cup \{x\}$, $C' = B \cup C - \{x\}$. We infer $A \cup \{x\} \in \mathscr{C}^*$, in contradiction with $r(B) < \infty$. ∎

5. Injection Designs and Extremal Problems

Here we suppose $X = N^d$. The injection geometry $\mathscr{A} = \mathscr{A}_0 \cup \cdots \cup \mathscr{A}_r$ is called an *injection design* of type $\{l_0, \ldots, l_r\}$ if $0 \leqslant l_0 < l_1 < \cdots < l_r$ and for all $A \in \mathscr{A}_i, |A| = l_i$ holds, $0 \leqslant i \leqslant r$.

In the case $d = 1$ an injection design is just a matroid design, i.e., a matroid in which flats of the same rank have the same size (cf. [11]). Clearly if $\mathscr{A} = \mathscr{A}_0 \cup \cdots \cup \mathscr{A}_r$ is an injection design then so is $\mathscr{A}_0 \cup \cdots \cup \mathscr{A}_i$, $1 \leqslant i < r$, too. It is called the truncation of $\mathscr{A}$. Among the examples given in the first section only Example 1.2 is not an injection design.

An easy computation and induction on i gives

$$|\mathscr{A}_i| = \prod_{j=0}^{i-1} \frac{(n - l_j)^d}{l_i - l_j}. \tag{1}$$

Note that F(i) implies

$$|A \cap A'| \in \{l_0, \ldots, l_{r-1}\} \qquad \text{for } A \neq A' \in \mathscr{A}_r. \tag{2}$$

If $\mathscr{F}$ is a family of l_r-element injective subsets of N^d satisfying (2) then F is called an $\{l_0, \ldots, l_r\}$*-system.*

For $d=1$ it was proved in [6] that for $n>n_0(l_r)$

$$|\mathscr{F}| \leqslant \prod_{0 \leqslant i<r} \frac{n-l_i}{l_r-l_i}, \tag{3}$$

moreover,

$$|\mathscr{F}| < c(l_r) n^{r-1} \qquad \text{unless} \left| \bigcap \mathscr{F} \right| = l_0 \tag{4}$$

Here we use (4) to prove the following generalization of (3).

THEOREM 5.1. *For an arbitrary* $\{l_0, \dots, l_r\}$*-system* F *on* N^d

$$|\mathscr{F}| \leqslant \prod_{0 \leqslant j<r} \frac{(n-l_j)^d}{l_r-l_j} \qquad \textit{holds provided } n>n_0(l_r). \tag{5}$$

Moreover, in case of equality $\mathscr{F}$ *is an injection design.*

Proof. We prove (5) by induction on l_r. Suppose first $l_0=0$. For $x \in N^d$ define $\mathscr{F}(x)=\{F-\{x\}: x \in F \in \mathscr{F}\}$. Denote by Y the set of points of N^d which together with x form an injective set. That is, Y is obtained from N^d by omitting the points of the hyperplanes through x. Thus Y can be considered as $(N-1)^d$ and $\mathscr{F}(x)$ is an $\{l_1-1, \dots, l_r-1\}$-system on Y. By induction we have

$$|\mathscr{F}(x)| \leqslant \prod_{1 \leqslant j<r} \frac{((n-1)-(l_j-1))^d}{(l_r-1)-(l_j-1)} = \prod_{1 \leqslant j<r} \frac{(n-l_j)^d}{l_r-l_j}. \tag{6}$$

Summing (6) over all $x \in N^d$, we infer

$$l_r |\mathscr{F}| = \sum_{x \in N^d} |\mathscr{F}(x)| \leqslant n^d \prod_{1 \leqslant j<r} \frac{(n-l_j)^d}{l_r-l_j},$$

which is equivalent to (5).

Suppose next $l_0>0$. If $|\cap \mathscr{F}|<l_0$, then (4) implies $|\mathscr{F}|<c(l_r)(n^d)^{r-1}$, yielding (5) for $n>n_0(l_r)$.

Thus we may assume that there is an $A \subset N^d$, $|A|=l_0$ such that $A \subset F$ holds for all $F \in \mathscr{F}$. Therefore A is injective. Thus $\mathscr{F}(A) \overset{\text{def}}{=} \{F-A: F \in \mathscr{F}\}$ can be considered as a $\{0, l_1-l_0, \dots, l_r-l_0\}$-system on $(N-l_0)^d$, and (5) follows by induction. The characterization of equality is merely technical, it is left to the reader. ∎

For $n>n_0(l_r)$, (5) implies that if $\mathscr{A}=\mathscr{A}_0 \cup \cdots \cup \mathscr{A}_r$ is an injection design with flats of size $l_0, \dots, l_r$ then $\mathscr{A}_r$ is not extendible as an $\{l_0, \dots, l_r\}$-system. Our next result shows that if $d \geqslant 2$ then this is true even without the assumption $n>n_0(l_r)$. In [8] the same is conjectured for $d=1$ and is proved for all known examples of matroid designs.

PROPOSITION 5.2. *If $d \geqslant 2$ and $\mathscr{A}_r$ is the family of hyperflats of an injection design $\mathscr{A}$ of type $(l_0, ..., l_r)$ and $B \notin \mathscr{A}, |B| = l_r$, B is injective, then there exists $A \in \mathscr{A}_r$ with $|B \cap A_r| \notin \{l_0, ..., l_r\}$.*

Proof. Apply induction on l_r. First we settle the case $l_0 = 0, r = 1$. Then $\mathscr{A}_r$ is a partition of N^d, $\mathscr{A}_0 = \{\varnothing\}$. Thus $B \neq \varnothing$. Consequently $B \cap A \neq \varnothing$ for some $A \in \mathscr{A}_r$. Now $B \notin \mathscr{A}_r$ implies $0 < |B \cap A| < l_1$, as desired.

In the general case suppose that B is a counterexample. Assume first that $|A \cap B| \in \{l_0, ..., l_r\}$ holds for all $A \in \mathscr{A}$. If $l_0 > 0$ and $A_0 \in \mathscr{A}_0$, then $A_0 \subset B$ follows. Consider $\mathscr{A}(A_0) = \{A - A_0 : A \in \mathscr{A}\}$ and $B - A_0$. Then the induction hypothesis together with $|B \cap A| = |(B - A_0) \cap (A - A_0)| + l_0$ yields the contradiction.

Thus there exists a maximal $i, 0 \leqslant i < r$ and $A \in \mathscr{A}_i$ so that $|A \cap B| \notin \{l_0, ..., l_r\}$ holds. Let $A_1, ..., A_m$ be the collection of flats of rank $i+1$ in $\mathscr{A}$ which contain A. Since $\mathscr{A}$ is an injection design, $m = (n - l_i)^d/(l_{i+1} - l_i)$ and by the maximality of i, B should intersect all of the pairwise disjoint sets $A_j - A$, $1 \leqslant j \leqslant m$, which form an injection design of the type $\{0, l_{i+1} - l_i\}$ on some Y of the form $(N - l_i)^d$. Since $B \cap Y \stackrel{\text{def}}{=} B_0$ is injective, $|B_0| \leqslant n - l_i$. We infer $n - l_i \geqslant m = (n - l_i)^d/(l_{i+1} - l_i)$ which is possible only if $d = 2$, $n = l_{i+1}$, and $|B_0| = n - l_i$. However, this implies $|B - Y| \geqslant l_i$, i.e., by injectivity $A \subset B$, contradicting the choice of A. ∎

6. REPRESENTATION OF INJECTION GEOMETRIES OVER FIELDS

Suppose $m \geqslant r > 1$ and $V_1, ..., V_d$ are disjoint copies of the m-dimensional vector space over a fixed field K (K may be infinite). Set $V = V_1 \oplus V_2 \oplus \cdots \oplus V_d$. For a subspace W of V let $\pi_i(W)$ be its canonical projection onto V_i. The subspace W is called *transversal* if $\dim \pi_i(W) = \dim W$ holds for $1 \leqslant i \leqslant d$.

It is easy to see that for $r \leqslant m$ the collection of all transversal subspaces of dimension at most r is an injection geometry of rank r.

For a subset A of V let $\langle A \rangle$ denote the smallest subspace containing it. Clearly $r(A) = \dim \langle A \rangle$ if $\langle A \rangle$ is transversal and ∞ otherwise.

Suppose $\mathscr{A}$ is an injection geometry on X. It is called *representable* over the field K if for m sufficiently large (it may be much larger than r) one can find a subset $Y \subset V$ and a 1–1 correspondence φ between the elements of X and Y so that $r(A) = r(\varphi(A))$ holds for all $A \subset X$.

Note that for $d = 1$ we get back the usual notion of matroids representable over K.

7. Concluding Remarks

Instead of all injective subsets of X, we could have considered an an arbitrary hereditary family (complex) $\mathscr{F}$ of subsets of X in the axioms. The objects obtained in this way can be called *$\mathscr{F}$-geometries* or *squashed geometries*. The equivalence of the different systems of axioms can be proved in the same way. For example, if $X_1 \cup X_2 \cup \cdots \cup X_r$ is a partition of X and $\mathscr{F}$ is the family of all partial transversals, $\mathscr{F} = \{F \subset X: |F \cap X_i| \leqslant 1\}$, we obtain *transversal geometries*, which were first considered in a slightly less general form in [7].

In continuation of the present work, in [14] other equivalent systems of axioms (base, circuit, closure) for *$\mathscr{F}$-geometries* are given.

In a forthcoming paper we consider various constructions of *squashed designs*, in particular, injection designs. Recently much work has been done in examination of structures related to matroids. Let us mention, as an example, greedoids which already have a long literature (cf. Korte & Lovász [10], Björner [2]).

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