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COMMUNICATION

A PROBABILISTIC PROOF FOR THE LYM-INEQUALITY

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A new short proof is given for the fundamental inequality concerning antichains. A proof which uses elementary probability theory and not chains or cyclic permutations.

Let $\mathscr{F}$ be any antichain consisting of subsets of $X=\{1,2,\ldots,n\}$ (i.e. no member of $\mathscr{F}$ contains another one). Sperner [3] proved that in this case $|\mathscr{F}|\leqslant\binom{n}{n/2}$ holds. This was refined by Lubell [1], Yamamoto [4] and Meshalkin [2] to

$$\sum_{F\in\mathscr{F}} 1\Big/\binom{n}{|F|}\leqslant 1, \quad \text{whenever } \mathscr{F} \text{ is an antichain on } X. \tag{1}$$

Here we give a short, inductive argument yielding (1). First note that (1) is evident if $n=1$, and also if $X\in\mathscr{F}$ (in the latter case necessarily $\mathscr{F}=\{X\}$ holds). Now assume (1) is true for $n-1$, $\mathscr{F}$ is an antichain and $X\notin\mathscr{F}$.

Let x be a random variable which takes the values $1,\ldots,n$; each with probability $1/n$. Let us define $\mathscr{F}(x)=\{F\in\mathscr{F}: x\notin F\}$. For any function $g(x)$, we denote by $E(g(x))$ its expectation. As $\mathscr{F}(x)$ is an antichain on X-$\{x\}$, for $\mathscr{F}(x)$ (1) holds with $n-1$ instead of n. We infer ($p(F\in\mathscr{F}(x))$ denotes the probability that $F\in\mathscr{F}$ belongs to the random family $\mathscr{F}(x)$, thus it equals $(n-|F|)/n$):

$$1\geqslant E\Bigg(\sum_{F\in\mathscr{F}(x)} 1\Big/\binom{n-1}{|F|}\Bigg)$$
$$=\sum_{F\in\mathscr{F}} p(F\in\mathscr{F}(x))\Big/\binom{n-1}{|F|}=\sum_{F\in\mathscr{F}}\frac{n-|F|}{n}\Big/\binom{n-1}{|F|}=\sum_{F\in\mathscr{F}} 1\Big/\binom{n}{|F|},$$

as desired.

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