JOURNAL OF COMBINATORIAL THEORY, Series A **34**, 41–45 (1983)

On the Trace of Finite Sets

PETER FRANKL*

CNRS, Paris, France and TATA Institute, Bombay, India

Communicated by the Managing Editors

Received January 25, 1981

For a family $\mathscr{F}$ of subsets of an n-set X we define the trace of it on a subset Y of X by $\mathrm{T}_{\mathscr{F}}(Y)=\{F\cap Y : F\in\mathscr{F}\}$. We say that $(m,n)\to(r,s)$ if for every $\mathscr{F}$ with $|\mathscr{F}|\geqslant m$ we can find a $Y\subset X\,|Y|=s$ such that $|\mathrm{T}_{\mathscr{F}}(Y)|\geqslant r$. We give a unified proof for results of Bollobàs, Bondy, and Sauer concerning this arrow function, and we prove a conjecture of Bondy and Lovász saying $(\lfloor n^2/4\rfloor+n+2,n)\to(3,7)$, which generalizes Turán's theorem on the maximum number of edges in a graph not containing a triangle.

1. INTRODUCTION

Let $\mathscr{F}$ be a family of subsets of $X=\{1,2,\ldots,n\}$. For a subset Y of X we set $\mathrm{T}_{\mathscr{F}}(Y)=\{F\cap Y : F\in\mathscr{F}\}$. Note that in $\mathrm{T}_{\mathscr{F}}(Y)$ we take every set only once. We call $\mathrm{T}_{\mathscr{F}}(Y)$ the trace of $\mathscr{F}$ on Y.

DEFINITION. The arrow relation $(m,n)\to(r,s)$ means that whenever $|\mathscr{F}|\geqslant m$, we can find $Y\subset X$, $|Y|=s$ such that $|\mathrm{T}_{\mathscr{F}}(Y)|\geqslant r$.

Bondy [1] proved that

$$(m,n)\to(m,n-1)\qquad\text{if}\quad m\leqslant n.\tag{1}$$

Bollobàs (see [6]) proved that

$$(m,n)\to(m-1,n-1)\qquad\text{if}\quad m\leqslant\lceil\tfrac{3}{2}n\rceil.\tag{2}$$

Sauer [8] proved that

$$(m,n)\to(2^s,s)\qquad\text{if}\quad m>\sum_{i=0}^{s-1}\binom{n}{i}.\tag{3}$$

Let us remark that these bounds are easily seen to be best possible.

* Present address: U.E.R. Math. Université Paris VII, 45–55 Place Jussieu, 75005 Paris.

0097-3165/83/010041–05$03.00/0

2. Results

Recall that a family, $\mathscr{F}$ of sets is *hereditary* if $G \subset F \in \mathscr{F}$ implies $G \in \mathscr{F}$. Our main result is the following:

THEOREM 1. *$(m, n) \to (r, s)$ holds if, whenever $\mathscr{F}$ is a hereditary family of subsets of $X = \{1, \dots, n\}$ and $|\mathscr{F}| = m$, there exists a set $Y \subset X$, $|Y| = s$ such that $|\mathrm{T}_{\mathscr{F}}(Y)| \geqslant r$.*

To show the effectiveness of this theorem we now deduce from it the three results mentioned in the Introduction.

For (3) just note that a hereditary family $\mathscr{F}$ with $|\mathscr{F}| > \sum_{i=0}^{s-1} \binom{n}{i}$ necessarily contains a set Y with $|Y| = s$ and then $|\mathrm{T}_{\mathscr{F}}(Y)| = 2^s$. As every nonempty hereditary family contains the empty set, $|\mathscr{F}| \leqslant n$ implies that for some $x \in X$ the singleton $\{x\}$ is not in $\mathscr{F}$, i.e., $|\mathrm{T}_{\mathscr{F}}(X - \{x\})| = |\mathscr{F}|$ proving (1).

If $|\mathscr{F}| \leqslant \lceil \frac{3}{2}n \rceil$, and $\{x\} \notin \mathscr{F}$ for some x, then again $\mathrm{T}_{\mathscr{F}}(X - \{x\}) = \mathscr{F}$. But if $\mathscr{F}$ contains the empty set and all the singletons, then there must be an $x \in X$ which is not covered by any two-element set in $\mathscr{F}$ (otherwise, $|\mathscr{F}| \geqslant 1 + n + \lceil n/2 \rceil > \lceil \frac{3}{2}n \rceil$), thus by the hereditary property $\{x\}$ is the only member of $\mathscr{F}$ containing x, i.e., $|\mathrm{T}_{\mathscr{F}}(X - \{x\})| = |\mathscr{F}| - 1$, yielding (2).

Let us recall Turàn's theorem for graphs without triangles:

THEOREM 2 (Turàn [9]). *If G is a simple graph on n vertices and without a triangle (i.e., 3 edges $\{x, y\}$, $\{y, z\}$, and $\{x, z\}$), then G has at most $\lfloor n^2/4 \rfloor$ edges.*

Bondy and Lovàsz conjectured that the following generalization of Turàn's theorem is true:

THEOREM 3. *If $m > \lfloor n^2/4 \rfloor + n + 1$, then $(m, n) \to (3, 7)$.*

To see that (4) generalizes Theorem 2 define for the simple graph G the family $\mathscr{F}(G)$ consisting of its edges, vertices, and the empty set. Now $|G| > \lfloor n^2/4 \rfloor$ yields $|\mathscr{F}(G)| > \lfloor n^2/4 \rfloor + n + 1$. Thus (4) guarantees the existence of 3 vertices x, y, z such that $|\mathrm{T}_{\mathscr{F}(G)}(\{x, y, z\})| \geqslant 7$. As $\mathscr{F}(G)$ contains only sets of cardinality 2 or less, these 7 sets are $\varnothing$, $\{x\}$, $\{y\}$, $\{z\}$, and the triangle $\{x, y\}$, $\{x, z\}$, $\{y, z\}$.

On the other hand, Theorem 3 is an immediate consequence of Theorem 1: we may assume $\mathscr{F}$ is hereditary. If it contains only sets of cardinality not exceeding 2, then (4) is just equivalent to Turàn's theorem. If $F \in \mathscr{F}$, $|F| = 3$, however, then $|\mathrm{T}_{\mathscr{F}}(F)| = 8 > 7$. Q.E.D.

We shall apply Theorem 1 to prove the following:

THEOREM 4. *Let t be a positive integer. If $m \leqslant \lceil n(2^t - 1)/t \rceil$, then*

$$(m, n) \to (m - 2^{t-1} + 1, n - 1). \tag{5}$$

Note that for $t = 1, 2$, (5) yields (1) and (2), respectively.

To prove (5) we need the Kruskal–Katona theorem. Define the antilexicographic ordering of subsets of $\{1, 2, \ldots, n\}$ by

$$A < B \qquad \text{iff} \quad A \subset B \qquad \text{or} \quad \max_{i \in A - B} i < \max_{i \in B - A} i.$$

For integers k, m let $\mathscr{F}(m, n)$, $(\mathscr{F}(m, k, n))$ denote the first m sets (k-subsets) in the antilexicographic ordering, respectively.

THEOREM 5 (Kruskal [5], Katona [4], for a simple proof see Daykin [2]). *Let $\mathscr{F}$ be a family of m sets each of cardinality k. Then for $0 < l < k$ the number of l-sets contained in some member of $\mathscr{F}$ is at least as much as that for $\mathscr{F}(m, k, n)$.*

We shall use the following easy corollary (cf. [3]).

COROLLARY. *Let $\mathscr{F}$ be a hereditary family, $|\mathscr{F}| = m$. Then for every monotone nonincreasing function $f(x)$ we have*

$$\sum_{F \in \mathscr{F}} f(|F|) \geqslant \sum_{F \in \mathscr{F}(m,n)} f(|F|). \tag{6}$$

3. THE PROOF OF THEOREM 1

Let us suppose the arrow relation $(m, n) \to (r, s)$ is false. Let $\mathscr{F}$ be a counterexample for which $\sum_{F \in \mathscr{F}} |F|$ is minimal.

Suppose $\mathscr{F}$ is not hereditary. Then we can find $F_0 \in \mathscr{F}$ and $i \in X$ such that $i \in F_0$, but $(F_0 - \{i\}) \notin \mathscr{F}$. Let us define the following transformation:

$$\begin{aligned} H(E) &= E - \{i\}, && \text{if} \quad i \in E, \quad (E - \{i\}) \notin \mathscr{F}, \\ &= E, && \text{otherwise}, \\ H(\mathscr{F}) &= \{H(F) \colon F \in \mathscr{F}\}. \end{aligned}$$

Obviously, $|\mathscr{F}| = |H(\mathscr{F})|$ and the sets of the two families differ only in the element i. Moreover, $H(F_0) = F_0 - \{i\}$ yielding

$$\sum_{F \in \mathscr{F}} |F| > \sum_{G \in H(\mathscr{F})} |G|.$$

The minimal choice of $\mathscr{F}$ implies the existence of $Y \subset X$ with $|Y| = s$, $|\mathrm{T}_{H(\mathscr{F})}(Y)| \geqslant r$. We want to prove the theorem by establishing the contradiction $|\mathrm{T}_{\mathscr{F}}(Y)| \geqslant r$. As for $i \notin Y$, we have $\mathrm{T}_{\mathscr{F}}(Y) = \mathrm{T}_{H(\mathscr{F})}(Y)$, we assume $i \in Y$. We divide the 2^s subsets of Y into 2^{s-1} pairs $(Z, Z \cup \{i\})$, where $Z \subseteq Y - \{i\}$. We state

$$|\mathrm{T}_{H(\mathscr{F})}(Y) \cap \{Z, Z \cup \{i\}\}| \leqslant |\mathrm{T}_{\mathscr{F}}(Y) \cap \{Z, Z \cup \{i\}\}|. \tag{7}$$

If the left-hand side is zero, (7) is trivial. If it is 1, then (7) follows from: $(H(F) \cap Y) \in \{Z, Z \cup \{i\}\}$ for every $F \subset X$, iff $F \cap Y \in \{Z, Z \cup \{i\}\}$. Thus we may assume $(Z \cup \{i\}) \in \mathrm{T}_{H(\mathscr{F})}(Y)$, i.e., for some F we have $H(F) \cap Y = Z \cup \{i\}$. In particular, $i \in H(F)$ which means $F = H(F)$ and $(F - \{i\}) \in \mathscr{F}$, by the definition of the operation H. We infer that $F \cap Y = Z \cup \{i\}$ and $(F - \{i\}) \cap Y = Z$ are both in $\mathrm{T}_{\mathscr{F}}(Y)$, proving (7).

Now summing up (7) for all $Z \subseteq (Y - \{i\})$ gives

$$\begin{aligned} |\mathrm{T}_{H(\mathscr{F})}(Y)| &= \sum_{Z \subseteq Y - \{i\}} |\mathrm{T}_{H(\mathscr{F})}(Y) \cap \{Z, Z \cup \{i\}\}| \\ &\leqslant \sum_{Z \subseteq Y - \{i\}} |\mathrm{T}_{\mathscr{F}}(Y) \cap \{Z, Z \cup \{i\}\}| = |\mathrm{T}_{\mathscr{F}}(Y)|, \end{aligned}$$

i.e., $|\mathrm{T}_{\mathscr{F}}(Y)| \geqslant |\mathrm{T}_{H(\mathscr{F})}(Y)| \geqslant s$, the desired contradiction.

4. The Proof of Theorem 4

By Theorem 1, we may assume indirectly that we have a hereditary counterexample $\mathscr{F}$, which means that every element of X is contained in at least 2^{t-1} members of $\mathscr{F}$ but $|\mathscr{F}| \leqslant \lceil n(2^t - 1)/t \rceil$. Let $\mathrm{L}(i)$ be the link of $i \in X$, that is to say, $\mathrm{L}(i) = \{E \subseteq (X - \{i\}) : (E \cup \{i\}) \in \mathscr{F}\}$. Now $\mathrm{L}(i)$ is a hereditary family with $|\mathrm{L}(i)| \geqslant 2^{t-1}$. We want to apply the corollary of Section 2 with $f(x) = 1/(x+1)$ as a nonincreasing function. Note that the first 2^{t-1} sets in the antilexicographic order are just all the subsets of $\{1, 2, \ldots, t-1\}$. We infer

$$\begin{aligned} \sum_{A \in \mathrm{L}(i)} 1/(|A|+1) &\geqslant \sum_{F \in \mathscr{F}(|\mathrm{L}(i)|, n)} 1/(|F|+1) \geqslant \sum_{F \in \mathscr{F}(2^{t-1}, n)} 1/(|F|+1) \\ &= \sum_{i=0}^{t-1} \binom{t-1}{i} \Big/ (i+1) = \sum_{j=1}^{t} \frac{1}{t} \binom{t}{j} = (2^t - 1)/t. \end{aligned}$$

Using this inequality and $\varnothing \in \mathscr{F}$, we deduce

$$|\mathscr{F}| = 1 + \sum_{F \in \mathscr{F}} \sum_{i \in F} 1/|F| = 1 + \sum_{i \in X} \sum_{i \in F \in \mathscr{F}} 1/|F|$$

$$= 1 + \sum_{i \in X} \sum_{A \in L(i)} 1/(|A| + 1) \geqslant 1 + n \frac{2^t - 1}{t}$$

which gives the result.

Remark. If t divides n, then Theorem 4 is best possible. To see this, let $X = Y_1 \cup Y_2 \cup \cdots \cup Y_{n/t}$ with $|Y_i| = t$ and define $\mathscr{F} = \{F \subset X: \exists i, 1 \leqslant i \leqslant n/t, F \subseteq Y_i\}$.

References

1. J. A. Bondy, Induced subsets, *J. Combin. Theory Ser. B* **12** (1972), 201–202.
2. D. E. Daykin, A simple proof of the Kruskal–Katona theorem, *J. Combin. Theory Ser. A* **17** (1974), 252–253.
3. G. O. H. Katona, Optimization for order ideals under a weight assignment, *in* "Problèmes Comb. et Théorie des Graphes," pp. 257–8, Colloque Int., CNRS, Paris, 1978.
4. Gy. Katona, A theorem of finite sets, *in* "Theory of Graphs," pp. 187–207. Proc. Coll., Tihany, Hungary 1966.
5. J. B. Kruskal, The number of simplices in a complex, *in* "Math. Opt. Techn.," pp. 251–278, Univ. of Californiaat Berkeley and Los Angeles, 1963.
6. L. Lovász, "Combinatorial Problems and Exercises," North-Holland, Amsterdam, 1979 (Problem 13/10).
7. L. Lovász, Personal communication, 1975.
8. N. Sauer, On the density of families of sets, *J. Combin. Theory Ser. A* **13** (1972), 145–147.
9. P. Turán, Eine Extremalaufgabe aus der Graphentheorie, *Mat. Lapok* **48** (1941), 436–452 (in Hungarian).