



Some recent results on Ramsey-type numbers

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ABSTRACT

In this paper we survey the authors' recent results on quantitative extensions of Ramsey theory. In particular, we discuss our recent results on Folkman numbers, induced bipartite Ramsey graphs, and explicit constructions of Ramsey graphs.

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1. Introduction

Ramsey theory can be loosely described as the study of structure which is preserved under finite decomposition. Its underlying philosophy is captured succinctly by the statement “Complete disorder is impossible”. Since the publication of the seminal paper of Ramsey [31] in 1930, the subject has grown with increasing vitality. For more information about Ramsey theory, see e.g., [21,22].

Ramsey Theorem ([31]). Let a_1, \dots, a_r and l be given. Then, there exists an integer n with the property that if the l -subsets of an n -set are colored with r colors, say c_1, \dots, c_r , then for some i , $1 \leq i \leq r$, there is an a_i -set, all of whose l -subsets have color c_i .

In particular, Ramsey's theorem implies that for every integer k and r there exists an n such that any r -coloring of the edges of K_n yields a monochromatic clique K_k (the smallest such n is called a *Ramsey number*). In Sections 1.1 and 1.2 we discuss some extensions of this statement. In Section 1.3 we give constructive bounds on off-diagonal Ramsey numbers. In Section 1.4 we present a generalization of induced Ramsey-type problems.

1.1. Edge Folkman numbers

For $k < l$, let $\mathcal{F}_e(r, k, l)$ be the family of K_l -free graphs with the property that if $G \in \mathcal{F}_e(r, k, l)$, then every r -coloring of the edges of G yields a monochromatic copy of K_k . Folkman [15] showed that $\mathcal{F}_e(2, k, l) \neq \emptyset$ for any $k < l$. The general case, i.e., $\mathcal{F}_e(r, k, l) \neq \emptyset$, $r \geq 2$, was settled positively by Nešetřil and Rödl [29]. Let

$$f_e(r, k, l) = \min\{|V(G)| : G \in \mathcal{F}_e(r, k, l)\}$$

be an *edge Folkman number*. The problem of determining the numbers $f_e(r, k, l)$, which in general includes the classical Ramsey numbers, is not easy.

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Theorem 1.4 (Dudek and Rödl [12]). For a given natural number r there exists a constant $c = c(r)$ such that for every k the vertex Folkman number satisfies

$$f_v(r, k, k + 1) \leq ck^2 \log^4 k.$$

It would be interesting to improve this bound. Perhaps an easier problem is bounding $f_v(r, k, l)$ when $l = ck$ for some constant $c > 1$. Łuczak et al. [37] showed that $f_v(r, k, r(k - 1)) \leq r(k - 1) + k^2 + 1$. Subsequently, Kolev and Nenov [25] proved that $f_v(r, k, r(k - 1)) \leq r(k - 1) + 3k + 1$. Complementing their result one can prove the following statement.

Theorem 1.5 (Dudek and Rödl [12]). For a given natural number r and an arbitrarily small $\varepsilon > 0$ there exists a constant $c = c(r, \varepsilon)$ such that for every k the vertex Folkman number satisfies

$$f_v(r, k, \lceil(2 + \varepsilon)k\rceil) \leq ck.$$

We were unable to find any nontrivial lower bound on $f_v(r, k, k + 1)$. It would be interesting to decide if the ratio $\frac{f_v(r, k, k+1)}{k}$ tends to infinity together with k . Here we propose the following problems.

Problem 1.6. Let an integer $r, r \geq 2$, be given. Is it true that

$$\lim_{k \rightarrow \infty} \frac{f_v(r, k, k + 1)}{k} = \infty?$$

Problem 1.7. Is it true that for each $\varepsilon > 0$ and a given $r \geq 2$

$$\lim_{k \rightarrow \infty} \frac{f_v(r, k, (1 + \varepsilon)k)}{k} < \infty?$$

1.3. Explicit Ramsey graphs

In 1947 Erdős proved that there are graphs on n vertices which contain neither a clique nor an independent set of size $(2+o(1)) \log_2 n$. This was one of the first applications, of the probabilistic method in combinatorics, a method with which one can prove the existence of finite structures, without finding a concrete definition of it. Therefore, Erdős asked for an explicit construction of such a graph, possibly with a constant larger than 2. This problem is still open. Frankl and Wilson [18] found an explicit construction for graphs that do not contain a clique or independent set of size $2^{\Theta(\sqrt{\log n \log \log n})}$. This has been recently superseded by Barak et al. [4] who gave a “strongly explicit construction”¹ of a graph which contains neither a clique nor an independent set of size $2^{(\log n)^{o(1)}}$.

In this section we discuss a nonsymmetric version of this problem. For two positive integers s and m the Ramsey number $R(s, m)$ is the least integer R such that every graph on R vertices contains a clique of size s or an independent set of size m . The best known lower bounds on Ramsey numbers are proven by probabilistic methods which do not give explicit constructions. All known constructions give worse bounds. For example, Kim [23] showed that

$$R(3, m) = \Theta\left(\frac{m^2}{\log m}\right),$$

while the best constructive lower bound (found by Alon [1]) is

$$R(3, m) \geq \Omega\left(m^{\frac{3}{2}}\right). \tag{1}$$

In [6] the lower bound (1) was proved by another algebraic argument. One can prove the following.

Theorem 1.8 (Kostochka et al. [26]). There are three families of explicitly defined graphs that give the following constructive bounds

$$R(4, m) \geq \Omega\left(m^{8/5}\right), \quad R(5, m) \geq \Omega\left(m^{5/3}\right), \quad R(6, m) \geq \Omega\left(m^2\right).$$

Note that the best known lower bounds [23,36] (which are nonconstructive) are $R(4, m) \geq \Omega\left((m/\log m)^{2.5}\right)$, $R(5, m) \geq \Omega\left((m/\log m)^3\right)$, and $R(6, m) \geq \Omega\left((m/\log m)^{3.5}\right)$.

Another construction was considered in [2]. That construction gives polynomial lower bounds on $R(s, m)$ with degree slowly increasing with s . However, for small values of s (such as $s = 3, 4, 5, 6$) that does not give anything interesting.

¹ An algorithm that decides whether $\{x, y\}$ is an edge in polynomial time over the size of the encoding of x and y , i.e., in time $\text{polylog}(n)$.

1.4. Bipartite graphs with the ε -density property

For a given graph G we say that R is its *induced Ramsey graph* if for any 2-coloring of the edges of R there is a monochromatic copy of G which is an induced subgraph of R . We write $R \xrightarrow{\text{ind}} G$ to denote this fact. It is well known that every graph G has a Ramsey graph R as was proved by Erdős et al. [14], Deuber [8], and Rödl [32]. This variant of Ramsey’s theorem immediately raises a numerical problem. For a given graph G , let $r_{\text{ind}}(G)$ denote the smallest integer n for which G has a Ramsey graph of order n . In 1975 Erdős and Rödl stated the following problem (see, e.g., [5]).

Problem 1.9. If G has n vertices, is it true that $r_{\text{ind}}(G) < c^n$ for some absolute constant c ?

This problem remains open. A weaker upper bound $r_{\text{ind}}(G) < c^{n \log^2 n}$ was given by Kohayakawa et al. [24]. A different proof of the same upper bound was recently found by Fox and Sudakov [16]. Note that if true, the exponential upper bound in Problem 1.9 is best possible since it is known that $r_{\text{ind}}(K_n) \geq 2^{n/2}$. Finally, let us note that for G bipartite the answer to Problem 1.9 is affirmative. This follows from the proof of Rödl [32].

Here we prove a stronger result. For a given bipartite graph G we write $R \xrightarrow{\varepsilon \text{ ind}} G$ and say that a bipartite graph R has the *induced ε -density property* if every subgraph of R with at least $\varepsilon|E(R)|$ edges contains a copy of G which is an induced subgraph of R .

Theorem 1.10. For every ε , $0 < \varepsilon < 1$, there is a constant $c = c(\varepsilon)$ such that for each n there exists a bipartite graph of order 2^{cn} such that $R \xrightarrow{\varepsilon \text{ ind}} G$ for every bipartite graph $G = (V_1, V_2, E)$ with $|V_1|, |V_2| \leq n$.

Our proof is deterministic. In Section 2 we give an explicit construction of the graph R from Theorem 1.10. Clearly, setting $\varepsilon = \frac{1}{k}$ in Theorem 1.10 yields the following corollary.

Corollary 1.11. For every integer k there is a constant $c = c(k)$ such that for each n there exists a bipartite graph of order 2^{cn} such that any k -coloring of its edges yields a monochromatic and induced copy of every $G = (V_1, V_2, E)$ with $|V_1|, |V_2| \leq n$.

2. Proof of Theorem 1.10

In order to prove Theorem 1.10 we will need the following result from combinatorics of finite sets obtained independently by Sauer [33], Shelah [34], and Vapnik and Chervonenkis [38].

Lemma 2.1 ([33,34,38]). Let \mathcal{F} be a family of subsets of an n -element set X satisfying

$$|\mathcal{F}| > \sum_{i=0}^{k-1} \binom{n}{i}. \tag{2}$$

Then, there is a set $K \subseteq X$ of size k such that for every subset $S \subseteq K$ there is $F \in \mathcal{F}$ such that $S = F \cap K$.

Now we prove Theorem 1.10. Let ε , $0 < \varepsilon < 1$, be given. The existence of a constant $c = c(\varepsilon)$ will follow from the proof (see inequality (4) below). Let $N = cn$. For simplicity we will always assume that n is sufficiently large (i.e., $n > n_0(\varepsilon)$) and that $\frac{\varepsilon N}{4}$ is an integer.

Define $R = (W_1, W_2, F)$ to be a bipartite graph with $|W_1| = N$ and $|W_2| = \binom{N}{N/2}$. Every vertex of W_2 is connected to all vertices of precisely one $N/2$ -subset of W_1 . Consequently, $\deg_R(w) = N/2$ for every $w \in W_2$, and hence, $|F| = \binom{N}{N/2} (N/2)$.

Clearly, the order of R is bounded by

$$|W_1| + |W_2| \leq N + \binom{N}{\frac{N}{2}} \leq 2^N.$$

It remains to show that R has the induced ε -density property for any bipartite graph $G = (V_1, V_2, E)$ with $|V_1|, |V_2| \leq n$.

Let $R' = (W_1, W_2, F')$ be a subgraph of R with at least $\varepsilon|F| = \varepsilon \binom{N}{N/2} (N/2)$ edges. Then, there are at least $\frac{\varepsilon}{2} \binom{N}{N/2}$ vertices in W_2 of degree $\deg_{R'}(w) > \frac{\varepsilon N}{4}$. Otherwise there would be at most $\frac{\varepsilon}{2} \binom{N}{N/2} - 1$ vertices in W_2 of degree higher than $\frac{\varepsilon N}{4}$, which leads to the following contradiction

$$|F'| < \left(\frac{\varepsilon}{2} \binom{N}{N/2} - 1 \right) N/2 + \binom{N}{N/2} \frac{\varepsilon N}{4} < \varepsilon \binom{N}{N/2} N/2 \leq |F'|.$$

Among the vertices of degree higher than $\frac{\varepsilon N}{4}$ we find at least

$$\frac{\frac{\varepsilon}{2} \binom{N}{N/2}}{\frac{N}{2} - \frac{\varepsilon N}{4}} = \frac{\varepsilon}{(1 - \frac{\varepsilon}{2})N} \binom{N}{N/2}$$

vertices of degree j , for some fixed j , $\frac{\varepsilon N}{4} < j \leq N/2$. Let $U \subseteq W_2$ be the set of these vertices. Clearly, $\deg_{R'}(u) = j$ for every $u \in U$, and

$$|U| \geq \frac{\varepsilon}{(1 - \frac{\varepsilon}{2})N} \binom{N}{N/2}.$$

For $u \in U$ let $J_u = N_R(u) - N_{R'}(u) \subseteq W_1$. Clearly, $|J_u| = N/2 - j$ for every $u \in U$. Consequently, there exists a set $J \in \binom{W_1}{N/2-j}$ such that for at least

$$\frac{|U|}{\binom{N}{N/2-j}} > \frac{\frac{\varepsilon}{(1-\frac{\varepsilon}{2})N} \binom{N}{N/2}}{\binom{N}{N/2-\varepsilon N/4}} \tag{3}$$

vertices $u \in U$ the equality $J_u = J$ holds. Let $P = P_J$ be the set of all these vertices. Since all vertices $u \in P$ have the same $J_u = J$, they have distinct neighborhoods $N_{R'}(u)$. Later, we will apply to these neighborhoods Lemma 2.1.

To approximate the lower bound of (3) we will use the entropy function

$$H(p) = -p \log_2 p - (1 - p) \log_2(1 - p).$$

Let $c = c(\varepsilon)$ be a sufficiently large number so that

$$H(1/2) - H(1/2 - \varepsilon/4) > H(2/c) \tag{4}$$

holds. Since, for a fixed $p \in (0, 1)$ we have $\binom{N}{\lfloor pN \rfloor} = 2^{N(H(p)+o(1))}$ (see, e.g., [3]), it follows using (3) and (4) that for $N = cn$ large enough

$$\begin{aligned} |P| &> \frac{\frac{\varepsilon}{(1-\frac{\varepsilon}{2})N} \binom{N}{N/2}}{\binom{N}{N/2-\varepsilon N/4}} = 2^{N(H(1/2)-H(1/2-\varepsilon/4)+o(1))} > 2^{N(H(2/c)+o(1))} \\ &= 2n \binom{N}{2n-1} \geq \sum_{i=0}^{2n-1} \binom{N}{i} \geq \sum_{i=0}^{2n-1} \binom{N-j}{i}. \end{aligned}$$

Let

$$\mathcal{F} = \{N_{R'}(u) : u \in P\} = \{N_R(u) - J_u : u \in P\}.$$

Note that $|\mathcal{F}| = |P|$. Lemma 2.1 yields that there exists a set $K \subseteq W_1 \setminus J$, $|K| = 2n$, such that for every subset $S \subseteq K$ there is a vertex $u \in P$ such that $S = N_{R'}(u) \cap K$.

Now we are going to show that the subgraph of R' induced on $P \cup W_1$ contains an induced copy of every bipartite graph $G = (V_1, V_2, E)$ with $|V_1|, |V_2| \leq n$. We may assume that $|V_1| = |V_2| = n$. For technical reasons, which will be clarified later, it will be convenient to assume that no two vertices in V_2 have the same neighborhood. If this is not the case, then we can enlarge V_1 by adding at most n new vertices each adjacent with precisely one vertex from V_2 . Denote this new bipartite graph by $H = (U_1, U_2, I)$, where $|U_1| = 2n$ and $|U_2| = n$. Clearly H contains an induced copy of G . Therefore, in order to finish the proof of Theorem 1.10, it is enough to show that R' contains an induced copy of $H = (U_1, U_2, I)$.

We will find $U'_1 \subseteq W_1$ and $U'_2 \subseteq P$ such that the subgraph of R' induced on $U'_1 \cup U'_2$ will be isomorphic to H . Let $U'_1 = K$ and μ be any one-to-one correspondence between U_1 and U'_1 . For every vertex $u \in U_2$ consider its neighborhood $N_H(u) \subseteq U_1$. Since $\mu(N_H(u))$ is a subset of K , there is a $w \in P$ such that $\mu(N_H(u)) = N_{R'}(w) \cap K$. Let U'_2 be the set of all those vertices w . Note that $|U'_2| = |U_2|$ since all vertices in U_2 have different neighborhoods. This shows that $R'[U'_1 \cup U'_2]$ is isomorphic to H .

This completes the proof of Theorem 1.10.

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