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Notes

On the number of nonnegative sums

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ABSTRACT

A short proof is presented for the following statement. If X is a set of n real numbers summing up to 0 and $n \geqslant (3/2)k^3$ then at least $\binom{n-1}{k-1}$ of the subset sums involving k numbers are nonnegative.

1. Introduction

Let $X := \{x_1, x_2, \dots, x_n\}$ be a set of not necessarily distinct real numbers listed in decreasing order and satisfying $x_1 + \cdots + x_n = 0$. Let $[n] := \{1, 2, \dots, n\}$. For a set $S \subset [n]$ we define

$$x(S) := \sum_{i \in S} x_i.$$

For $1 \leqslant k \leqslant n$, define the family of nonnegative k-sums

$$\mathcal{P}(X, k) := \bigl\{ S \subset [n]\colon\ |S| = k,\ x(S) \geqslant 0 \bigr\}.$$

A longstanding conjecture of Manickam, Miklós and Singhi is as follows.

Conjecture 1.1. *(Cf. [4,5].)*

$$\bigl|\mathcal{P}(X, k)\bigr| \geqslant \binom{n-1}{k-1} \tag{1}$$

holds for all X and $n \geqslant 4k$.

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Manickam and Singhi [5] proved (1) for all n that are divisible by k. However, the general case proved unexpectedly difficult and only limited progress was made (cf. [1] for detailed reference). In a recent paper Alon, Huang and Sudakov [1] made a breakthrough by establishing the validity of the conjecture for $n \geqslant 33k^2$ thus significantly improving the previous superexponential lower bound. The aim of the present note is to provide a short proof of a somewhat weaker result still giving a polynomial lower bound for n.

Theorem 1.2. *Let $n \geqslant (3/2)k^3$. Then one of the following must hold.*

(i) *All k-subsets of $[n]$ containing 1 are in $\mathcal{P}(X,k)$, or*
(ii) $|\mathcal{P}(X,k)| \geqslant 2\binom{n-k^2}{k-1} > \binom{n-1}{k-1}$.

2. Proof of the theorem

Let us define the following k pairwise disjoint $(k-1)$-element sets $S_i := \{n - i(k-1) + 1, \dots, n-(i-1)(k-1)\}$ for $i = 1, \dots, k$. By monotonicity of the x_j's the sum $x(S_1)$ is the smallest among all sums involving $k-1$ elements of X. Consequently, if $x_1 + x(S_1) \geqslant 0$ holds then the case (i) follows.

Suppose that $x_r + x(S_r) \geqslant 0$ holds for some $k \geqslant r > 1$. Define

$$R := [n] - \big([r] \cup S_1 \cup \dots \cup S_{r-1}\big).$$

By the monotonicity of x_i's, for all $(k-1)$-element sets $Q \subset R$ and all $1 \leqslant j \leqslant r$ the sum $x_j + x(Q)$ is nonnegative. Thus

$$\big|\mathcal{P}(X,k)\big| \geqslant \big|\mathcal{P}\big([r] \cup R, k\big)\big| \geqslant r\binom{n-r-(r-1)(k-1)}{k-1} \geqslant 2\binom{n-k^2}{k-1},$$

yielding case (ii).

From now on, we can assume that $x_r + x(S_r) < 0$ holds for each $r = 1, \dots, k$. We prove that case (ii) holds again. Let $T = S_1 \cup \dots \cup S_k$. We have

$$x\big([k]\big) + x(T) \leqslant 0 \quad \text{with } T \subset [n] - [k],\ |T| = k^2 - k. \tag{2}$$

Define $t := \lfloor (n - |T|)/k \rfloor$. Let Y consist of the first kt elements of X and note that Y is disjoint from $\widetilde{T} = \{x_i \colon i \in T\}$. We have $|Y| = kt \geqslant n - |T| - k + 1 = n - k^2 + 1$. If all the elements of Y are nonnegative, then using $n \geqslant (3/2)n^3$, we obtain

$$\big|\mathcal{P}(X,k)\big| \geqslant \big|\mathcal{P}(Y,k)\big| \geqslant \binom{n-k^2+1}{k} \geqslant 2\binom{n-k^2}{k-1}.$$

If there are negative elements in Y then the monotonicity of the x_i's implies that every $x_j \in X - Y - \widetilde{T}$ is negative. Also, (2) gives that $x(T) \leqslant 0$, so $x(X) = 0$ implies $x(Y) \geqslant 0$.

Now we are ready to apply a simple but very useful averaging argument due to Katona [3]. Let $Y' = [kt]$, so that $Y = \{x_i \colon i \in Y'\}$. Let $Y' = P_1 \cup \dots \cup P_t$ be an arbitrary partition of Y' into k-element sets. We claim that at least two of the P_i's are in $\mathcal{P}(Y,k)$. Indeed, $x(Y) \geqslant 0$ gives that there exists a P_j with $x(P_j) \geqslant 0$. Using (2) and $x(P_j) \leqslant x([k])$ we obtain

$$x\big([n]\big) = 0 \geqslant x\big([k]\big) + x(T) \geqslant x(P_j) + x(T) \geqslant x(P_j) + x\big([n] - Y'\big).$$

This gives $x(Y' - P_j) \geqslant 0$, so there must be another $P_i \in \mathcal{P}(Y,k)$.

Thus we have shown that, in an arbitrary partition of Y into t k-sets, at least 2 members of the partition have nonnegative sum. By Katona's argument this implies

$$\big|\mathcal{P}(Y,k)\big| \geqslant \frac{2}{t}\binom{|Y|}{k} = 2\binom{|Y|-1}{k-1},$$

leading to

$$\left|\mathcal{P}(X,k)\right| \geqslant \left|\mathcal{P}(Y,k)\right| \geqslant 2\binom{|Y|-1}{k-1} = 2\binom{tk-1}{k-1} \geqslant 2\binom{n-k^2}{k-1}.$$

An easy calculation shows that

$$2\binom{n-k^2}{k-1} > \binom{n-1}{k-1}$$

holds for $n \geqslant (3/2)k^3$, completing the proof of the theorem.

3. Some remarks

Although our results are somewhat weaker than those of Alon, Huang and Sudakov [1], the proof is considerably simpler. In [1] instead of conclusion (ii) an exact result is proven. Let us mention that their proof is basically the same as the new proof given for the Hilton–Milner Theorem in [2]. To keep the paper short, we contented ourselves with the slightly weaker assertion (ii). Note that the core of our proof is the following fact.

Fact 3.1. *Suppose that T is a subset of $[n]-[k]$, $|T| < n-3k$, satisfying*

$$x\big([k]\big) + x(T) \leqslant 0.$$

Then $|\mathcal{P}(X-T)| \geqslant 2\binom{n-|T|-k}{k-1}$ holds.

To obtain a quadratic bound—matching that of [1], one would need the size of T from (2) to be linear in k, which does not seem to be easy to obtain. However, we hope to return to this problem with some new bounds characterizing sequences with $|\mathcal{P}(X,k)| = O(n^{k-1})$.

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