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Improved bounds for Erdős' Matching Conjecture

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ABSTRACT

The main result is the following. Let $\mathcal{F}$ be a family of k-subsets of an n-set, containing no $s+1$ pairwise disjoint edges. Then for $n \geqslant (2s+1)k - s$ one has $|\mathcal{F}| \leqslant \binom{n}{k} - \binom{n-s}{k}$. This upper bound is the best possible and confirms a conjecture of Erdős dating back to 1965. The proof is surprisingly compact. It applies a generalization of Katona's Intersection Shadow Theorem.

1. Introduction and notation

Let $[n] := \{1, 2, \dots, n\}$ and let $\mathcal{F} \subset \binom{[n]}{k}$, $n \geqslant k \geqslant 1$. The *matching number* $\nu(\mathcal{F})$ is the maximum number of pairwise disjoint members (edges) of $\mathcal{F}$. One of the classical problems of extremal set theory is to determine $\max|\mathcal{F}|$, for $\nu(\mathcal{F})$ fixed. Here are two easy constructions.

$$\mathcal{A}(k,s) := \binom{[k(s+1)-1]}{k}, \qquad \left|\mathcal{A}(k,s)\right| = \binom{k(s+1)-1}{k},$$
$$\mathcal{A}(n,1,s) := \left\{A \in \binom{[n]}{k}\colon A \cap [s] \neq \emptyset\right\}, \qquad \left|\mathcal{A}(n,1,s)\right| = \binom{n}{k} - \binom{n-s}{k}.$$

The Matching Conjecture. *(See Erdős [4] (1965).) If $\mathcal{F} \subset \binom{[n]}{k}$, $\nu(\mathcal{F}) = s$ and n is at least $k(s+1)-1$ then*

$$|\mathcal{F}| \leqslant \max\left\{\binom{k(s+1)-1}{k}, \binom{n}{k} - \binom{n-s}{k}\right\} \tag{1}$$

holds.

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The case $s=1$ is the classical Erdős–Ko–Rado Theorem [6]. For $k=1$ the conjecture holds trivially and for $k=2$ it was proved by Erdős and Gallai [5]. Erdős [4] proved (1) for $n>n_0(k,s)$. In [3] the bound on $n_0(k,s)$ was lowered to $2sk^3$. Recently, Huang, Loh and Sudakov [12] improved it to $3sk^2$, which was slightly improved in [9]. On the other hand Füredi and the author proved $n_0(k,s)\leqslant cks^2$, however their result was never published. The aim of the present paper is to provide a completely new argument proving a bound simultaneously improving all known bounds.

Theorem 1.1. *Let $\mathcal{F}\subset\binom{[n]}{k}$, $\nu(\mathcal{F})=s$ and $n\geqslant(2s+1)k-s$ then*

$$|\mathcal{F}|\leqslant\binom{n}{k}-\binom{n-s}{k} \tag{2}$$

with equality if and only if $\mathcal{F}$ is isomorphic to $\mathcal{A}(n,1,s)$.

One of the principal tools in proving (2) is an extension of Katona's *Intersection Shadow Theorem* [13]. For a family $\mathcal{F}\subset\binom{[n]}{k}$ let us define its shadow $\partial\mathcal{F}$ by

$$\partial\mathcal{F}:=\left\{G\in\binom{[n]}{k-1}:\ \exists F\in\mathcal{F},\ G\subset F\right\}.$$

Theorem 1.2. *Let $\mathcal{F}\subset\binom{[n]}{k}$, $\nu(\mathcal{F})=s$, then*

$$s|\partial\mathcal{F}|\geqslant|\mathcal{F}| \tag{3}$$

holds.

Let us note that for $s=1$ the inequality (3) is a special case of Katona's Intersection Theorem. The proof of Theorem 1.2 is by double induction on n and k—just imitating the original proof of Katona [13]. The starting case is $\mathcal{A}(k,s)$, that is all k-subsets of an n-set where $n=k(s+1)-1$. For $\mathcal{A}(k,s)$ one has $\partial\mathcal{A}(k,s)=\binom{[k(s+1)-1]}{k-1}$ and $s\binom{k(s+1)-1}{k-1}=\binom{k(s+1)-1}{k}$ showing that the factor s is the best possible. On the other hand it follows from the proof that (3) is strict unless $\mathcal{F}$ is isomorphic to $\mathcal{A}(k,s)$.

It is well known (cf. for example [7]) that in proving both theorems one can assume that $\mathcal{F}$ is stable. That is, for all $1\leqslant i<j\leqslant n$ and $F\in\mathcal{F}$, the conditions $i\notin F$, $j\in F$ imply that $F\cup\{i\}-\{j\}$ is in $\mathcal{F}$ as well. The only other ingredient of the proof is the following version of the König–Hall Theorem.

König–Hall Theorem. *(Cf. [14].) Let $\mathcal{G}$ be a bipartite graph with $\nu(\mathcal{G})=s$. Then there exists a subset T of the vertices with $|T|=s$, such that all edges of $\mathcal{G}$ are incident to at least one vertex of T.*

2. Proof of Theorem 1.2

Assume that $\mathcal{F}\subset\binom{[n]}{k}$ is a stable family with $\nu(\mathcal{F})\leqslant s$. Let us first prove the statement for all k and s with $(s+1)k-1\geqslant n$. Let us construct a bipartite graph with partite sets $\mathcal{F}$ and $\partial\mathcal{F}$ where we put an edge connecting F and G if and only if G is a subset of F. It is immediate that each $F\in\mathcal{F}$ has degree k, and each $G\in\partial\mathcal{F}$ has degree at most $n-|G|=n-k+1$. Since $sk\geqslant n-k+1$ for $n\leqslant(s+1)k-1$, (3) holds in the above range. Moreover, equality can hold only if $n=(s+1)k-1$ and each $G\in\partial\mathcal{F}$ has degree ks, so $G\cup\{y\}\in\mathcal{F}$ for $y\notin G\in\partial\mathcal{F}$. It follows that $G-\{x\}+\{y\}$ also should be a member of $\partial\mathcal{F}$ (for $x\in G$, $y\notin G$) so $\partial\mathcal{F}$ is the complete $(k-1)$-uniform hypergraph on $[(s+1)k-1]$ and $\mathcal{F}=\binom{[(s+1)k-1]}{k}$ follows.

From now on, we suppose that $n\geqslant(s+1)k$, $k\geqslant2$ and (3) holds for $n-1$ for both k and $k-1$. Let us use the usual notation $\mathcal{F}(\bar{n}):=\{F\in\mathcal{F}:\ n\notin F\}$, $\mathcal{F}(n):=\{F-\{n\}:\ F\in\mathcal{F},\ n\in F\}$. These are the two families for which we want to use the induction hypothesis. Here $\nu(\mathcal{F}(\bar{n}))\leqslant s$ is obvious. The inequality $\nu((\mathcal{F}(n)))\leqslant s$ follows from stability using the following standard argument (cf. [7]). If one has $s+1$ disjoint sets $F_i - \{n\} \in \mathcal{F}(n)$ (where $F_i \in \mathcal{F}$, $1 \leqslant i \leqslant s+1$), then $n-1 \geqslant (s+1)(k-1)+s$ implies that there are elements $1 \leqslant x_1 < \cdots < x_s \leqslant n-1$ disjoint to each F_i. Then stability implies that the sets $F_i - \{n\} \cup \{x_i\} \in \mathcal{F}$ (here $1 \leqslant i \leqslant s$) together with F_{s+1} form a matching of size $s+1$ in $\mathcal{F}$, a contradiction.

Note that $\partial\mathcal{F}(\bar{n})$ provides us with sets in $\partial\mathcal{F}$ which do not contain n. At the same time, adjoining n to any member of $\partial\mathcal{F}(n)$ provides us with a member of $\partial\mathcal{F}$ which contains n. This proves $|\partial\mathcal{F}| \geqslant |\partial\mathcal{F}(\bar{n})| + |\partial\mathcal{F}(n)|$. Using the induction hypothesis yields

$$s|\partial\mathcal{F}| \geqslant s\left|\partial\mathcal{F}(\bar{n})\right| + s\left|\partial\mathcal{F}(n)\right| \geqslant \left|\mathcal{F}(\bar{n})\right| + \left|\mathcal{F}(n)\right| = |\mathcal{F}|$$

as desired. □

3. A general inequality

The families $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{s+1}$ are called *nested* if $\mathcal{F}_{s+1} \subset \mathcal{F}_s \subset \cdots \subset \mathcal{F}_1$ holds. The families $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{s+1}$ are called *cross-dependent* if there is no choice of $F_i \in \mathcal{F}_i$ such that $F_1, \ldots, F_{s+1}$ are pairwise disjoint.

Theorem 3.1. *Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{s+1} \subset \binom{Y}{\ell}$, be nested, cross-dependent families, $|Y| \geqslant t\ell$. Suppose further $t \geqslant 2s+1$, then*

$$|\mathcal{F}_1| + |\mathcal{F}_2| + \cdots + |\mathcal{F}_s| + (s+1)|\mathcal{F}_{s+1}| \leqslant s\binom{|Y|}{\ell}. \tag{4}$$

Proof. Let us choose randomly (according to uniform distribution) t pairwise disjoint sets $B_1, \ldots, B_t \in \binom{Y}{\ell}$ and define $\mathcal{B} = \{B_1, \ldots, B_t\}$. Since the probability $p(B_j \in \mathcal{F}_i) = |\mathcal{F}_i|/\binom{|Y|}{\ell}$, the expected size $M(|\mathcal{B} \cap \mathcal{F}_i|)$ is $t|\mathcal{F}_i|/\binom{|Y|}{\ell}$. Let us prove a lemma.

Lemma 3.2. *For every choice of $\mathcal{B}$ one has*

$$|\mathcal{B} \cap \mathcal{F}_1| + \cdots + |\mathcal{B} \cap \mathcal{F}_s| + (s+1)|\mathcal{B} \cap \mathcal{F}_{s+1}| \leqslant st. \tag{5}$$

Proof. Define a bipartite graph $\mathcal{G}$ with partite sets $\mathcal{B}$ and $\{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{s+1}\}$ where we join B_j and $\mathcal{F}_i$ by an edge if and only if $B_j \in \mathcal{F}_i$. The fact that $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{s+1}$ are cross-dependent translates to $\nu(\mathcal{G}) \leqslant s$. Applying the König–Hall Theorem we can find a subset T of the vertices, $|T| = s$ such that all edges are incident to some element of T.

Let T have x elements in $\mathcal{B}$ and $s-x$ elements in $\{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{s+1}\}$. Let us estimate the total number of edges incident to T. For $\mathcal{F}_i$ there can be at most t incident edges. This gives an upper bound $(s-x)t$ for the $s-x$ vertices from $\{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{s+1}\}$. The x vertices in $\mathcal{B}$ can be adjacent to $(s+1)-(s-x) = x+1$ additional vertices each. This gives the upper bound

$$(s-x)t + x(x+1) = x^2 - (t-1)x + st. \tag{6}$$

So far we have not used that $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{s+1}$ are nested. If $B_j \in \mathcal{F}_{s+1}$ then $B_j \in \mathcal{F}_i$ follows for all $1 \leqslant i \leqslant s$ as well. That is, B_j has degree $s+1$ in $\mathcal{G}$. Consequently, $B_j \in T$.

Thus setting $b := |\mathcal{B} \cap \mathcal{F}_{s+1}|$, we infer $x \geqslant b$. Now (6) is a quadratic polynomial in x with main term x^2. Therefore the maximum of (6) in the range $b \leqslant x \leqslant s$ is attained either for $x = b$ or $x = s$. We infer

$$|\mathcal{G}| = |\mathcal{B} \cap \mathcal{F}_1| + \cdots + |\mathcal{B} \cap \mathcal{F}_{s+1}| \leqslant \max\left\{b^2 - (t-1)b + st, s^2 - (t-1)s + st\right\}.$$

To prove (5) we need to show that here the right hand side is at most $st - sb$. Let us check it separately for both terms. The inequality $b^2 - (t-1)b + st \leqslant st - sb$ is equivalent to $b(t-1-s-b) \geqslant 0$ which is true because of $b \leqslant s$, $t \geqslant 2s+1$. The inequality $s^2 - (t-1)s + st \leqslant st - sb$ is equivalent to $s(t-1-s-b) \geqslant 0$ which is true for the same reason. □

Let us return to the proof of the Theorem 3.1. Since the lemma holds for all choices of $\mathcal{B}$, the same inequality must hold for the expected values as well, yielding

$$\frac{t|\mathcal{F}_1|}{\binom{|Y|}{\ell}} + \cdots + \frac{t|\mathcal{F}_s|}{\binom{|Y|}{\ell}} + (s+1)\frac{t|\mathcal{F}_{s+1}|}{\binom{|Y|}{\ell}} \leqslant ts,$$

or equivalently

$$|\mathcal{F}_1| + \cdots + |\mathcal{F}_s| + (s+1)|\mathcal{F}_{s+1}| \leqslant s\binom{|Y|}{\ell}$$

as desired. $\square$

Remark. Changing the requirement $t \geqslant 2s+1$ one can prove similar inequalities where the coefficient of $|\mathcal{F}_{s+1}|$ is changing in function of t and s.

4. The proof of Theorem 1.1

Let $\mathcal{F} \subset \binom{[n]}{k}$ be a stable family with $\nu(\mathcal{F}) = s$, $n \geqslant (2s+1)k - s$. We want to prove $|\mathcal{F}| \leqslant |\mathcal{A}(n,1,s)|$. Let us write $\mathcal{A}$ for short instead of $\mathcal{A}(n,1,s)$ throughout the proof. Let us partition both families according to the intersection of their edges with $[s+1]$: For a subset $Q \subset [s+1]$ define

$$\mathcal{F}(Q) := \{F \in \mathcal{F}\colon\ F \cap [s+1] = Q\},$$
$$\mathcal{A}(Q) := \{A \in \mathcal{A}\colon\ A \cap [s+1] = Q\}.$$

Note that for $|Q| \geqslant 2$, $|\mathcal{A}(Q)| = \binom{n-s-1}{k-|Q|}$ implying $|\mathcal{F}(Q)| \leqslant |\mathcal{A}(Q)|$. For $1 \leqslant i \leqslant s$, $|\mathcal{A}(\{i\})| = \binom{n-s-1}{k-1}$ and $\mathcal{A}(\{s+1\}) = \mathcal{A}(\emptyset) = \emptyset$. Thus all we need is to show

$$\left|\mathcal{F}(\emptyset)\right| + \sum_{1 \leqslant 1 \leqslant s+1} \left|\mathcal{F}(\{i\})\right| \leqslant s\binom{n-s-1}{k-1}. \tag{7}$$

We prove (7) in two steps. First we prove

$$\left|\mathcal{F}(\emptyset)\right| \leqslant s\left|\mathcal{F}(\{s+1\})\right|. \tag{8}$$

As a matter of fact, for every $H \in \partial\mathcal{F}(\emptyset)$ stability of $\mathcal{F}$ implies $(H \cup \{s+1\}) \in \mathcal{F}(\{s+1\})$. Now (8) is a direct consequence of Theorem 1.2. Plugging (8) into (7) we see that the inequality to prove is

$$\left|\mathcal{F}(\{1\})\right| + \cdots + \left|\mathcal{F}(\{s\})\right| + (s+1)\left|\mathcal{F}(\{s+1\})\right| \leqslant s\binom{n-s-1}{k-1}. \tag{9}$$

To apply Theorem 3.1 set $\mathcal{F}_i := \{F - \{i\}\colon\ F \in \mathcal{F}(\{i\})\}$. Since $\mathcal{F}$ is stable, $\mathcal{F}_1, \ldots, \mathcal{F}_{s+1}$ are nested. Also, since $\nu(\mathcal{F}) = s$, $\mathcal{F}_1, \ldots, \mathcal{F}_{s+1}$ are cross-dependent. Setting $\ell = k-1$, $Y = [s+2, n]$, $|Y| = n-s-1 \geqslant (2s+1)(k-1)$, all conditions of Theorem 3.1 are satisfied for $t = 2s+1$. Thus (9) follows from (4), completing the proof.

In case of equality $\mathcal{F}(\emptyset) = \emptyset$ is immediate through Theorem 1.2. Then $\mathcal{F}(\{s+1\}) = \emptyset$ follows, leading to $\mathcal{F} \subset \mathcal{A}$. $\square$

5. Concluding remarks

The situation with Erdős' Matching Conjecture was dormant for two decades. There was a sudden increase of interest during the last two years. It was mainly caused by the fact that through the works of Alon, Frankl, Huang, Rödl, Ruciński, and Sudakov [2] and Alon, Huang and Sudakov [1] it was shown that the Matching Conjecture is relevant in the proof of some seemingly unrelated problems. This motivated the research of Huang, Loh and Sudakov [12] and Frankl, Rödl and Ruciński [10] improving the old bounds of Erdős [4] and Bollobás, Daykin and Erdős [3]. Also it led to the complete solution of the Matching Conjecture for 3-uniform hypergraphs (Łuczak and Mieczkowska [15] for large s, Frankl [8] for all s).

The present proof comes within a factor of two to of covering the full range, i.e., $n \leqslant (s+1)k-1$. However, a full solution does not seem possible along these lines. On the other hand some improvements are possible. Let us mention just two of them.

If $k \leqslant s+1$ then $[k] \subset [s+1]$ implies that $\nu(\mathcal{F}(\emptyset)) \leqslant s-1$. Using this fact the same proof yields that the Matching Conjecture is true already for $n \geqslant 2sk-s$ and even earlier for the case that k is substantially smaller than s.

For $\mathcal{F}(\emptyset)$ we used that its matching number is at most s. However, the much stronger statement $\nu(\partial\mathcal{F}(\emptyset)) \leqslant s$ follows from the stability of $\mathcal{F}$. Using this property and the same inductive argument, the factor $1/s$ can be replaced by the larger $\binom{k}{(k-1)s-1}$. The only reason that we did not prove and use this version is that for fixed s and k large, the ratio is approaching $1/s$ which does not permit an improvement of our bounds in general.

Let us conclude this paper by mentioning a Hilton–Milner-type extension of Erdős' Theorem. Hilton and Milner [11] determined the size of the largest intersecting subfamily $\mathcal{F} \subset \binom{[n]}{k}$ with the property $\bigcap \mathcal{F} = \emptyset$. We generalize their construction for all $s \geqslant 1$ by defining a family $\mathcal{H}$ with $\nu(\mathcal{H}) = s$ with the property that for every element $x \in [n]$ one still has $\nu(\mathcal{H}\{\bar{x}\}) = s$ (i.e., $\mathcal{H}$ is ν-*stable*). Let $x_0, \dots, x_{s-1}$ be elements and $T_1, \dots, T_s$ be disjoint k-subsets of $[n]$ such that $x_i \in T_i$, $i = 1, \dots, s-1$ but x_0 is not contained in any of T_i, $i = 1, \dots, s$. Define the family

$$\mathcal{H}(n,s,k) := \left\{ H \in \binom{[n]}{k} \colon \text{there is an } i,\ 0 \leqslant i \leqslant s-1,\ x_i \in H \right.$$
$$\left. \text{and then } H \cap (T_{i+1} \cup \cdots \cup T_s) \neq \emptyset \right\} \cup \{T_1, \dots, T_s\}.$$

Theorem 5.1. *If $\mathcal{F} \subset \binom{[n]}{k}$ satisfies $\nu(\mathcal{F}) = s$, $\nu(\mathcal{F}(\bar{x})) = s$ for every $x \in [n]$, $k \geqslant 4$ and $n \geqslant n_1(s,k)$ then $|\mathcal{F}| \leqslant |\mathcal{H}(n,s,k)|$ holds with equality if and only if $\mathcal{F}$ is isomorphic to $\mathcal{H}(n,s,k)$.*

The proof of this theorem together with a similar result for $k = 3$ will appear in a forthcoming paper.

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