Journal of Combinatorial Theory, Series A 120 (2013) 1578–1589

Contents lists available at SciVerse ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

The Katona theorem for vector spaces

Peter Frankl [a], Norihide Tokushige [b,1]

[a] Alfréd Rényi Institute of Mathematics, H-1364 Budapest, P.O. Box 127, Hungary
[b] College of Education, Ryukyu University, Nishihara, Okinawa 903-0213, Japan

ARTICLE INFO

Article history:
Received 18 October 2011
Available online 7 June 2013

Dedicated to Gyula O.H. Katona on his 70th birthday

Keywords:
Katona's t-intersection theorem
Avoiding just one intersection
Inclusion matrix
Linear algebra methods

ABSTRACT

We present a vector space version of Katona's t-intersection theorem (Katona, 1964 [12]). Let V be the n-dimensional vector space over a finite field, and let $\mathcal{F}$ be a family of subspaces of V. Suppose that $\dim(F \cap F') \geqslant t$ holds for all $F, F' \in \mathcal{F}$. Then we show that $|\mathcal{F}| \leqslant \sum_{k=d}^{n} \begin{bmatrix} n \\ k \end{bmatrix}$ for $n + t = 2d$, and $|\mathcal{F}| \leqslant \sum_{k=d+1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ d \end{bmatrix}$ for $n + t = 2d + 1$. We also consider the case when the condition $\dim(F \cap F') \geqslant t$ is replaced with $\dim(F \cap F') \neq t - 1$.

© 2013 Elsevier Inc. All rights reserved.

1. Introduction

In 1964, Katona published his t-intersection theorem [12], which is one of the most basic results in extremal set theory. It has been extended in many ways, one of them being a result concerning a set-system avoiding just one intersection due to Frankl and Füredi [6]. In this article, we show vector space versions of these results using the linear algebra method.

We begin by recalling Katona's original theorem. Let $X = \{1, 2, \ldots, n\}$ and let $\binom{X}{k}$ denote the set of all k-element subsets of X. Let

$$\mathcal{P}(X) = \bigcup_{k=0}^{n} \binom{X}{k}$$

be the power set of X. We say that a family of subsets $\mathcal{F} \subset \mathcal{P}(X)$ is t-intersecting if $|F \cap F'| \geqslant t$ holds for all $F, F' \in \mathcal{F}$. Let us define a t-intersecting family $\mathcal{K}(n,t)$ of subsets as follows. For $n + t = 2d$, let $\mathcal{K}(n,t) = \bigcup_{k=d}^{n} \binom{X}{k}$. For $n + t = 2d + 1$, choose an $(n-1)$-element subset $Y \subset X$, and set $\mathcal{K}(n,t) = (\bigcup_{k=d+1}^{n} \binom{X}{k}) \cup \binom{Y}{d}$. Then Katona's t-intersection theorem states the following.

E-mail addresses: peter.frankl@gmail.com (P. Frankl), hide@edu.u-ryukyu.ac.jp (N. Tokushige).

[1] The second author was supported by JSPS KAKENHI 20340022 and 25287031.

0097-3165/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jcta.2013.05.007

Theorem 1. *(See [12].) Let $1 \leqslant t \leqslant n$ and let $\mathcal{F} \subset \mathcal{P}(X)$ be t-intersecting. Then $|\mathcal{F}| \leqslant |\mathcal{K}(n,t)|$. Moreover if $t > 1$ then equality holds iff $\mathcal{F}$ is isomorphic to $\mathcal{K}(n,t)$.*

For a family of subsets $\mathcal{F}$ of X and $0 \leqslant u \leqslant n$ we define the u-th shadow $\Delta_u(\mathcal{F})$ of $\mathcal{F}$ by

$$\Delta_u(\mathcal{F}) = \left\{ G \in \binom{X}{u} \colon G \subset F \text{ for some } F \in \mathcal{F} \right\}.$$

The following result is a key tool for the original proof of Theorem 1.

Theorem 2. *(See [12].) Let $1 \leqslant t \leqslant k \leqslant n$ and let $\mathcal{F} \subset \binom{X}{k}$ be t-intersecting. Then, for $k - t \leqslant u \leqslant k$, we have*

$$\left|\Delta_u(\mathcal{F})\right| / |\mathcal{F}| \geqslant \binom{2k-t}{u} \Big/ \binom{2k-t}{k}.$$

Now we present vector space versions of the above theorems. Fix the q-element field $\mathbb{F}_q$ and let V be the n-dimensional vector space over this field. Let $\left[{V \atop k}\right]$ denote the set of all k-dimensional subspaces of V, let $\left[{n \atop k}\right] = \left|\left[{V \atop k}\right]\right| = \prod_{i=0}^{k-1} \frac{q^{n-i}-1}{q^{k-i}-1}$, and let

$$\mathcal{L}(V) = \bigcup_{k=0}^{n} \begin{bmatrix} V \\ k \end{bmatrix}$$

be the lattice of subspaces of V with respect to inclusion. We say that a family of subspaces $\mathcal{F} \subset \mathcal{L}(V)$ is t-intersecting if $\dim(F \cap F') \geqslant t$ holds for all $F, F' \in \mathcal{F}$. For $0 \leqslant u \leqslant n$ we define the u-th shadow $\Delta_u[\mathcal{F}]$ of $\mathcal{F}$ by

$$\Delta_u[\mathcal{F}] = \left\{ G \in \begin{bmatrix} V \\ u \end{bmatrix} \colon G \subset F \text{ for some } F \in \mathcal{F} \right\}.$$

Then the corresponding result to Theorem 2 is as follows.

Theorem 3. *Let $1 \leqslant t \leqslant k \leqslant n$ and let $\mathcal{F} \subset \left[{V \atop k}\right]$ be t-intersecting. Then, for $k - t \leqslant u \leqslant k$, we have*

$$\left|\Delta_u[\mathcal{F}]\right| / |\mathcal{F}| \geqslant \begin{bmatrix} 2k-t \\ u \end{bmatrix} \Big/ \begin{bmatrix} 2k-t \\ k \end{bmatrix}.$$

Let us define a t-intersecting family $\mathcal{K}[n,t]$ of subspaces as follows. For $n + t = 2d$, let $\mathcal{K}[n,t] = \bigcup_{k=d}^{n} \left[{V \atop k}\right]$. For $n + t = 2d + 1$, choose an $(n-1)$-dimensional subspace $W \subset V$, and set $\mathcal{K}[n,t] = \left(\bigcup_{k=d+1}^{n} \left[{V \atop k}\right]\right) \cup \left[{W \atop d}\right]$. Using Theorem 3 we will obtain the following vector space version of Katona's theorem.

Theorem 4. *Let $1 \leqslant t \leqslant n$ and let $\mathcal{F} \subset \mathcal{L}(V)$ be t-intersecting. Then $|\mathcal{F}| \leqslant |\mathcal{K}[n,t]|$. Moreover if $t > 1$ then equality holds iff $\mathcal{F}$ is isomorphic to $\mathcal{K}[n,t]$.*

We say that a family of subsets $\mathcal{F} \subset \mathcal{P}(X)$ is $(t-1)$-avoiding if $|F \cap F'| \neq t - 1$ holds for all *distinct* $F, F' \in \mathcal{F}$. Notice that if $\mathcal{F}$ is t-intersecting then it is $(t-1)$-avoiding. In 1975, Erdős [4] asked what happens if in Theorem 1 we weaken the condition "t-intersecting" to "$(t-1)$-avoiding." Define a $(t-1)$-avoiding family $\mathcal{K}^*(n,t-1)$ of subsets of X by $\mathcal{K}^*(n,t-1) = \mathcal{K}(n,t) \cup \bigcup_{k<t-1} \binom{X}{k}$. In [5], Frankl conjectured that this construction gives the maximum possible size for $n > n_0(t)$, and he proved this for the case $t = 2$ (1-avoiding families) for all n. This conjecture was solved by Frankl and Füredi in 1984 [6] using the so-called "linear algebra method." We present the corresponding vector space version. To state our result, we need some definitions. We say that a family of subspaces $\mathcal{F} \subset \mathcal{L}(V)$ is $(t-1)$-avoiding if $\dim(F \cap F') \neq t - 1$ holds for all distinct $F, F' \in \mathcal{F}$. Define a $(t-1)$-avoiding family $\mathcal{K}^*[n,t-1]$ of subspaces of V by $\mathcal{K}^*[n,t-1] = \mathcal{K}[n,t] \cup \bigcup_{k<t-1} \left[{V \atop k}\right]$.

Theorem 5. *Let $t \geqslant 1$, $n > n_0(t)$, and let $\mathcal{F} \subset \mathcal{L}(V)$ be $(t-1)$-avoiding. Then $|\mathcal{F}| \leqslant |\mathcal{K}^*[n, t-1]|$. Moreover if $t > 1$ then equality holds iff $\mathcal{F}$ is isomorphic to $\mathcal{K}^*[n, t-1]$.*

Since a t-intersecting family is always a $(t-1)$-avoiding family, the following result is an obvious extension of Theorem 3 (for the case $k \geqslant 2t-1$), which will be used to prove Theorem 5.

Theorem 6. *Let $t \geqslant 1$, $n \geqslant k \geqslant 2t-1$, and let $\mathcal{F} \subset \left[{V \atop k}\right]$ be $(t-1)$-avoiding. Then, for $k-t \leqslant u \leqslant k$, we have*

$$\left|\Delta_u[\mathcal{F}]\right|/|\mathcal{F}| \geqslant \begin{bmatrix} 2k-t \\ u \end{bmatrix} \Big/ \begin{bmatrix} 2k-t \\ k \end{bmatrix}.$$

In [6] the corresponding set-system version of Theorem 6 is conjectured to be true but it is proved only under the assumption of $k > k_0(t)$. This is because the proof relies on a result of Frankl and Singhi [10] stating that every k-uniform, $(t-1)$-avoiding family of subsets is $(k-t)$-independent, provided $k > k_0(t)$. (We will define "$(k-t)$-independence" in Section 2.) This proof, in turn, uses a divisibility property of integers which requires $k > k_0(t)$. On the other hand, we will use some basic properties of the cyclotomic polynomials to show that every k-uniform, $(t-1)$-avoiding family of subspaces is $(k-t)$-independent provided $k \geqslant 2t-1$ (Lemma 5). In this sense, Theorem 6 is an example where a vector space version of a theorem has a stronger result than a set-system version, with a simpler proof.

Finally we mention the maximum size of k-uniform, $(t-1)$-avoiding families. As for the case $k \geqslant 2t-1$, we only have the following weaker bound, which is stated in [8] without a proof. (In [8] they claimed that Theorem 7 follows from their Theorem 1.1, but this is true only for t-intersecting families.)

Theorem 7. *Let $t \geqslant 1$, $n \geqslant k \geqslant 2t-1$, and let $\mathcal{F} \subset \left[{V \atop k}\right]$ be $(t-1)$-avoiding. Then $|\mathcal{F}| \leqslant \left[{n \atop k-t}\right]$.*

Frankl and Graham [8] conjecture that if $k \geqslant 2t$ then the upper bound can be improved to $\left[{n-t \atop k-t}\right]$. (Theorem 7 for the case $k = 2t-1$ is almost sharp as described below.) On the other hand, Frankl and Füredi [7] obtained the sharp upper bound $\binom{n-t}{k-t}$ for the corresponding set-system version, provided $k \geqslant 2t$ and $n > n_0(k)$. The proof technique used in [7] is more combinatorial, and different from that in [6].

For the case $k \leqslant 2t-1$ we will derive the following result from Theorem 7.

Theorem 8. *Let $t \geqslant 1$, $2t-1 \geqslant k > t-1$, $n \geqslant k$, and let $\mathcal{F} \subset \left[{V \atop k}\right]$ be $(t-1)$-avoiding. Then $|\mathcal{F}| \leqslant \left[{n \atop t-1}\right]\left[{2k-t \atop k}\right]/\left[{2k-t \atop t-1}\right]$.*

Theorem 8 is asymptotically tight as $n \to \infty$ for fixed t, k. We show the tightness (Theorem 9 in Section 4) using a packing result of Rödl [14].

We will use the linear algebra method to prove our results. The proofs are similar to those in [6], but we will follow the formulation in the Babai–Frankl book [2]. The key idea is an independence of row vectors of the inclusion matrix. This idea was already used by Frankl and Graham in [8], and we could use their results but we choose to give direct and elementary proofs for self-completeness.

This paper is organized as follows. In Section 2 we prepare some basic tools for the linear algebra method, and prove Theorem 3 and Theorem 4 (the Katona theorem for vector spaces). Then in Section 3 we consider families avoiding just one intersection, and prove Theorem 5 and Theorem 6. In Section 4 we focus on uniform families and prove Theorem 7 and Theorem 8.

2. The Katona theorem for vector spaces

In this section, we prepare some basic tools for the linear algebra method, and prove Theorem 3 and Theorem 4.

Let V be the n-dimensional vector space over $\mathbb{F}_q$. For $0 \leqslant i \leqslant k \leqslant n$, $\mathcal{F} \subset \left[{V \atop k}\right]$, and $\mathcal{G} \subset \left[{V \atop i}\right]$, define the inclusion matrix $M(\mathcal{F}, \mathcal{G})$ as follows. This is an $|\mathcal{F}| \times |\mathcal{G}|$ matrix whose (F, G)-entry $m(F, G)$, where $F \in \mathcal{F}$ and $G \in \mathcal{G}$, is defined by

$$m(F,G) = \begin{cases} 1 & \text{if } F \supset G, \\ 0 & \text{if } F \not\supset G. \end{cases}$$

For $\mathcal{F} \subset \left[{V \atop k}\right]$ and $0 \leqslant j \leqslant i \leqslant k$, simple counting yields

$$M\left(\mathcal{F}, \begin{bmatrix} V \\ i \end{bmatrix}\right) M\left(\begin{bmatrix} V \\ i \end{bmatrix}, \begin{bmatrix} V \\ j \end{bmatrix}\right) = \begin{bmatrix} k-j \\ i-j \end{bmatrix} M\left(\mathcal{F}, \begin{bmatrix} V \\ j \end{bmatrix}\right). \tag{1}$$

In fact, the (F, J)-entry of (1), where $F \in \mathcal{F}$ and $J \in \left[{V \atop j}\right]$, counts

$$\#\left\{I \in \begin{bmatrix} V \\ i \end{bmatrix}: J \subset I \subset F\right\}.$$

In particular, (1) shows the following.

Lemma 1. *Let $0 \leqslant j \leqslant i \leqslant k$ and $\mathcal{F} \subset \left[{V \atop k}\right]$. Then $\operatorname{colsp} M\left(\mathcal{F}, \left[{V \atop j}\right]\right)$ is contained in $\operatorname{colsp} M\left(\mathcal{F}, \left[{V \atop i}\right]\right)$, where $\operatorname{colsp} M$ denotes the column space of M over $\mathbb{Q}$.*

We say that $\mathcal{F} \subset \left[{V \atop k}\right]$ is s-independent if the rows of $M\left(\mathcal{F}, \left[{V \atop s}\right]\right)$ are linearly independent over $\mathbb{Q}$, that is, the inclusion matrix has full row-rank. In this case, $|\mathcal{F}| \leqslant \left[{n \atop s}\right]$ immediately follows.

Lemma 2. *(See [8].) Let $0 \leqslant s \leqslant u \leqslant k$ and let $\mathcal{F} \subset \left[{V \atop k}\right]$ be s-independent. Then*

$$\left|\Delta_u[\mathcal{F}]\right| / |\mathcal{F}| \geqslant \begin{bmatrix} k+s \\ u \end{bmatrix} \Big/ \begin{bmatrix} k+s \\ k \end{bmatrix}. \tag{2}$$

Proof. Let $A \oplus B = V$ denote the direct sum, that is, $A \cap B = \{\mathbf{0}\}$ and $\operatorname{span}\{A, B\} = V$. For each line $x \in \left[{V \atop 1}\right]$ choose $W = W_x \in \left[{V \atop n-1}\right]$ so that $x \oplus W = V$. Let

$$\mathcal{F}_x = \left\{G \in \begin{bmatrix} W \\ k-1 \end{bmatrix}: x \oplus G \in \mathcal{F}\right\} \subset \begin{bmatrix} W \\ k-1 \end{bmatrix}.$$

Claim 1. *$\mathcal{F}_x \subset \left[{W \atop k-1}\right]$ is s-independent, that is, $\operatorname{rank} M\left(\mathcal{F}_x, \left[{W \atop s}\right]\right) = |\mathcal{F}_x|$.*

We postpone the proof of Claim 1, and we first prove the lemma by induction on k assuming Claim 1. Inequality (2) trivially holds for the following three cases: $s = 0$, $u = s$, and $u = k$. So let $1 \leqslant s < u < k$ and assume that (2) is true for $k - 1$. By Claim 1 we can apply the induction hypothesis to $\mathcal{F}_x \subset \left[{W \atop k-1}\right]$, and we get

$$\left|\Delta_{u-1}[\mathcal{F}_x]\right| \geqslant |\mathcal{F}_x| \frac{\left[{(k-1)+s \atop u-1}\right]}{\left[{(k-1)+s \atop (k-1)}\right]}. \tag{3}$$

By counting $\#\left\{(x, F) \in \left[{V \atop 1}\right] \times \mathcal{F}: x \subset F\right\}$ in two ways, namely, by counting the number of edges in the corresponding bipartite graph from each side, we have

$$\sum_{x \in \left[{V \atop 1}\right]} |\mathcal{F}_x| = \begin{bmatrix} k \\ 1 \end{bmatrix} |\mathcal{F}|. \tag{4}$$

Similarly by counting $\#\{(x,G)\in\left[{V\atop 1}\right]\times\Delta_u[\mathcal{F}]:\ x\subset G\}$, we have

$$\sum_{x\in\left[{V\atop 1}\right]}\left|\Delta_{u-1}[\mathcal{F}_x]\right|=\begin{bmatrix}u\\1\end{bmatrix}\left|\Delta_u[\mathcal{F}]\right|. \tag{5}$$

Using (5), (3), and (4), we get

$$\begin{aligned}\left|\Delta_u[\mathcal{F}]\right| &\overset{(5)}{=} \frac{1}{\left[{u\atop 1}\right]}\sum_x\left|\Delta_{u-1}[\mathcal{F}_x]\right| \overset{(3)}{\geqslant} \frac{1}{\left[{u\atop 1}\right]}\sum_x|\mathcal{F}_x|\frac{\left[{k-1+s\atop u-1}\right]}{\left[{k-1+s\atop k-1}\right]}\\ &\overset{(4)}{=} \frac{1}{\left[{u\atop 1}\right]}\cdot\begin{bmatrix}k\\1\end{bmatrix}|\mathcal{F}|\cdot\frac{\left[{k-1+s\atop u-1}\right]}{\left[{k-1+s\atop k-1}\right]}=|\mathcal{F}|\frac{\left[{k+s\atop u}\right]}{\left[{k+s\atop k}\right]}.\end{aligned}$$

This shows that (2) is true for k as well, and completes the induction.

So all we need is to prove Claim 1. Fix $x\in\left[{V\atop 1}\right]$ and let $W\in\left[{V\atop n-1}\right]$ be such that $x\oplus W=V$. Divide $\left[{V\atop s}\right]$ into two parts $\left[{V\atop s}\right]=\mathcal{C}\cup\mathcal{D}$, where $\mathcal{C}$ is the set of s-dimensional subspaces of V not containing x, and the remaining part is $\mathcal{D}=\{x\oplus T:\ T\in\left[{W\atop s-1}\right]\}$. (Then $|\mathcal{C}|=q^s\left[{n-1\atop s}\right]$ and $|\mathcal{D}|=\left[{n-1\atop s-1}\right]$.) Let

$$\mathcal{F}^x=\{F\in\mathcal{F}:\ x\subset F\}\subset\begin{bmatrix}V\\k\end{bmatrix}.$$

We divide the columns of $M(\mathcal{F}^x,\left[{V\atop s}\right])$ into two blocks:

$$M\left(\mathcal{F}^x,\begin{bmatrix}V\\s\end{bmatrix}\right)=\left(M(\mathcal{F}^x,\mathcal{C})\middle|M(\mathcal{F}^x,\mathcal{D})\right). \tag{6}$$

A subspace $S\in\left[{V\atop s}\right]$ can be represented by an $s\times n$ matrix in reduced echelon form with no zero rows (see, e.g., [3]), and let $\mathrm{ref}(S)$ denote the matrix. We can associate $\mathcal{D}$ with matrices for which leading 1 in the last row occurs in the last column. Then there is a natural bijection from $\mathcal{D}$ to $\left[{W\atop s-1}\right]$ by taking the $(s-1)\times(n-1)$ principal minor of $\mathrm{ref}(S)$. Thus we may assume that

$$M(\mathcal{F}^x,\mathcal{D})=M\left(\mathcal{F}^x,\begin{bmatrix}W\\s-1\end{bmatrix}\right).$$

This together with Lemma 1 gives

$$\operatorname{colsp}M(\mathcal{F}^x,\mathcal{D})=\operatorname{colsp}M\left(\mathcal{F}^x,\begin{bmatrix}W\\s-1\end{bmatrix}\right)\subset\operatorname{colsp}M\left(\mathcal{F}^x,\begin{bmatrix}W\\s\end{bmatrix}\right). \tag{7}$$

If $S\in\mathcal{C}$ then the $s\times(n-1)$ principal minor of $\mathrm{ref}(S)$ determines a subspace in $\left[{W\atop s}\right]$. This gives a map $\varphi:\mathcal{C}\to\left[{W\atop s}\right]$, and for each $S\in\left[{W\atop s}\right]$ we have $|\varphi^{-1}(S)|=q^s$ because $\varphi(S)=\varphi(S')$ iff $\mathrm{ref}(S)$ and $\mathrm{ref}(S')$ differ only in the last column. Thus columns corresponding to S and S' in $M(\mathcal{F}^x,\mathcal{C})$ are the same iff $\varphi(S)=\varphi(S')$, and $M(\mathcal{F}^x,\mathcal{C})$ can be viewed as q^s copies of $M(\mathcal{F}^x,\left[{W\atop s}\right])$. Hence we have

$$\operatorname{colsp}M(\mathcal{F}^x,\mathcal{C})=\operatorname{colsp}M\left(\mathcal{F}^x,\begin{bmatrix}W\\s\end{bmatrix}\right). \tag{8}$$

By (7) and (8) with (6), it follows $\operatorname{colsp}M(\mathcal{F}^x,\left[{V\atop s}\right])\subset\operatorname{colsp}M(\mathcal{F}^x,\left[{W\atop s}\right])$, and

$$\operatorname{rank}M\left(\mathcal{F}^x,\begin{bmatrix}V\\s\end{bmatrix}\right)\leqslant\operatorname{rank}M\left(\mathcal{F}^x,\begin{bmatrix}W\\s\end{bmatrix}\right). \tag{9}$$

The opposite inequality is trivial, thus we have equality in (9). On the other hand, since $\mathcal{F}$ is s-independent, $\mathcal{F}^x$ is also s-independent and $\operatorname{rank}M(\mathcal{F}^x,\left[{V\atop s}\right])=|\mathcal{F}^x|$. Thus equality in (9) yields that $\operatorname{rank}M(\mathcal{F}^x,\left[{W\atop s}\right])=|\mathcal{F}^x|$. Finally, noting that $|\mathcal{F}^x|=|\mathcal{F}_x|$ and $M(\mathcal{F}^x,\left[{W\atop s}\right])=M(\mathcal{F}_x,\left[{W\atop s}\right])$, we have $\operatorname{rank}M(\mathcal{F}_x,\left[{W\atop s}\right])=|\mathcal{F}_x|$ as needed. This completes the proof of Claim 1 and Lemma 2. $\square$

Lemma 3. *(See [8].) Let $1 \leqslant t \leqslant k$ and let $\mathcal{F} \subset \begin{bmatrix} V \\ k \end{bmatrix}$ be t-intersecting. Then $\mathcal{F}$ is $(k-t)$-independent.*

Proof. Let

$$f(x) = \prod_{t \leqslant i < k} \begin{bmatrix} x-i \\ 1 \end{bmatrix} = \prod_{t \leqslant i < k} \frac{q^{x-i}-1}{q-1}. \tag{10}$$

By setting $y = q^x$, we can rewrite $f(x)$ as a polynomial $g(y)$ of degree $k-t$ in $\mathbb{Q}[y]$, that is,

$$f(x) = g(y) = \prod_{t \leqslant i < k} \frac{q^{-i}y-1}{q-1}.$$

Let $\phi_s(y) = \prod_{i=0}^{s-1} \frac{q^{-i}y-1}{q^{s-i}-1}$. Then $\phi_0(y), \ldots, \phi_{k-t}(y)$ form a basis of the vector space (over $\mathbb{Q}$) of polynomials of degree $k-t$ with variable y. Thus we can determine $a_0, a_1, \ldots, a_{k-t} \in \mathbb{Q}$ uniquely so that

$$g(y) = \sum_{s=0}^{k-t} a_s \phi_s(y).$$

In other words, noting that $\phi_s(y) = \prod_{i=0}^{s-1} \frac{q^{x-i}-1}{q^{s-i}-1} = \begin{bmatrix} x \\ s \end{bmatrix}$, we can determine $a_0, \ldots, a_{k-t}$ so that

$$f(x) = \sum_{s=0}^{k-t} a_s \begin{bmatrix} x \\ s \end{bmatrix}. \tag{11}$$

Now define an $|\mathcal{F}| \times |\mathcal{F}|$ matrix A by

$$A = \sum_{s=0}^{k-t} a_s M\left(\mathcal{F}, \begin{bmatrix} V \\ s \end{bmatrix}\right) M\left(\mathcal{F}, \begin{bmatrix} V \\ s \end{bmatrix}\right)^T.$$

For $F, F' \in \mathcal{F}$, the (F, F')-entry of A is

$$\sum_{s=0}^{k-t} a_s \# \left\{ W \in \begin{bmatrix} V \\ s \end{bmatrix} : W \subset F \cap F' \right\} = \sum_{s=0}^{k-t} a_s \begin{bmatrix} \dim(F \cap F') \\ s \end{bmatrix}.$$

This equals $f(\dim(F \cap F'))$ by (11). Moreover, using the t-intersecting property with (10), we have

$$f\left(\dim(F \cap F')\right) = \begin{cases} 0 & \text{if } F \neq F', \\ f(k) \neq 0 & \text{if } F = F'. \end{cases}$$

Thus A is a diagonal matrix with no zero diagonal entries, and $\operatorname{rank} A = |\mathcal{F}|$.

On the other hand, it follows from Lemma 1 that the $\operatorname{colsp} M(\mathcal{F}, \begin{bmatrix} V \\ s \end{bmatrix})$ is contained in $\operatorname{colsp} M(\mathcal{F}, \begin{bmatrix} V \\ k-t \end{bmatrix})$ for $0 \leqslant s < k-t$, and so $\operatorname{colsp} A$ is contained in $\operatorname{colsp} M(\mathcal{F}, \begin{bmatrix} V \\ k-t \end{bmatrix})$. This gives $\operatorname{rank} M(\mathcal{F}, \begin{bmatrix} V \\ k-t \end{bmatrix}) \geqslant \operatorname{rank} A = |\mathcal{F}|$. Thus $M(\mathcal{F}, \begin{bmatrix} V \\ k-t \end{bmatrix})$ has full row-rank, namely, $\mathcal{F}$ is $(k-t)$-independent. □

Proof of Theorem 3. By Lemma 3, $\mathcal{F}$ is $(k-t)$-independent. So letting $s = k-t$ in Lemma 2, we get the desired inequality. □

Proof of Theorem 4. We start with the following simple counting fact.

Claim 2. *Let $A \in \begin{bmatrix} V \\ a \end{bmatrix}$. Then $\#\{B \in \begin{bmatrix} V \\ n-a \end{bmatrix} : A \oplus B = V\} = q^{a(n-a)}$.*

Proof. We may assume that $\mathrm{ref}(A) = (\,O \quad I_a\,)$. Then $A \oplus B = V$ gives $\mathrm{ref}(B) = (\,I_{n-a} \quad *\,)$, and there are $q^{a(n-a)}$ ways for choosing the $*$ part. $\square$

Let $\mathbf{G} = \mathbf{G}(a, n-a)$ be a bipartite graph with the vertex partition $V(\mathbf{G}) = \begin{bmatrix} V \\ a \end{bmatrix} \cup \begin{bmatrix} V \\ n-a \end{bmatrix}$ and the edge set $E(\mathbf{G}) = \{(A, B)\colon\ A \oplus B = V\}$. Then, by Claim 2, this is a $q^{a(n-a)}$-regular graph. For a vertex subset $\mathring{A} \subset \begin{bmatrix} V \\ a \end{bmatrix}$, let

$$N_{\mathbf{G}}(\mathring{A}) = \left\{ B \in \begin{bmatrix} V \\ n-a \end{bmatrix} \colon\ (A, B) \in E(\mathbf{G}) \text{ for some } A \in \mathring{A} \right\}$$

denote the neighborhood of $\mathring{A}$. We count the number of edges between $\mathring{A}$ and $N_{\mathbf{G}}(\mathring{A})$ in two ways. Then this number is exactly $q^{a(n-a)}|\mathring{A}|$ on one hand, and at most $q^{a(n-a)}|N_{\mathbf{G}}(\mathring{A})|$ on the other hand. Namely, we have the Hall condition:

$$|\mathring{A}| \leqslant \left|N_{\mathbf{G}}(\mathring{A})\right|.$$

Thus the bipartite graph has a perfect matching, which can be stated as follows.

Lemma 4. *There is a bijection* $\psi : \begin{bmatrix} V \\ a \end{bmatrix} \to \begin{bmatrix} V \\ n-a \end{bmatrix}$ *such that* $A \oplus \psi(A) = V$ *holds for all* $A \in \begin{bmatrix} V \\ a \end{bmatrix}$.

We will use $\psi(A)$ as a “complement” of A here, and also in the proof of Theorem 8 later. (Notice that the orthogonal space $A^{\perp}$ does not necessarily satisfy $A \oplus A^{\perp} = V$. The authors thank one of the referees for notifying this fact.)

Let $\mathcal{F}_k = \mathcal{F} \cap \begin{bmatrix} V \\ k \end{bmatrix}$, $f_k = |\mathcal{F}_k|$, $d = \lfloor (n+t)/2 \rfloor$, and $a = k - t + 1$.

Claim 3. $|\Delta_a[\mathcal{F}_k]| + |\mathcal{F}_{n-a}| \leqslant \begin{bmatrix} n \\ n-a \end{bmatrix}$ *for* $t \leqslant k \leqslant d$.

Proof. Let $G \in \Delta_a[\mathcal{F}_k]$ and $G \subset F \in \mathcal{F}_k$. Using Lemma 4 let $H = \psi(G) \in \begin{bmatrix} V \\ n-a \end{bmatrix}$. Since $V = G \oplus H \subset F + H \subset V$ we have $n = \dim(F + H)$ and

$$\dim(F \cap H) = \dim F + \dim H - \dim(F + H) = k + (n-a) - n = t - 1.$$

Then it follows from the t-intersecting property of $\mathcal{F}$ that $H = \psi(G) \notin \mathcal{F}_{n-a}$. This gives $\psi(\Delta_a[\mathcal{F}_k]) \cap \mathcal{F}_{n-a} = \emptyset$, and

$$\left|\Delta_a[\mathcal{F}_k]\right| + |\mathcal{F}_{n-a}| = \left|\psi\left(\Delta_a[\mathcal{F}_k]\right)\right| + |\mathcal{F}_{n-a}| \leqslant \begin{bmatrix} n \\ n-a \end{bmatrix}$$

as desired. $\square$

We notice for later use in the proof of Theorem 5 in Section 3 that the proof above did not use the full t-intersecting property, but only the $(t-1)$-avoiding property.

Let $t \leqslant k < d$. Applying Theorem 3 with $u = a = k - t + 1$, we have

$$\left|\Delta_a[\mathcal{F}_k]\right| \geqslant |\mathcal{F}_k| \begin{bmatrix} 2k-t \\ a \end{bmatrix} \Big/ \begin{bmatrix} 2k-t \\ k \end{bmatrix} = \frac{q^k - 1}{q^a - 1} f_k.$$

Since $\frac{q^k-1}{q^a-1} \geqslant 1$ iff $a \leqslant k$, that is, $t \geqslant 1$, it follows that $|\Delta_a[\mathcal{F}_k]| \geqslant f_k$ with equality holding iff $\mathcal{F}_k = \emptyset$ or $t = 1$. Then we can infer from Claim 3 that

$$f_k + f_{n-a} \leqslant \left|\Delta_a[\mathcal{F}_k]\right| + |\mathcal{F}_{n-a}| \leqslant \begin{bmatrix} n \\ n-a \end{bmatrix}. \tag{12}$$

(This is true for $k = d$ as well but we will not use this case.) Moreover if $t > 1$ then $f_k + f_{n-a} = \begin{bmatrix} n \\ n-a \end{bmatrix}$ iff $f_k = 0$ and $f_{n-a} = \begin{bmatrix} n \\ n-a \end{bmatrix}$ where $t \leqslant k < d$.

First consider the case $n + t = 2d$. Applying (12) for $k = t, t+1, \ldots, d-1$ we have

$$\begin{aligned}|\mathcal{F}| &= \sum_{k=0}^{n} f_k = f_n + \sum_{k=t}^{n-1} f_k \\ &= f_n + \big((f_t + f_{n-1}) + (f_{t+1} + f_{n-2}) + \cdots + (f_{d-1} + f_d)\big) \\ &\leqslant \begin{bmatrix} n \\ n \end{bmatrix} + \left(\begin{bmatrix} n \\ n-1 \end{bmatrix} + \begin{bmatrix} n \\ n-2 \end{bmatrix} + \cdots + \begin{bmatrix} n \\ d \end{bmatrix}\right) = \big|\mathcal{K}[n,t]\big|.\end{aligned}$$

If $t > 1$ then equality holds iff $f_k = 0$ for $0 \leqslant k < d$ and $f_k = \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ for $d \leqslant k \leqslant n$, namely, $\mathcal{F}$ is isomorphic to $\mathcal{K}[n,t]$.

Next consider the case $n + t - 1 = 2d$. Since $\mathcal{F}_d \subset \left[\begin{smallmatrix} V \\ d \end{smallmatrix}\right]$ is a t-intersecting family with $2d - t < n < 2d$, we can use a result in [11] to get

$$f_d = |\mathcal{F}_d| \leqslant \begin{bmatrix} 2d-t \\ d \end{bmatrix} = \begin{bmatrix} n-1 \\ d \end{bmatrix}. \tag{13}$$

Moreover if $t > 1$ then equality holds iff $\mathcal{F}_d = \left[\begin{smallmatrix} W \\ d \end{smallmatrix}\right]$ for some $(n-1)$-dimensional subspace $W \subset V$. Using (12) for $k = t, t+1, \ldots, d-1$ and using (13) for $k = d$, we have

$$\begin{aligned}|\mathcal{F}| &= \sum_{k=0}^{n} f_k = f_n + \sum_{k=t}^{n-1} f_k + f_d \\ &= f_n + \big((f_t + f_{n-1}) + (f_{t+1} + f_{n-2}) + \cdots + (f_{d-1} + f_{d+1})\big) + f_d \\ &\leqslant \begin{bmatrix} n \\ n \end{bmatrix} + \left(\begin{bmatrix} n \\ n-1 \end{bmatrix} + \begin{bmatrix} n \\ n-2 \end{bmatrix} + \cdots + \begin{bmatrix} n \\ d+1 \end{bmatrix}\right) + \begin{bmatrix} n-1 \\ d \end{bmatrix} = \big|\mathcal{K}[n,t]\big|.\end{aligned}$$

If $t > 1$ then equality holds iff $f_k = 0$ for $0 \leqslant k < d$, $f_k = \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ for $d+1 \leqslant k \leqslant n$, and $\mathcal{F}_d = \left[\begin{smallmatrix} W \\ d \end{smallmatrix}\right]$ for some $(n-1)$-dimensional subspace W, namely, $\mathcal{F}$ is isomorphic to $\mathcal{K}[n,t]$. This completes the proof of Theorem 4. □

3. Avoiding just one intersection

In this section we prove Theorem 5 and Theorem 6.

Lemma 5. *Let $t \geqslant 1$, $k \geqslant 2t-1$, and let $\mathcal{F} \subset \left[\begin{smallmatrix} V \\ k \end{smallmatrix}\right]$ be $(t-1)$-avoiding. Then $\mathcal{F}$ is $(k-t)$-independent.*

Proof. We proceed as in Lemma 3 but using a different $f(x)$, that is,

$$f(x) = \frac{(q^{k-1} - q^x)(q^{k-2} - q^x)\cdots(q^t - q^x)}{(q^{k-t} - 1)(q^{k-t-1} - 1)\cdots(q - 1)}. \tag{14}$$

As we did in the proof of Lemma 3 we can write $f(x) = \sum_{s=0}^{k-t} a_s \left[\begin{smallmatrix} x \\ s \end{smallmatrix}\right]$ for some $a_0, \ldots, a_{k-t} \in \mathbb{Q}$. Define an $|\mathcal{F}| \times |\mathcal{F}|$ matrix A by

$$A = \sum_{s=0}^{k-t} a_s M\left(\mathcal{F}, \begin{bmatrix} V \\ s \end{bmatrix}\right) M\left(\mathcal{F}, \begin{bmatrix} V \\ s \end{bmatrix}\right)^T.$$

Then, for $F, F' \in \mathcal{F}$, the (F, F')-entry of A is $\sum_{s=0}^{k-t} a_s \left[\begin{smallmatrix} \dim(F \cap F') \\ s \end{smallmatrix}\right] = f(\dim(F \cap F'))$.

By (14), we have $f(x) = 0$ for $x = t, t+1, \ldots, k-1$, and

$$\begin{aligned}f(k) &= \prod_{j=1}^{k-t} \frac{q^{k-j} - q^k}{q^j - 1} = \prod_{j=1}^{k-t} \frac{q^{k-j}(1 - q^j)}{q^j - 1} \\ &= (-1)^{k-t} q^{(k-1)+(k-2)+\cdots+t} = (-1)^{k-t} q^{(k-t)(k-t+1)/2}.\end{aligned} \tag{15}$$

For the remaining values except $t-1$, namely, for $x = 0, 1, \dots, t-2$, we have $f(x) = q^{x(k-t)} \begin{bmatrix} k-x-1 \\ k-t \end{bmatrix}$ and

$$\begin{bmatrix} k-x-1 \\ k-t \end{bmatrix} = \begin{bmatrix} k-x-1 \\ t-x-1 \end{bmatrix} = \frac{(q^{k-x-1}-1)\cdots(q^{k-t+1}-1)}{(q^{t-x-1}-1)\cdots(q-1)} \in \mathbb{Z}[q]. \tag{16}$$

Recall that $q^n - 1 = \prod_{j|n} \Phi_j(q)$, where $\Phi_j(q) \in \mathbb{Z}[q]$ is the j-th cyclotomic polynomial. Let us look at the RHS of (16). The numerator contains $\Phi_{k-t+1}(q)$ as a factor coming from $q^{k-t+1} - 1$. On the other hand, $j = t-x-1$ is the maximum j such that $\Phi_j(q)$ appears in the denominator as a factor. Using $x \geqslant 0$ and $k \geqslant 2t-1$ we have $t-x-1 \leqslant t-1 < k-t+1$. So $\Phi_{k-t+1}(q)$ does not appear in the denominator. Since cyclotomic polynomials are pairwise relatively prime, it follows from (16) that $\Phi_{k-t+1}(q)$ divides $\begin{bmatrix} k-x-1 \\ k-t \end{bmatrix}$, namely,

$$\Phi_{k-t+1}(q) \mid f(x) \quad \text{for } x = 0, 1, \dots, t-2.$$

But $\Phi_{k-t+1}(q)$ does not divide $f(k)$ in $\mathbb{Z}[q]$ by (15). Note also that $f(t-1)$ never appears in A because of the $(t-1)$-avoiding property. Consequently it follows that $f(\dim(F \cap F')) \in \mathbb{Z}[q]$ and

$$\begin{cases} \Phi_{k-t+1}(q) \mid f\big(\dim\big(F \cap F'\big)\big) & \text{if } F \neq F', \\ \Phi_{k-t+1}(q) \nmid f\big(\dim\big(F \cap F'\big)\big) & \text{if } F = F'. \end{cases}$$

This means that A is a diagonal matrix with no zero diagonal entries in the residue ring $\mathbb{Z}[q]/(\Phi_{k-t+1}(q))$, and thus $\operatorname{rank} A = |\mathcal{F}|$. On the other hand, it follows from (1) and the definition of A that $\operatorname{colsp} A \subset \operatorname{colsp} M\big(\mathcal{F}, \begin{bmatrix} V \\ k-t \end{bmatrix}\big)$. Therefore we have

$$|\mathcal{F}| = \operatorname{rank} A \leqslant \operatorname{rank} M\left(\mathcal{F}, \begin{bmatrix} V \\ k-t \end{bmatrix}\right) \leqslant |\mathcal{F}|.$$

Thus $\operatorname{rank} M\big(\mathcal{F}, \begin{bmatrix} V \\ k-t \end{bmatrix}\big) = |\mathcal{F}|$, namely, $\mathcal{F}$ is $(k-t)$-independent. $\square$

Proof of Theorem 6. This follows from Lemma 2 and Lemma 5. $\square$

Proof of Theorem 5. Let $\mathcal{F}_k = \mathcal{F} \cap \begin{bmatrix} V \\ k \end{bmatrix}$, $f_k = |\mathcal{F}_k|$, $d = \lfloor (n+t)/2 \rfloor$, and $a = k-t+1$. Let $t \leqslant k < d$. By Theorem 6, we have

$$\big|\Delta_a[\mathcal{F}_k]\big| \geqslant |\mathcal{F}_k| \begin{bmatrix} 2k-t \\ a \end{bmatrix} \Big/ \begin{bmatrix} 2k-t \\ k \end{bmatrix} = \frac{q^k - 1}{q^a - 1} f_k,$$

and so $|\Delta_a[\mathcal{F}_k]| \geqslant f_k$ with equality holding iff $\mathcal{F}_k = \emptyset$ or $t = 1$. Then we can infer from Claim 3 (see the notice right after the proof of Claim 3) that

$$f_k + f_{n-a} \leqslant \frac{q^k - 1}{q^a - 1} f_k + f_{n-a} \leqslant \begin{bmatrix} n \\ n-a \end{bmatrix}. \tag{17}$$

Moreover if $t > 1$ then $f_k + f_{n-a} = \begin{bmatrix} n \\ n-a \end{bmatrix}$ iff $f_k = 0$ and $f_{n-a} = \begin{bmatrix} n \\ n-a \end{bmatrix}$ where $t \leqslant k < d$. For $k < t$ we will use a trivial upper bound $f_k \leqslant \begin{bmatrix} n \\ k \end{bmatrix}$.

Case 1. $n + t = 2d$.

We have

$$|\mathcal{F}| = \sum_{k=0}^{n} f_k = f_n + \sum_{k=t}^{n-1} f_k + \sum_{k=0}^{t-1} f_k$$

$$= f_n + \big((f_t + f_{n-1}) + (f_{t+1} + f_{n-2}) + \cdots + (f_{d-1} + f_d)\big) + \sum_{k=0}^{t-1} f_k. \tag{18}$$

First suppose that $f_{t-1}=0$. Then applying (17) for $k=t,t+1,\dots,d-1$ we have

$$|\mathcal{F}|\leqslant\begin{bmatrix}n\\n\end{bmatrix}+\left(\begin{bmatrix}n\\n-1\end{bmatrix}+\begin{bmatrix}n\\n-2\end{bmatrix}+\cdots+\begin{bmatrix}n\\d\end{bmatrix}\right)+\sum_{k=0}^{t-2}\begin{bmatrix}n\\k\end{bmatrix}=\left|\mathcal{K}^*[n,t-1]\right|.$$

If $t>1$ then equality holds iff $f_k=\left[{n \atop k}\right]$ for $0\leqslant k<t-1$, $f_k=0$ for $t-1\leqslant k<d$ and $f_k=\left[{n \atop k}\right]$ for $d\leqslant k\leqslant n$, namely, $\mathcal{F}$ is isomorphic to $\mathcal{K}^*[n,t-1]$.

Next suppose that $f_{t-1}\neq 0$, that is, there is an $F_0\in\mathcal{F}_{t-1}$. Since $\mathcal{F}$ is $(t-1)$-avoiding, no subspace containing F_0 can be a member of $\mathcal{F}$, which implies that

$$f_k\leqslant\begin{bmatrix}n\\k\end{bmatrix}-\begin{bmatrix}n-(t-1)\\k-(t-1)\end{bmatrix}$$

for $k\geqslant t$. In particular we have

$$f_d\leqslant N-M,$$

where $N=\left[{n \atop d}\right]$, $M=\left[{n-(t-1) \atop d-(t-1)}\right]$. Setting $k=d-1$ in (17) we have

$$\alpha f_{d-1}+f_d\leqslant N,$$

where $\alpha=\frac{q^{d-1}-1}{q^{d-t}-1}\geqslant 1$. So $f_{d-1}\leqslant\frac{1}{\alpha}(N-f_d)$. Thus we have

$$\begin{aligned}f_{d-1}+f_d&\leqslant\frac{1}{\alpha}(N-f_d)+f_d=\frac{1}{\alpha}N+\left(1-\frac{1}{\alpha}\right)f_d\\&\leqslant\frac{1}{\alpha}N+\left(1-\frac{1}{\alpha}\right)(N-M)=N-\left(1-\frac{1}{\alpha}\right)M.\end{aligned}$$

Hence we have

$$f_{t-1}+f_{d-1}+f_d\leqslant\begin{bmatrix}n\\t-1\end{bmatrix}+\begin{bmatrix}n\\d\end{bmatrix}-\frac{q^{d-1}-q^{d-t}}{q^{d-1}-1}\begin{bmatrix}n-(t-1)\\d-(t-1)\end{bmatrix}.$$

The RHS is less than $\left[{n \atop d}\right]$ for $n>n_0(t)$. Using this with (17) for $k=t,t+1,\dots,d-2$ we can infer from (18) that $|\mathcal{F}|<|\mathcal{K}^*[n,t-1]|$.

Case 2. $n+t-1=2d$.

If $F,F'\in\left[{V \atop d}\right]$ then $\dim(F\cap F')\geqslant t-1$. Since $\mathcal{F}$ is $(t-1)$-avoiding, $\mathcal{F}_d\subset\left[{V \atop d}\right]$ is actually t-intersecting. So we can use a result in [11] to get

$$f_d=|\mathcal{F}_d|\leqslant\begin{bmatrix}n-1\\d\end{bmatrix}.\tag{19}$$

Moreover if $t>1$ then equality holds iff $\mathcal{F}_d=\left[{W \atop d}\right]$ for some $(n-1)$-dimensional subspace $W\subset V$. Write $|\mathcal{F}|$ as follows:

$$\begin{aligned}|\mathcal{F}|&=\sum_{k=0}^{n}f_k=f_n+\sum_{k=t}^{n-1}f_k+\sum_{k=0}^{t-1}f_k\\&=f_n+\big((f_t+f_{n-1})+(f_{t+1}+f_{n-2})+\cdots+(f_{d-1}+f_{d+1})\big)+f_d+f_{t-1}+\sum_{k=0}^{t-2}f_k.\end{aligned}$$

First suppose that $f_{t-1}=0$. We use (17) for $k=t,t+1,\dots,d-1$, (19) for $k=d$, and $f_k\leqslant\left[{n \atop k}\right]$ for the remaining k. In this way we get $|\mathcal{F}|\leqslant|\mathcal{K}^*[n,t-1]|$. Moreover if $t>1$ then equality holds iff $f_k=\left[{n \atop k}\right]$ for $0\leqslant k<t-1$, $f_k=0$ for $t-1\leqslant k<d$, $f_k=\left[{n \atop k}\right]$ for $d+1\leqslant k\leqslant n$, and $\mathcal{F}_d=\left[{W \atop d}\right]$ for some $(n-1)$-dimensional subspace W, namely, $\mathcal{F}$ is isomorphic to $\mathcal{K}^*[n,t-1]$.

Next suppose that $f_{t-1} \neq 0$. Then we can argue as in Case 1 to conclude that $f_{t-1} + f_{d-1} + f_{d+1} < \left[{n \atop d+1}\right]$ for $n > n_0(t)$, which gives $|\mathcal{F}| < |\mathcal{K}^*[n, t-1]|$. This completes the proof of Theorem 5. □

If we change the definition of a $(t-1)$-avoiding family $\mathcal{F}$ so that $\dim(F \cap F') \neq t-1$ is required for *all* $F, F' \in \mathcal{F}$, then $f_{t-1} = 0$ follows from this new definition. In this case the above proof shows that Theorem 5 holds without assuming $n > n_0(t)$.

4. Uniform families

In this section we prove Theorem 7 and Theorem 8. Then we will show that Theorem 8 is asymptotically sharp using a packing result of Rödl.

Proof of Theorem 7. This is a direct consequence of Lemma 5. □

In the rest of this section we follow the proof in [7]. Recall the bijection ψ from Lemma 4.

Proof of Theorem 8. Let $b = 2t - k - 1$. For $B \in \left[{V \atop b}\right]$ let

$$\mathcal{F}(B) := \left\{ C \in \begin{bmatrix} \psi(B) \\ k-b \end{bmatrix} : \ B \oplus C \in \mathcal{F} \right\}.$$

Then we have

$$\sum_{B \in \left[{V \atop b}\right]} \left|\mathcal{F}(B)\right| = \begin{bmatrix} k \\ b \end{bmatrix} |\mathcal{F}|. \tag{20}$$

Let $\tilde{k} = k - b = 2k - 2t + 1$ and $\tilde{t} - 1 = (t-1) - b = k - t$. Then $\mathcal{F}(B)$ is a $\tilde{k}$-uniform, $(\tilde{t}-1)$-avoiding family with $\tilde{k} = 2\tilde{t} - 1$. (In fact if there are $C_1, C_2 \in \mathcal{F}(B)$ such that $\dim(C_1 \cap C_2) = \tilde{t} - 1$, then $F_i = B \oplus C_i \in \mathcal{F}$ $(i = 1, 2)$ but $\dim(F_1 \cap F_2) = b + (\tilde{t} - 1) = t - 1$, a contradiction.) Thus, by Theorem 7, we have

$$\left|\mathcal{F}(B)\right| \leqslant \begin{bmatrix} n-b \\ \tilde{k} - \tilde{t} \end{bmatrix} = \begin{bmatrix} n-b \\ k-t \end{bmatrix}. \tag{21}$$

Now it follows from (20) and (21) that

$$|\mathcal{F}| \leqslant \begin{bmatrix} n \\ b \end{bmatrix} \begin{bmatrix} n-b \\ k-t \end{bmatrix} \Big/ \begin{bmatrix} k \\ b \end{bmatrix} = \begin{bmatrix} n \\ t-1 \end{bmatrix} \begin{bmatrix} 2k-t \\ k \end{bmatrix} \Big/ \begin{bmatrix} 2k-t \\ t-1 \end{bmatrix}$$

as needed. □

The bound in Theorem 8 is asymptotically sharp. Namely, we have the following.

Theorem 9. *Let $t \geqslant 1$ and $k > t-1$ be fixed. Then for every $\epsilon > 0$ there is an n_0 such that for all $n > n_0$ and $V = \mathbb{F}_q^n$ there is a $(t-1)$-avoiding family $\mathcal{F} \subset \left[{V \atop k}\right]$ with $|\mathcal{F}| > (1-\epsilon)\left[{n \atop t-1}\right]\left[{2k-t \atop k}\right]/\left[{2k-t \atop t-1}\right]$.*

To prove Theorem 9 we need the following variant of the packing theorem of Rödl [14].

Theorem 10. *Let r and s be fixed. Then for every $\epsilon > 0$ there is an n_0 such that for all $n > n_0$ and $V = \mathbb{F}_q^n$ there is a family $\mathcal{H} \subset \left[{V \atop r}\right]$ which satisfies $\dim(H \cap H') < s$ for all $H, H' \in \mathcal{H}$ and $|\mathcal{H}| > (1-\epsilon)\left[{n \atop s}\right]/\left[{r \atop s}\right]$.*

Proof of Theorem 9. By Theorem 10 we can take a family $\mathcal{SS} \subset \left[{V \atop 2k-t}\right]$ with $\dim(S \cap S') < t-1$ for all $S, S' \in \mathcal{SS}$ and $|\mathcal{SS}| \sim \left[{n \atop t-1}\right]/\left[{2k-t \atop t-1}\right]$ as $n \to \infty$. Let $\mathcal{F} = \Delta_k(\mathcal{SS})$. Then $\mathcal{F}$ is $(t-1)$-avoiding and $|\mathcal{F}| = \left[{2k-t \atop k}\right]|\mathcal{SS}|$ because $k > t-1$. Thus $\mathcal{F}$ satisfies the desired properties. □

Finally we remark that Theorem 10 is derived from the following result stating that almost regular hypergraphs have almost perfect matchings. This result was originally obtained by Frankl and Rödl [9] and we use a stronger version given by Pippenger (see [1] or [13]).

Theorem 11. *(See [9,1,13].) Let* $\mathcal{F} \subset \binom{X}{k}$ *satisfy the following.*

(1) *There is* D *such that* $\#\{F \in \mathcal{F}\colon x \in F\} = D$ *for all* $x \in X$.
(2) *For all* $\{x, y\} \in \binom{X}{2}$, $\#\{F \in \mathcal{F}\colon \{x, y\} \subset F\} = o(D)$ *as* $D \to \infty$.

Then there exist pairwise disjoint $F_1, \ldots, F_m \in \mathcal{F}$ *with* $m \sim |X|/k$ *(as* $D \to \infty$ *and hence* $|X| \to \infty$*).*

Proof of Theorem 10. Let $X = \begin{bmatrix} V \\ s \end{bmatrix}$ and $k = \begin{bmatrix} r \\ s \end{bmatrix}$. Define $\mathcal{F} := \left\{\begin{bmatrix} R \\ s \end{bmatrix}\colon R \in \begin{bmatrix} V \\ r \end{bmatrix}\right\} \subset \binom{X}{k}$. Then $\mathcal{F}$ is D-regular, where $D = \begin{bmatrix} n-s \\ r-s \end{bmatrix}$. Moreover, for a pair $\{x, y\} \subset X$, we have

$$\#\bigl\{F \in \mathcal{F}\colon \{x, y\} \subset F\bigr\} \leqslant \begin{bmatrix} n-s-1 \\ r-s-1 \end{bmatrix} = o(D).$$

In fact if $n \to \infty$ for fixed r and s, then $D \to \infty$ and $\begin{bmatrix} n-s-1 \\ r-s-1 \end{bmatrix}/D = \frac{q^{r-s}-1}{q^{n-s}-1} \to 0$, namely, $\begin{bmatrix} n-s-1 \\ r-s-1 \end{bmatrix} = o(D)$. Thus, by Theorem 11, we have a matching $F_1, \ldots, F_m \in \mathcal{F}$ with $m \sim \begin{bmatrix} n \\ s \end{bmatrix}/\begin{bmatrix} r \\ s \end{bmatrix}$. For $1 \leqslant i \leqslant m$ we can write $F_i = \begin{bmatrix} R_i \\ s \end{bmatrix}$. Then $\mathcal{H} := \{R_1, \ldots, R_m\} \subset \begin{bmatrix} V \\ r \end{bmatrix}$ satisfies the desired properties of Theorem 10. □

Acknowledgments

The authors thank the referees for their careful reading and many helpful comments.

References

[1] N. Alon, J. Spencer, The Probabilistic Method, second edition, 2000. A Wiley–Interscience publication.
[2] L. Babai, P. Frankl, Linear Algebra Methods in Combinatorics with Applications to Geometry and Computer Science. Preliminary Version 2, September 1992.
[3] P.J. Cameron, Combinatorics: Topics, Techniques, Algorithms, Cambridge University Press, 1994.
[4] P. Erdős, Problems and results in graph theory and combinatorial analysis, in: Proceedings of the Fifth British Combinatorial Conference, Univ. Aberdeen, Aberdeen, 1975, in: Congr. Numer., vol. XV, Utilitas Math., Winnipeg, MB, 1976, pp. 169–192.
[5] P. Frankl, An intersection problem for finite sets, Acta Math. Acad. Sci. Hungar. 30 (1977) 371–373.
[6] P. Frankl, Z. Füredi, On hypergraphs without two edges intersecting in a given number of vertices, J. Combin. Theory Ser. A 36 (1984) 230–236.
[7] P. Frankl, Z. Füredi, Forbidding just one intersection, J. Combin. Theory Ser. A 39 (1985) 160–176.
[8] P. Frankl, R.L. Graham, Intersection theorems for vector spaces, European J. Combin. 6 (1985) 183–187.
[9] P. Frankl, V. Rödl, Near perfect coverings in graphs and hypergraphs, European J. Combin. 6 (1985) 317–326.
[10] P. Frankl, N.M. Singhi, Linear dependencies among subsets of a finite set, European J. Combin. 4 (1983) 313–318.
[11] P. Frankl, R.M. Wilson, The Erdős–Ko–Rado theorem for vector spaces, J. Combin. Theory Ser. A 43 (1986) 228–236.
[12] G.O.H. Katona, Intersection theorems for systems of finite sets, Acta Math. Acad. Sci. Hungar. 15 (1964) 329–337.
[13] N. Pippenger, J. Spencer, Asymptotic behavior of the chromatic index for hypergraphs, J. Combin. Theory Ser. A 51 (1989) 24–42.
[14] V. Rödl, On a packing and covering problem, European J. Combin. 6 (1985) 69–78.