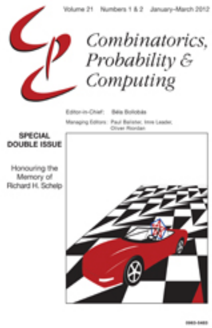

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On the Maximum Number of Edges in a Triple System Not Containing a Disjoint Family of a Given Size

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In 1965 Erdős conjectured a formula for the maximum number of edges in a k-uniform n-vertex hypergraph without a matching of size s. We prove this conjecture for $k = 3$ and all $s \geqslant 1$ and $n \geqslant 4s$.

1. Introduction

A *k-uniform hypergraph*, or *k-graph* for short, is a pair $H = (V, E)$, where $V := V(H)$ is a finite set of vertices and $E := E(H) \subseteq \binom{V}{k}$ is a family of k-element subsets of V. Whenever convenient we will identify H with $E(H)$. *A matching* in H is a set of disjoint edges of H. The number of edges in a matching is called *the size* of the matching. The size of the largest matching in a k-graph H is denoted by $\nu(H)$. A matching is *perfect* if its size equals $|V|/k$.

In this paper we study the relation between $|E(H)|$ and $\nu(H)$.

Definition. Let integers k, s, and n be such that $k \geqslant 2$ and $0 \leqslant s \leqslant n/k$. Define $m^s(k, n)$ to be the smallest integer m such that every n-vertex k-graph H with $|E(H)| \geqslant m$ contains a matching of size s. In other words,

$$m^s(k, n) = \min\{m \ : \ |E(H)| \geqslant m \Longrightarrow \nu(H) \geqslant s\}.$$

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It is quite easy to see that, for graphs, $m^{n/2}(2,n) = \binom{n-1}{2} + 1$. For k-graphs an analogous result is true: $m^{n/k}(k,n) = \binom{n-1}{k} + 1$. Indeed, the example of the clique on $n-1$ vertices plus an isolated vertex yields the lower bound. For the upper bound, let k divide n, and let H be an arbitrary n-vertex k-graph with at least $\binom{n-1}{k} + 1$ edges. Then, the complement H^c of H has fewer than $\binom{n-1}{k-1}$ edges, and therefore has to miss at least one perfect matching of $K_n^{(k)}$. (This can be seen by considering the expected number of edges in common with a random perfect matching.)

Erdős and Gallai [4] proved that, for all $1 \leqslant s \leqslant n/2$,

$$m^s(2,n) = \max\left\{\binom{2s-1}{2}, \binom{n}{2} - \binom{n-s+1}{2}\right\} + 1. \tag{1.1}$$

A few years later Erdős [3] conjectured a generalization to all $k \geqslant 2$ and $1 \leqslant s \leqslant n/k$:

$$m^s(k,n) = \max\left\{\binom{ks-1}{k}, \binom{n}{k} - \binom{n-s+1}{k}\right\} + 1. \tag{1.2}$$

The two competing k-graphs yielding the lower bound are $K_{ks-1}^{(k)} \cup (n-ks+1)K_1$, that is, the clique on $ks-1$ vertices appended by $n-ks+1$ isolated vertices, and $K_n^{(k)} - K_{n-s+1}^{(k)}$, a k-graph obtained from the complete k-graph on n vertices by deleting all edges of a fixed clique of order $n-s+1$. Equivalently, $K_n^{(k)} - K_{n-s+1}^{(k)}$ is the k-graph consisting of all k-element sets intersecting a given subset of vertices of size $s-1$.

Trivially, the conjecture is true for $s = 1$, but already for $s = 2$ it is equivalent to the celebrated Erdős–Ko–Rado theorem. For larger s, the conjecture has been confirmed by Erdős himself [3], but only for n sufficiently large with respect to k and s. Later, Bollobás, Daykin and Erdős [2], and Frankl and Füredi [6] improved the lower bound on n, to $2k^3s$ and $100ks^2$, respectively. Recently, Huang, Loh, and Sudakov [7] improved this bound further to $3k^2s$. We refer the reader to the survey paper [5]. Here we prove the Erdős conjecture for $k = 3$ and all $s \geqslant 1$ and $n \geqslant 4s$.

Theorem 1.1. *For all $s \geqslant 1$ and $n \geqslant 4s$, if H is a 3-uniform hypergraph with $|V(H)| = n$ and $\nu(H) \leqslant s-1$, then $|H| \leqslant \binom{n}{3} - \binom{n-s+1}{3}$. In other words,*

$$m^s(3,n) = \binom{n}{3} - \binom{n-s+1}{3} + 1.$$

Note that for $k = 3$ and $n \geqslant 4s$ the maximum in (1.2) is achieved by the second term. The actual transition point is around $(3.486)s$, so the question remains whether Theorem 1.1 can be extended to all n for which $\binom{3s-1}{3} \leqslant \binom{n}{3} - \binom{n-s+1}{3}$.

The proof of Theorem 1.1 relies on the technique of *shifting* and is presented in Section 2.

1.1. Minimum degree versus perfect matching

There are several results relating the minimum degree of a k-uniform hypergraph to the existence of a perfect matching (see, *e.g.*, [11]). It was shown in [1] that in order to determine asymptotically a minimum degree guaranteeing the existence of a perfect matching in a k-graph on n vertices, it suffices to prove a fractional version of Erdős's conjecture for $(k-1)$-graphs with $n-1$ vertices and $s = \frac{n}{k}$. For $k = 3$ this can be easily deduced from Theorem 1.1 and we obtain the following corollary.

Corollary 1.2. *If H is a 4-uniform hypergraph with the number of vertices n divisible by 4 and with $\delta(H) \geqslant (1+o(1))\frac{37}{64}\binom{n-1}{3}$, then H contains a perfect matching.*

This improves a previous bound of $(1+o(1))\frac{42}{64}\binom{n-1}{3}$ due to Markström and Ruciński [10]. In [1] the fractional version of Erdős's conjecture is proved, by quite different methods, also for 4-graphs with $s = \frac{n}{5}$. This yields an analogue of Corollary 1.2 for 5-uniform hypergraphs with $\frac{369}{625}\binom{n-1}{4}$ in place of $\frac{37}{64}\binom{n-1}{3}$. (Both thresholds are asymptotically best possible.)

Recently, the result of Corollary 1.2 has also been proved, by quite different methods, by Khan [8] as well as by Lo and Markström [9].

2. Proof of Theorem 1.1

The proof is by induction on s, with the case $s = 1$ completely trivial. Before we show the induction's step, let us recall the operation of shifting. Consider a hypergraph H with the vertex set $V(H)$ ordered linearly, say $V(H) = \{1, 2, \ldots, n\}$. Given $1 \leqslant i < j \leqslant n$ and an edge $e \in H$, we define the (i, j)-shift $S_{ij}(e)$ of e as follows:

$$S_{ij}(e) = \begin{cases} (e \setminus \{j\}) \cup \{i\} & \text{if } i \notin e, j \in e, (e \setminus \{j\}) \cup \{i\} \notin H, \\ e & \text{otherwise.} \end{cases} \tag{2.1}$$

We define $S_{ij}(H) = \{S_{ij}(e) \,:\, e \in H\}$. We call H *shifted* if $S_{ij}(H) = H$ for all $1 \leqslant i < j \leqslant n$. Note that shifting preserves the size of a hypergraph and that it does not increase the size of a largest matching. Formally, $|S_{ij}(H)| = |H|$ and $\nu(S_{ij}(H)) \leqslant \nu(H)$. Therefore, in what follows we may assume that H is shifted.

Let us fix $s \geqslant 2$ and assume that Theorem 1.1 is true for all $s' < s$. In the proofs of the next two claims we are going to use the following notation: for all $v \in V(H)$, let

$$H_{\not\ni v} = \{e \in H : v \notin e\}, \quad H_{\ni v} = \{e \in H : v \in e\}.$$

We will first show that it suffices to restrict the proof of Theorem 1.1 to the sole case of $n = 4s$.

Claim 2.1. *For all $s \geqslant 2$ and $n \geqslant 4s+1$, if Theorem 1.1 holds for $n-1$ then it also holds for n.*

Proof. Let H be a shifted 3-uniform hypergraph on n vertices with $\nu(H) \leqslant s-1$. Then, clearly, $|V(H_{\not\ni n})| = n-1$, $\nu(H_{\not\ni n}) \leqslant s-1$ and so, by our assumption that Theorem 1.1 holds for $n-1$, we have $|H_{\not\ni n}| \leqslant \binom{n-1}{3} - \binom{n-s}{3}$.

Let $H' = \{e \setminus \{n\} : e \in H_{\ni n}\}$. We claim that $\nu(H') \leqslant s-1$. Suppose not. Then there are s disjoint edges in H' and, because H is shifted, each of them forms an edge of H with any vertex $v \in V(H)$. There are, however, at least $n - 2s \geqslant 2s+1 \geqslant s$ vertices v available and a matching of size s exists in H, a contradiction.

Since $\nu(H') \leqslant s-1$, by the Erdős–Gallai theorem (see (1.1)), $|H'| \leqslant \binom{n-1}{2} - \binom{n-s}{2}$. Hence,

$$|H| = |H_{\not\ni n}| + |H'| \leqslant \binom{n-1}{3} - \binom{n-s}{3} + \binom{n-1}{2} - \binom{n-s}{2} = \binom{n}{3} - \binom{n-s+1}{3}.$$

□

We can therefore assume that $n = 4s$ throughout the rest of the proof.

Claim 2.2. *If* $\{1, 3s-1, 3s\} \in H$ *and* $\nu(H) \leqslant s-1$, *then* $|H| \leqslant \binom{n}{3} - \binom{n-s+1}{3}$.

Proof. Suppose $\nu(H_{\not\ni 1}) = s-1$. Then, because H is shifted, there is a matching M of size $s-1$ in $H_{\not\ni 1}$ such that $V(M) = \{2, 3, \ldots, 3s-2\}$. This matching, together with the edge $\{1, 3s-1, 3s\}$, forms a matching of size s in H, which contradicts the assumption that $\nu(H) \leqslant s-1$. Consequently, $\nu(H_{\not\ni 1}) \leqslant s-2$ and, hence, by induction on s, $|H_{\not\ni 1}| \leqslant \binom{n-1}{3} - \binom{n-s+1}{3}$. On the other hand, trivially, $|H_{\ni 1}| \leqslant \binom{n-1}{2}$ and we conclude that

$$|H| \leqslant |H_{\not\ni 1}| + |H_{\ni 1}| \leqslant \binom{n-1}{3} + \binom{n-1}{2} - \binom{n-s+1}{3} = \binom{n}{3} - \binom{n-s+1}{3}. \quad \square$$

In view of Claim 2.2, we assume that $\{1, 3s-1, 3s\} \notin H$. As a consequence, no edge of H intersects the set $\{3s-1, 3s, \ldots, 4s\}$ in 2 or 3 vertices. Let

$$F_0 = \left\{ e \in \binom{[4s]}{3} : |e \cap \{3s-1, 3s, \ldots, 4s\}| \geqslant 2 \right\}.$$

Then the complement H^c of H contains F_0 and

$$|F_0| = \binom{s+2}{3} + \binom{s+2}{2}(3s-2).$$

Hence, in order to prove Theorem 1.1, it is enough to verify that

$$|H^c \setminus F_0| \geqslant \binom{3s+1}{3} - |F_0| = \frac{1}{6}(17s^3 - 24s^2 - 5s + 12) := W(s). \tag{2.2}$$

Consider an auxiliary bipartite graph $B = (X, Y, E(B))$, where $X = \{2s+1, \ldots, 3s\}$, $Y = [s] := \{1, \ldots, s\}$ and, for $w \in X$ and $i \in Y$, the pair $\{w, i\} \in E(B)$ if $\{i, 2s+1-i, w\} \in H$. Since $\nu(H) \leqslant s-1$, the graph B does not have a perfect matching, and by Hall's theorem, there is a set $T \subseteq X$, $1 \leqslant t := |T| \leqslant s$, such that its neighbourhood $N := N_B(T)$ has size $|N| = t-1$. Because H is shifted, we may assume that $T = \{3s-t+1, \ldots, 3s\}$.

Because H is shifted, if $\{x, y, z\} \notin H$ and $x \leqslant u$, $y \leqslant v$, and $z \leqslant w$, with all u, v, w distinct, then also $\{u, v, w\} \notin H$. In such a case we say that triple $\{u, v, w\} \notin H$ is *forbidden* by triple $\{x, y, z\} \notin H$.

By the definition of the set N, for every $w \in T$ and $i \in [s] \setminus N$, we have $\{i, 2s+1-i, w\} \notin H$. For each $i = 1, \ldots, s$, let A_i be the set of all triples of $H^c \setminus F_0$ forbidden by $\{i, 2s+1-i, 3s-t+1\}$, that is,

$$A_i = \{1 \leqslant u < v < w \leqslant 4s : i \leqslant u, 2s+1-i \leqslant v \leqslant 3s-2, 3s-t+1 \leqslant w\}.$$

Then

$$|H^c \setminus F_0| \geqslant \left| \bigcup_{i \in [s] \setminus N} A_i \right|. \tag{2.3}$$

Since our goal is to prove (2.2), it will be sufficient to show that $|\bigcup_{i \in [s] \setminus N} A_i| \geqslant W(s)$. We will consider two cases, $t = 1$ and $t \geqslant 2$, with the latter split into two subcases, $t = s$ and $2 \leqslant t \leqslant s - 1$.

Case $t = 1$. In this case $N = \emptyset$, and hence none of the triples $\{i, 2s + 1 - i, 3s\}$, $i = 1, \ldots, s$, belong to H. In order to estimate $|\bigcup_{i \in [s] \setminus N} A_i|$, we count triples which are *not* forbidden by any of the triples $\{i, 2s + 1 - i, 3s\}$.

For each $w = 3s, 3s + 1, \ldots, 4s$ only the pairs $\{i, j\}$, $i = 1, \ldots, s - 1$, $j = i + 1, \ldots, 2s - i$, can form an edge with w, altogether, at most $s(s - 1)$ edges for each w. This means that

$$\left| \bigcup_{i \in [s] \setminus N} A_i \right| \geqslant (s + 1)\left[\binom{3s - 2}{2} - s(s - 1) \right] = \frac{1}{2}(7s^3 - 6s^2 - 7s + 6),$$

and inequality (2.2) can be easily verified for every $s \geqslant 1$.

Case $t \geqslant 2$. In order to calculate $|A_i|$, we define four segments of vertices:

- $A = \{i, \ldots, 2s - i\}$, $a := |A| = 2s - 2i + 1$,
- $B = \{2s - i + 1, \ldots, 3s - t\}$, $b := |B| = s - t + i$,
- $C = \{3s - t + 1, \ldots, 3s - 2\}$, $c := |C| = t - 2$,
- $D = \{3s - 1, \ldots, 4s\}$, $d := |D| = s + 2$.

Observe that

$$A_i = \{1 \leqslant u < v < w \leqslant 4s : u \in A \cup B \cup C, v \in B \cup C, w \in C \cup D\},$$

and, moreover,

- $ab(c + d)$ triples in A_i satisfy $u \in A, v \in B$,
- $a\big(cd + \binom{c}{2}\big)$ triples in A_i satisfy $u \in A, v \in C$,
- $\binom{b}{2}(c + d)$ triples in A_i satisfy $u \in B, v \in B$,
- $b\big(cd + \binom{c}{2}\big)$ triples in A_i satisfy $u \in B, v \in C$,
- $\binom{c}{2}d + \binom{c}{3}$ triples in A_i satisfy $u \in C, v \in C$.

Hence,

$$|A_i| = ab(c + d) + a\left(cd + \binom{c}{2} \right) + \binom{b}{2}(c + d) + b\left(cd + \binom{c}{2} \right) + \binom{c}{2}d + \binom{c}{3}.$$

After plugging in the formulas for a, b, c, d and collecting together all terms involving a, we obtain the following formula:

$$|A_i| = a[b(c + d) + q] + r_i = (2s - 2i + 1)[(s - t + i)(s + t) + q] + r_i, \tag{2.4}$$

where

$$q = \binom{c}{2} + cd = \binom{t - 2}{2} + (t - 2)(s + 2)$$

and

$$r_i = \binom{b}{2}(c+d) + bq + \binom{c}{2}d + \binom{c}{3}.$$

Subcase $t = s$. For $t = s$ the above formula simplifies to

$$|A_i| = (2s-2i+1)(2si+q) + si(i-1) + qi + \binom{s-2}{2}(s+2) + \binom{s-2}{3},$$

which is a quadratic function of i with the main term $-3si^2$. So, the minimum is achieved at either $i = 1$ or $i = s$.

We have

$$|A_1| = \frac{1}{3}(11s^3 - 12s^2 - 5s + 6)$$

and

$$|A_s| = \frac{1}{6}(19s^3 - 18s^2 - 7s + 6).$$

It can be easily checked that for $s \geqslant 1$ both these quantities are greater than or equal to $W(s)$, and so (2.2) holds.

Subcase $2 \leqslant t \leqslant s-1$. In this case, we refine our estimates by considering unions $|A_i \cup A_j|$. Given $1 \leqslant i < j \leqslant s$, observe that

$$A_i \cap A_j = \{1 \leqslant u < v < w \leqslant 4s : j \leqslant u, 2s+1-i \leqslant v \leqslant 3s-2, 3s-t+1 \leqslant w\},$$

and thus, the formula for $|A_i \cap A_j|$ can be obtained from that for $|A_i|$ by replacing the set $A = \{i, \ldots, 2s-i\}$ with $\tilde{A} = \{j, \ldots, 2s-i\}$, in other words, replacing $a = |A|$ with $\tilde{a} = |\tilde{A}| = 2s - i - j + 1$.

Hence,

$$|A_i \cap A_j| = (2s-i-j+1)[(s+i-t)(s+t)+q] + r_i,$$

and consequently

$$|A_i \cup A_j| = |A_i| + |A_j| - |A_i \cap A_j| = Q(i,j,t) + |A_j|,$$

where

$$Q(i,j,t) = (j-i)[(s+i-t)(s+t)+q]. \tag{2.5}$$

We are going to minimize Q with respect to all three variables: i, j, and t. Note that Q depends also on s, but we suppress this dependence here. Since $|[s] \setminus N| = s-t+1$, there are $i, j \in [s] \setminus N$ such that $s-t+1 \leqslant j \leqslant s$ and $1 \leqslant i \leqslant j-s+t$. For every $2 \leqslant t \leqslant s-1$, we estimate

$$\left|\bigcup_{i\in[s]\setminus N} A_i\right| \geqslant \max_{i,j\in[s]\setminus N} |A_i \cup A_j| \geqslant \min\{|A_i \cup A_j| \ : \ s-t+1 \leqslant j \leqslant s,\ 1 \leqslant i \leqslant j-s+t\}. \tag{2.6}$$

Since $Q(i, j, t)$ is a quadratic function of i with a negative coefficient at i^2, we have in the given range of i

$$Q(i, j, t) \geqslant \min\{Q(1, j, t), Q(j - s + t, j, t)\} \geqslant Q(j - s + t, j, t),$$

where the last inequality comes from direct comparison, *i.e.*,

$$Q(1, j, t) - Q(j - s + t, j, t) = (j + t - s - 1)(q + s + t) \geqslant 0,$$

because $j \geqslant s - t + 1$. Consequently, combining (2.2)–(2.6),

$$|H^c \setminus F_0| \geqslant Q(j - s + t, j, t) + |A_j| = (s - t)[j(s + t) + q] + |A_j| := P(j, t)$$

for some $j = s - t + 1, \ldots, s$. (Again, we suppress the dependence of P on s.)

Using (2.4) we can check that $P(j, t)$ is a quadratic function of j with a negative coefficient at j^2, and so

$$P(j, t) \geqslant \min\{P(s - t + 1, t), P(s, t)\}.$$

Our plan is to express both $f(t) := P(s, t)$ and $g(t) := P(s - t + 1, t)$ as polynomials (of degree 3) in t and show that their minima over t, $2 \leqslant t \leqslant s - 1$, are still at least as large as the right-hand side of (2.2). After collecting all terms we obtain

$$f(t) = -\frac{1}{3}t^3 - \frac{1}{2}(5s - 1)t^2 + \left(3s^2 + \frac{3}{2}s + \frac{5}{6}\right)t + 3s^3 - 5s^2 - 2s + 1,$$

and

$$g(t) = -\frac{4}{3}t^3 - 2(s - 1)t^2 + \left(4s^2 - \frac{2}{3}\right)t + 3s^3 - 6s^2 - s + 2.$$

Since the second derivatives with respect to t satisfy $f''(t) = -2t + 1 - 5s < 0$ and $g''(t) = -8t - 4s + 4 < 0$, both functions are concave and the minima are attained at $t = 2$ or $t = s - 1$. It remains to compare $f(2), f(s - 1), g(2)$, and $g(s - 1)$ against $W(s)$, the right-hand side of (2.2). We have

$$\begin{aligned} f(2) &= 3s^3 + s^2 - 9s + 2, \\ f(s - 1) &= \frac{1}{6}(19s^3 - 43s + 6), \\ g(2) &= 3s^3 + 2s^2 - 9s - 2, \\ \text{and} \quad g(s - 1) &= \frac{1}{6}(22s^3 - 70s + 36). \end{aligned}$$

It can be easily checked that

$$\min\{f(2), f(s - 1), g(2), g(s - 1)\} - W(s) \geqslant 0$$

for every $s \geqslant 2$. This completes the proof of (2.2) and, therefore, also the proof of Theorem 1.1. □

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