JOURNAL OF COMBINATORIAL THEORY (A) 19, 208–213 (1975)

The Proof of a Conjecture of G. O. H. Katona

PETER FRANKL

Eötvös L. University, Budapest

Communicated by the Managing Editors

Received January 14, 1975

The following conjecture of G. O. H. Katona is proved. Let X be a finite set of cardinality n, and $\mathscr{A}$ a family of subsets of X. Let us suppose that for any two members A, B of $\mathscr{A}$ we have $|A \cup B| \leqslant n - r$, $|A \cap B| \geqslant 1$, r is a positive integer, $r \leqslant n$. Then

$$|\mathscr{A}| \leqslant \sum_{i=0}^{(n-1-r)/2} \binom{n-1}{i}$$

for odd, and

$$|\mathscr{A}| \leqslant \sum_{i=0}^{(n-2-r)/2} \binom{n-1}{i} + \binom{n-2}{(n-2-r)/2}$$

for even values of $n - r$.

1. Introduction

Let X be a finite set of cardinality n. Let $\mathscr{A}$ be a system of subsets of X. Let r and s be positive integers $r, s \leqslant n$.

Katona has proved in [1] that if for any sets A, B belonging to $\mathscr{A}$ we have $|A \cup B| \leqslant n - r$, then

$$|\mathscr{A}| \leqslant \sum_{i=0}^{(n-r)/2} \binom{n}{i} \tag{*}$$

for even, and

$$|\mathscr{A}| \leqslant \sum_{i=0}^{(n-r-1)/2} \binom{n}{i} + \binom{n-1}{(n-r-1)/2} \tag{**}$$

for odd values of $n - r \geqslant 0$.

In his survey paper [2] he raised the problem what can we say about $|\mathscr{A}|$ if we know that for any two sets A, B belonging to $\mathscr{A}$ we have

$$|A \cup B| \leqslant n - r, \qquad |A \cap B| \geqslant s, \qquad r, s \leqslant n.$$

He conjectured that for $s = 1$,

$$|\mathscr{A}| \leqslant \sum_{i=0}^{(n-1-r)/2} \binom{n-1}{i}$$

for odd, and

$$|\mathscr{A}| \leqslant \sum_{i=0}^{(n-2-r)/2} \binom{n-1}{i} + \binom{n-2}{(n-2-r)/2}$$

for even values of $n - r$.

Katona has shown that for the following systems equality can be attained.

(a) $n - r = 2t - 1$

Let x be an arbitrary element of X, and let $\mathscr{A}$ consist of exactly those subsets of X which contain x, and have cardinality less than or equal to t.

(b) $n - r = 2t$

Let x and y be two different elements of X. Let $\mathscr{A}$ consist of the subsets of X which contain x and have cardinality less than or equal to t or which contain both x and y and have cardinality $t + 1$.

In this paper we prove this conjecture and show that every optimal system is of the above form, unless $r = 1$.

We need two results, the first of which is due to Kleitman [3] while the second is due to Katona [1].

I. Let $\mathscr{A}$, $\mathscr{B}$ be two families of subsets of X which satisfy

$$A' \subset A \in \mathscr{A} \quad \text{implies} \quad A' \in \mathscr{A},$$
$$B' \supset B \in \mathscr{B} \quad \text{implies} \quad B' \in \mathscr{B}.$$

Then we have

$$|\mathscr{A} \cap \mathscr{B} = \{C \mid C \in \mathscr{A} \wedge C \in \mathscr{B}\}| \leqslant (|\mathscr{A}|\,|\mathscr{B}|)/2^n. \tag{1}$$

II. Let $\mathscr{A}$ be a family of l-element subsets of X. Let

$$\mathscr{A}_g = \{B \mid |B| = g \wedge \exists A \in \mathscr{A} \mid B \subset A\}.$$

If any two sets belonging to $\mathscr{A}$ have at least k elements in common then we have

$$\frac{|\mathscr{A}_g|}{|\mathscr{A}|} \geqslant \frac{\binom{2l-k}{g}}{\binom{2l-k}{l}} \qquad (g+k \geqslant l, g \leqslant l). \tag{2}$$

2. The Results

Theorem. *Let X be a finite set of cardinality n, and let $\mathscr{A}$ be a family of subsets of X. Let us suppose that for any two sets A, B belonging to $\mathscr{A}$ we have $|A \cup B| \leqslant n-r$, $|A \cap B| \geqslant 1$, $r \leqslant n$.*

Then

$$|\mathscr{A}| \leqslant \sum_{i=0}^{(n-1-r)/2} \binom{n-1}{i}$$

for odd, and

$$|\mathscr{A}| \leqslant \sum_{i=0}^{(n-2-r)/2} \binom{n-1}{i} + \binom{n-2}{(n-2-r)/2}$$

for even values of $n-r$.

For $r \neq 1$, equality holds only for the families given in the Introduction.

Proof. We separate the two cases $n-r$ is odd, $n-r$ is even.

Case a ($n-r$ is odd). Let us define two further families:

$$\mathscr{A}^* = \{B \mid \exists A \in \mathscr{A}, A \subset B\},$$
$$\mathscr{A}_* = \{C \mid \exists A \in \mathscr{A}, C \subset A\}.$$

If B, B' belong to $\mathscr{A}^*$, and C, C' belong to $\mathscr{A}_*$, then we have

$$|B \cap B'| \geqslant 1, \qquad |C \cup C'| \leqslant n-r.$$

We define these families for the other case, too. By (*) and by (**) we have

$$|\mathscr{A}^*| \leqslant 2^{n-1}, \qquad |\mathscr{A}_*| \leqslant \sum_{i=0}^{(n-r-1)/2} \binom{n}{i} + \binom{n-1}{(n-r-1)/2}.$$

As $\mathscr{A} \subset \mathscr{A}_* \cap \mathscr{A}^*$, applying (1) we get

$$| \mathscr{A} | \leqslant | \mathscr{A}_* \cap \mathscr{A}^* | \leqslant \tfrac{1}{2} \left(\sum_{i=0}^{(n-r-1)/2} \binom{n}{i} + \binom{n-1}{(n-r-1)/2} \right)$$

$$= \tfrac{1}{2} \left(\binom{n}{0} + \sum_{i=1}^{(n-r-1)/2} \left[\binom{n-1}{i} + \binom{n-1}{i-1} \right] + \binom{n-1}{(n-r-1)/2} \right)$$

$$= \sum_{i=0}^{(n-r-1)/2} \binom{n-1}{i},$$

as asserted. Equality can hold only if equality holds in (**). It can happen only if for some $x \in X$ we have

$$\mathscr{A}_* = \{A \mid | A | \leqslant (n-r)/2\} \cup \{A \mid x \in A, | A | = (n-r+1)/2\}, \qquad r > 1.$$

It follows that the members of the second term in this union belong to $\mathscr{A}$ and consequently every set belonging to $\mathscr{A}$ has to contain x (they are nondisjoint with the sets in the second term of the above union), and the uniqueness follows.

Case b ($n - r$ is even). Let a_k denote the number of k-element subsets of X belonging to $\mathscr{A}$. By the Erdös–Ko–Rado theorem (see [3]) we have $a_k \leqslant \binom{n-1}{k-1}$ for $k \leqslant n/2$. Hence

$$\sum_{i=0}^{(n-r)/2} a_i \leqslant \sum_{i=0}^{(n-r-2)/2} \binom{n-1}{i}. \tag{3'}$$

Let us suppose that $\mathscr{A}$ has maximal cardinality, then we have

$$\sum_{i=(n-r+2)/2}^{n} a_i \geqslant \binom{n-2}{(n-2-r)/2}. \tag{3}$$

Let $\mathscr{A}_k$ denote the family of k-element subsets of X belonging to $\mathscr{A}_*$, $| \mathscr{A}_K | = \bar{a}_k$. By (3) we have

$$\sum_{k=(n-r+2)/2}^{n} \bar{a}_k \geqslant \binom{n-2}{(n-2-r)/2}.$$

We shall next prove

$$\bar{a}_j + \frac{j-1+r}{j} \bar{a}_{n-r+1-j} \leqslant \binom{n}{j} \qquad (2j \leqslant n-r). \tag{4}$$

This inequality is essentially contained in [1]. Let

$$\mathscr{B}_j = \{B \mid |B| = n - j, \qquad \exists A \in \mathscr{A}_{n-r+1-j} \mid A \subset B\}.$$

Obviously we have $|\mathscr{A}_j| + |\mathscr{B}_j| \leqslant \binom{n}{j}$, and by (2),

$$|\mathscr{B}_j| \geqslant |\mathscr{A}_{n-r+1-j}| \, [(j - 1 + r)/j],$$

and (4) follows. As $2j \leqslant n - r$, so we can rewrite (4) ($j \neq 0$):

$$\bar{a}_j + \bar{a}_{n-r+1-j} \leqslant \binom{n}{j} - \frac{r-1}{j}\bar{a}_{n-r+1-j} \leqslant \binom{n}{j} - \frac{2(r-1)}{n-r}\bar{a}_{n-r+1-j}. \quad (5)$$

Summing (5) from $j = 0$ through $j = (n - r)/2$ we get ($\bar{a}_0 \leqslant 1 = \binom{n}{0}$):

$$\begin{aligned}
|\mathscr{A}_*| &= \sum_{j=0}^{n-r} \bar{a}_j \leqslant \sum_{j=0}^{(n-r)/2} (\bar{a}_j + \bar{a}_{n-r+1-j}) \\
&\leqslant \sum_{j=0}^{(n-r)/2} \binom{n}{j} - \frac{2(r-1)}{n-r} \sum_{j=(n-r+2)/2}^{n} \bar{a}_{n-r+1-j} \\
&\leqslant \sum_{j=1}^{(n-r)/2} \left[\binom{n-1}{j} + \binom{n-1}{j-1}\right] + 1 - \frac{2(r-1)}{n-r}\binom{n-2}{(n-2-r)/2} \\
&= 2\sum_{j=0}^{(n-r-2)/2} \binom{n-1}{j} + \binom{n-1}{(n-r)/2} - \frac{2(r-1)}{n-r}\binom{n-2}{(n-2-r)/2} \\
&= 2\left(\sum_{j=0}^{(n-r-2)/2} \binom{n-1}{j} + \binom{n-2}{(n-2-r)/2}\right). \qquad (6)
\end{aligned}$$

As $|\mathscr{A}^*| \leqslant 2^{n-1}$, the statement of the theorem follows as in Case a. If we have equality in (6) and $r > 1$ then we must have equality in (3), whence in (3′). Consequently $a_1 = 1$, i.e., for some $x \in X$, $\{x\} \in \mathscr{A}$. Hence for any $A \in \mathscr{A}$, $x \in A$ and the uniqueness of the optimal family follows from (**) as in Case a.

3. Remarks

The theorem for the special case $r = 1$ was first proved by Daykin and Lovász [5]. In this case the optimal families are not unique. This case is a trivial consequence of (1) as it was observed by Kleitman (personal communication).

We conclude this paper with a

Conjecture. Let X be a finite set of cardinality n. Let $\mathscr{A}$ be a family consisting of subsets of X, and let us suppose that for any to members A, B of $\mathscr{A}$ we have

$$|A \cup B| \leqslant n - 2, \qquad |A \cap B| \geqslant 2. \tag{7}$$

Let $X = Y \cup Z$, $|Y| = [n/2]$, $|Z| = [(n+1)/2]$ for $n = k$ and $n = k + 1$; $|Y| = [(n-2)/2]$, $|Z| = [(n+3)/2]$ for $n = k + 2$ and $n = k + 3$.

Let $\mathscr{C}$ be an optimal family of subsets of Y and $\mathscr{D}$ be an optimal family of subsets of Z which satisfy the first and the second condition, respectively in (7). Let us define the family

$$\mathscr{B} = \{B \subset X \mid B \cap Y \in \mathscr{C}, B \cap Z \in \mathscr{D}\}, \qquad \text{then} \qquad |\mathscr{B}| \geqslant |\mathscr{A}|.$$

References

1. G. Katona, Intersection theorems for systems of finite sets, *Acta Math. Acad. Sci. Hungar.* **15** (1964), 329–37.
2. G. O. H. Katona, Extremal problems for hypergraphs, *in* "Combinatorics," (M. Hall and J. H. van Lint, Eds.), Part 2, pp. 13–42, Mathematical Centre Tracts 55–57, Mathematisch Centrum, Amsterdam, 1974.
3. D. J. Kleitman, Families of non-disjoint subsets, *J. Combinatorial Theory* **1** (1966), 153–155.
4. P. Erdös, Chao-Ko, and R. Rado, Intersection theorems for systems of finite sets, *Quart. J. Math. Oxford Ser.* **12** (1961), 313–318.
5. D. E. Daykin and L. Lovász, On the number of values of a Boolean function, to appear.