A NOTE ON SUPERSATURATED SET SYSTEMS

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Abstract. A well known theorem of Erdős, Ko and Rado implies that any family $\mathscr{F}$ of k-element subsets of an n-element set with more than $\binom{n-t}{k-t}$ members must contain two members F and F' with $|F \cap F'| < t$, as long as n is sufficiently large with respect to k and t. We investigate how many such pairs $(F, F') \in \mathscr{F} \times \mathscr{F}$ there must be in any such family $\mathscr{F}$ with $|\mathscr{F}| = \alpha\binom{n-t}{k-t}$ and $\alpha > 1$.

1. Introduction

Let $1 \leq t < k \leq n$ be integers. Let X be a set with $|X| = n$ and let $\mathscr{F} \subset \binom{X}{k}$ be a family of k-element subsets of X. We say that $\mathscr{F}$ is *t-intersecting* if $|F \cap F'| \geq t$ for any F and $F' \in \mathscr{F}$. For example, the family $\mathscr{I}(T)$ of all $\binom{n-t}{k-t}$ subsets of X with k elements containing a fixed t-element set $T \subset X$ is t-intersecting.

The well known theorem of Erdős, Ko and Rado [5] states that, for any given $1 \leq t < k$, the largest possible size of a t-intersecting family $\mathscr{F} \subset \binom{X}{k}$ is $\binom{n-t}{k-t}$, provided that $|X| = n \geq n_0(k, t)$. The least value of $n_0(k, t)$ for which this holds, namely, $n_0(k, t) = (k-t+1)(t+1)$, was established by the first author [7] for every $t > 14$ and Wilson [11] generalized this to $t > 1$. The original paper [5] already contained this result for $t = 1$ (see also [8] for a new short proof). Thus, if $n \geq (k - t + 1)(t + 1)$ and $\mathscr{F} \subset \binom{X}{k}$ is such that $|\mathscr{F}| > \binom{n-t}{k-t}$, then $\mathscr{F}$ contains a pair of members F and F' with $|F \cap F'| < t$. In this note, we study how many pairs $(F, F') \in \mathscr{F} \times \mathscr{F}$ with $|F \cap F'| < t$ are guaranteed to exist in any family $\mathscr{F} \subset \binom{X}{k}$ with $|\mathscr{F}| = \alpha\binom{n-t}{k-t}$ and $\alpha > 1$. The case $t = 1$ is already very interesting: this problem was raised by Ahlswede [1] in 1980. Very recently, Das, Gan and Sudakov [4] addressed this problem for arbitrary t and k. The related problem in which we let $t = 1$ and consider general set systems $\mathscr{F} \subset 2^X$ and not only k-uniform hypergraphs $\mathscr{F} \subset \binom{X}{k}$ was solved independently by the first author [6] and Ahlswede [1] (see also Bollobás and Leader [3]). For a detailed, recent account on the relevant literature, the reader is referred to [4].

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In order to state our result precisely, we introduce some notation. As usual, let $[n] = \{1, \ldots, n\}$. Given a family of sets $\mathscr{F}$, let

$$\mathscr{F}_j^{(2)} = \{(F, F') \in \mathscr{F} \times \mathscr{F} : |F \cap F'| = j\} \tag{1}$$

and let $\mathscr{F}_{<t}^{(2)} = \bigcup_{j<t} \mathscr{F}_j^{(2)}$. Let

$$\mu(n, k, t, \alpha) = \min_{\mathscr{F}} \left|\mathscr{F}_{<t}^{(2)}\right|, \tag{2}$$

where the minimum is taken over all $\mathscr{F} \subset \binom{[n]}{k}$ with $|\mathscr{F}| = \left\lfloor \alpha\binom{n-t}{k-t} \right\rfloor$. Das, Gan and Sudakov [4] determined $\mu(n, k, t, \alpha)$ *precisely* for all k, t and α, for all $n \geq n_0(k, t, \alpha)$. We shall be able to handle a somewhat wider range of the parameters k, t and α, at the expense of giving *asymptotic estimates* for $\mu(n, k, t, \alpha)$ only. Our proofs are also of a different nature.

We need some further preparations to state our results. For any real number α, let

$$\left\{\begin{matrix} \alpha \\ 2 \end{matrix}\right\} = \binom{\lfloor \alpha \rfloor}{2} + (\alpha - \lfloor \alpha \rfloor)\lfloor \alpha \rfloor, \tag{3}$$

where, as usual, for any integer k, we let $\binom{x}{k} = (x)_k/k! = x(x-1)\ldots(x-k+1)/k!$ if $k \geq 0$ and we let $\binom{x}{k} = 0$ if $k < 0$. If α is an integer, then $\left\{\begin{smallmatrix} \alpha \\ 2 \end{smallmatrix}\right\} = \binom{\alpha}{2}$ and, for real α, if we write $\alpha = \lfloor \alpha \rfloor + x$, then $\left\{\begin{smallmatrix} \alpha \\ 2 \end{smallmatrix}\right\}$ grows linearly in x from $\binom{\lfloor \alpha \rfloor}{2}$ to $\binom{\lfloor \alpha \rfloor + 1}{2}$. Thus, $\left\{\begin{smallmatrix} \alpha \\ 2 \end{smallmatrix}\right\}$ is the 'simplest' piecewise linear function that coincides with $\binom{\alpha}{2}$ on integer values of α.

In what follows, the asymptotic notation is with respect to $n \to \infty$. For a function $f = f(n)$ and an eventually positive function $g = g(n)$, we write $f \ll g$ and $f = o(g)$ if $|f/g| \to 0$ as $n \to \infty$. In particular, $o(1)$ denotes functions $f = f(n)$ with $|f| \to 0$ as $n \to \infty$. When convenient, we shall write $f \sim g$ to mean that $f = (1 + o(1))g$ and we shall write $f \gtrsim g$ to mean that $f \geq (1 + o(1))g$.

Let σ be a real number such that, for some x_0, every interval $[x - x^\sigma, x]$ with $x \geq x_0$ contains a prime number. Baker, Harman and Pintz [2] have proved that one can take $\sigma = 0.525$. Fix a real number $0 < \eta < 1$. Let $t = t(n)$ and $k = k(n)$ be non-decreasing integer functions with $1 \leq t < k \leq n$. We say that (k, t) is an *η-admissible* pair of functions if either

(*i*) $k = k(n) \leq 1/\eta$ for every n

or

(*ii*) $k \to \infty$, $k \leq (1 - \eta)n^{1/2}$ and $t \ll k^{1/2}$ and $t \ll (n/k)^{1-\sigma}$.

We shall prove the following result.

Theorem 1. *Let $\eta > 0$ be a fixed real number. Let $k = k(n)$ and $t = t(n)$ be integer functions forming an η-admissible pair (k, t). Let $\alpha = \alpha(n)$ be a non-decreasing real function with $1+\eta \le \alpha \le \binom{n}{k}\binom{n-t}{k-t}^{-1}$. Then the following hold.*

(*i*) *If $k \ll n^{1/2}$, then*

$$\mu(n, k, t, \alpha) \sim 2\left\{{\alpha \atop 2}\right\}\binom{n-t}{k-t}^2. \tag{4}$$

(*ii*) *Suppose $k \ge cn^{1/2}$ for some $0 < c < 1$ and every large enough n and suppose $\alpha \ll n/k$. Then*

$$\mu(n, k, t, \alpha) \sim 2\left\{{\alpha \atop 2}\right\}\binom{n-t}{k-t}\binom{n-k-t}{k-t}. \tag{5}$$

Remark 2. Consider case (i) of Theorem 1, that is, let $k \ll n^{1/2}$.

(*a*) Suppose $\alpha \to \infty$. Then $2\left\{{\alpha \atop 2}\right\} \sim \alpha^2$. It follows that the right-hand side of (4) is $\sim \alpha^2\binom{n-t}{k-t}^2$. Therefore, (4) tells us that any $\mathscr{F} \subset \binom{[n]}{k}$ with $|\mathscr{F}|\binom{n-t}{k-t}^{-1} \to \infty$ is such that $|\mathscr{F}^{(2)}_{<t}| \sim |\mathscr{F}|^2$ if $k \ll n^{1/2}$.

(*b*) Note that, in this case, that is, when $k \ll n^{1/2}$, almost all pairs (A, B) in $\binom{[n]}{k} \times \binom{[n]}{k}$ are such that $A \cap B = \emptyset$. Remark (a) above tells us that any 'moderately large' $\mathscr{F}$ is such that $|\mathscr{F}^{(2)}_{<t}| \sim |\mathscr{F}|^2$. Naturally, these two facts go hand in hand.

Remark 3. Consider case (ii) of Theorem 1, that is, suppose $k = \Omega(n^{1/2})$. In this case, a positive fraction of the pairs $(A, B) \in \binom{[n]}{k} \times \binom{[n]}{k}$ are such that $|A \cap B| \ge t$. It is natural that $\mu(n, k, t, \alpha)$ should be 'smaller' in this case: note that the estimate in (5) is smaller than the estimate in (4) by a factor of $\binom{n-k-t}{k-t}\binom{n-t}{k-t}^{-1}$.

Remark 4. As mentioned before, Das, Gan and Sudakov [4] have determined $\mu(n, k, t, \alpha)$ precisely for all k, t and α, for all $n \ge n_0(k, t, \alpha)$. (They have also characterized the extremal systems.) Let us consider the case $t = 1$. In our notation, their result establishes $\mu(n, k, 1, \alpha)$ for all $n \ge n_1(k, \alpha)$ with $n_1(k, \alpha)$ of order $k^2\alpha(k + \alpha)$. Therefore, unfortunately, their result does not cover values of k of order larger than $n^{1/3}$ or α or order larger than $n^{1/2}$. Theorem 1, even though far from *determining* $\mu(n, k, 1, \alpha)$, covers a wider range of k and α. The case in which $t > 1$ is similar.

Remark 5. The proof of the fact that $\mu(n, k, t, \alpha)$ is *at most* the right-hand side of (4) and (5) asymptotically involves a construction that, as proved in [4], gives the exact value of $\mu(n, k, t, \alpha)$ for a certain range of the parameters. Their precise result covers, however, a somewhat more restricted range of k, t and α. Let us also remark that, in Theorem 1, since we suppose that (k, t) is an η-admissible pair, we suppose that $k \le (1 - \eta)n^{1/2} \le n^{1/2}$. However, this condition on k

is not required in the proof of the asymptotic upper bound in (5), which turns out to be valid for k of order larger than $n^{1/2}$, as long as $\alpha \ll n/k$ and $t \ll n^{1/2}$ (this can be read out of the proof below, given in Section 2.2).

2. Proof of Theorem 1

To prove Theorem 1, we establish the upper and lower bounds involved in (4) and (5) separately. We prove the lower bounds in Section 2.1 and we prove the upper bounds in Section 2.2.

2.1. **Proof of the lower bounds.** Let us first introduce some concepts and results that will be required. A set system $\mathscr{S} \subset \binom{[n]}{k}$ is a (θ, k, t)-*system* if

(*i*) $|\mathscr{S}| = \theta\binom{n}{t}\binom{k}{t}^{-1}$ and

(*ii*) $|S \cap S'| < t$ for all distinct S and $S' \in \mathscr{S}$.

A simple double counting argument shows that any $\mathscr{S} \subset \binom{[n]}{k}$ that satisfies (*ii*) is such that $|\mathscr{S}| \leq \binom{n}{t}\binom{k}{t}^{-1}$. Systems close to achieving this upper bound (that is, with $\theta \to 1$) are sometimes called *almost complete* (k,t)-*systems*. The following result [9, 10], which will be crucial in what follows, states that, roughly speaking, almost complete (k,t)-systems exist for any admissible pair of functions (k,t),

Lemma 6. *Let $\eta > 0$ be fixed and let $k = k(n)$ and $t = t(n)$ form an η-admissible pair of functions (k,t). Then, for any $\delta > 0$, there is n_0 such that, for any $n \geq n_0$, there is a (θ, k, t)-system with $\theta \geq 1 - \delta$.*

We are interested in the cardinality of $\mathscr{S}_j^{(2)}$ for (θ, k, t)-systems $\mathscr{S}$ (recall the notation introduced in (1)).

Fact 7. Let $\mathscr{S} \subset \binom{[n]}{k}$ be a (θ, k, t)-systems. Then, for any $0 \leq j \leq t$, we have

$$|\mathscr{S}_j^{(2)}| \leq \theta\frac{(n)_t(n-k)_{t-j}}{j!\big((k-j)_{t-j}\big)^2}. \tag{6}$$

Proof. We count the pairs (S, S') in $\mathscr{S}_j^{(2)}$ as follows. First choose $S \in \mathscr{S}$. Then we choose a j-subset J of S and a $(t-j)$-subset T' of $[n] \setminus S$. Our hypothesis (*ii*) implies that at most one member S' of $\mathscr{S}$ contains the t-element set $T = J \cup T'$. Note that, even if $\mathscr{S}$ contains such a set S', it may happen that $|S' \cap S| > j$; we are not interested in such S' at the moment, as we are currently concerned with pairs $(S, S') \in \mathscr{S}_j^{(2)}$. Suppose $T = J \cup T'$ determines $S' \in \mathscr{S}$ with $|S' \cap S| = j$, that is, $S' \cap S = J$. Each such S' is obtained in exactly $\binom{k-j}{t-j}$ ways: this is the number of $T' \subset S' \setminus S = S' \setminus J$ with $|T'| = t - j$ that, together with J, determine this particular S'. We conclude that there are at most $\binom{n-k}{t-j}\binom{k-j}{t-j}^{-1}$ such sets S' for any given J. We now sum over all choices of S and J and infer that

$$|\mathscr{S}_j^{(2)}| \leq |\mathscr{S}|\binom{k}{j}\binom{n-k}{t-j}\binom{k-j}{t-j}^{-1} = \theta\binom{n}{t}\binom{k}{t}^{-1}\binom{k}{j}\binom{n-k}{t-j}\binom{k-j}{t-j}^{-1} = \theta\frac{(n)_t(n-k)_{t-j}}{j!\,((k-j)_{t-j})^2}, \quad (7)$$

as required. □

We shall derive the lower bounds in (4) and (5) from the following lemma.

Lemma 8. *Let $1 \leq t < k \leq n-t$ and suppose a (θ, k, t)-systems $\mathscr{S} \subset \binom{[n]}{k}$ exists. Let $\mathscr{F} \subset \binom{[n]}{k}$ be given and set $\alpha = |\mathscr{F}|\binom{n-t}{k-t}^{-1}$. Then*

$$|\mathscr{F}_{<t}^{(2)}| \geq \frac{2}{\theta}\left\{\begin{matrix}\theta\alpha \\ 2\end{matrix}\right\}\binom{n-t}{k-t}\binom{n-k-t}{k-t}. \quad (8)$$

Proof. Let $\mathscr{F} \subset \binom{[n]}{k}$ be given and let α be as defined in the statement of our lemma. Fix a (θ, k, t)-system $\mathscr{S}$. We shall consider the quantity

$$\sum_{\pi} |\pi(\mathscr{S}) \cap \mathscr{F}|, \quad (9)$$

where π runs over all permutations $\pi\colon [n] \to [n]$. Observe that the number of times each $F \in \mathscr{F}$ is counted in (9) is

$$|\mathscr{S}|k!(n-k)! = \theta\binom{n}{t}\binom{k}{t}^{-1}k!(n-k)! = \theta(n)_t(k-t)!(n-k)!. \quad (10)$$

Consequently,

$$\sum_{\pi} |\pi(\mathscr{S}) \cap \mathscr{F}| = |\mathscr{F}||\mathscr{S}|k!(n-k)! = \theta\alpha\binom{n-t}{k-t}(n)_t(k-t)!(n-k)! = \theta\alpha n!. \quad (11)$$

We shall now use the following fact.

Fact 9. Let $x_1, \ldots, x_N$ be non-negative integers and let $a = N^{-1}\sum_{1\leq i\leq N} x_i$. Then

$$\frac{1}{N}\sum_{1\leq i\leq N}\binom{x_i}{2} \geq \left\{\begin{matrix}a \\ 2\end{matrix}\right\}. \quad (12)$$

Proof. It is straightforward to check that the left-hand side of (12) is minimized when all the x_i are as equal as possible, that is, when $x_i \in \{\lfloor a \rfloor, \lceil a \rceil\}$. Suppose $aN = \lfloor a \rfloor N + r$, where $0 \leq r < N$. The minimum of the left-hand side of (12) is achieved when $N - r$ of the x_i

are $\lfloor a \rfloor$ and the remaining r are $\lceil a \rceil$. Thus, this minimum is

$$\left(1-\frac{r}{N}\right)\binom{\lfloor a \rfloor}{2}+\frac{r}{N}\binom{\lceil a \rceil}{2}=\left\{ a \atop 2 \right\}, \tag{13}$$

as required. □

In view of (11), Fact 9 applied to the sequence of $n!$ numbers $|\pi(\mathscr{S}) \cap \mathscr{F}|$ gives that

$$\sum_{\pi}\binom{|\pi(\mathscr{S}) \cap \mathscr{F}|}{2} \geq \left\{ \theta\alpha \atop 2 \right\} n!. \tag{14}$$

We now note that

$$\begin{aligned}\sum_{\pi}\binom{|\pi(\mathscr{S}) \cap \mathscr{F}|}{2} &= \frac{1}{2}\sum_{j<t}\sum_{(S,S')\in\mathscr{S}_j^{(2)}}\sum_{(F,F')\in\mathscr{F}_j^{(2)}}\sum_{\pi}\mathbf{1}\{\pi(S)=F,\,\pi(S')=F'\} \\ &= \frac{1}{2}\sum_{j<t}|\mathscr{S}_j^{(2)}||\mathscr{F}_j^{(2)}|j!\big((k-j)!\big)^2(n-2k+j)!, \end{aligned}\tag{15}$$

which, by (6), is at most

$$\begin{aligned}&\frac{\theta}{2}\sum_{j<t}|\mathscr{F}_j^{(2)}|\frac{(n)_t(n-k)_{t-j}\big((k-j)!\big)^2 j!}{j!\,\big((k-j)_{t-j}\big)^2}(n-2k+j)! \\ &\qquad = \frac{\theta}{2}\sum_{j<t}|\mathscr{F}_j^{(2)}|(n)_t(n-k)_{t-j}(n-2k+j)!\big((k-t)!\big)^2. \end{aligned}\tag{16}$$

Multiplying and dividing the quantity in (16) by $n!$, we obtain

$$\begin{aligned}&\theta\frac{n!}{2}\sum_{j<t}|\mathscr{F}_j^{(2)}|\frac{(n)_t(n-k)_{t-j}(n-2k+j)!}{n!}\big((k-t)!\big)^2 \\ &\qquad = \theta\frac{n!}{2}\sum_{j<t}|\mathscr{F}_j^{(2)}|\frac{\big((k-t)!\big)^2}{(n-t)_{k-t}(n-k-t+j)_{k-t}} \\ &\qquad = \theta\frac{n!}{2}\binom{n-t}{k-t}^{-1}\sum_{j<t}|\mathscr{F}_j^{(2)}|\binom{n-k-t+j}{k-t}^{-1}. \end{aligned}\tag{17}$$

Comparing (14) and (17) we see that

$$\frac{\theta}{2}\binom{n-t}{k-t}^{-1}\sum_{j<t}|\mathscr{F}_j^{(2)}|\binom{n-k-t+j}{k-t}^{-1} \geq \left\{ \theta\alpha \atop 2 \right\}. \tag{18}$$

Since $\binom{n-k-t+j}{k-t} \geq \binom{n-k-t}{k-t}$, we get from (18) that

$$|\mathscr{F}_{<t}^{(2)}| = \sum_{j<t}|\mathscr{F}_j^{(2)}| \geq \frac{2}{\theta}\left\{ \theta\alpha \atop 2 \right\}\binom{n-t}{k-t}\binom{n-k-t}{k-t}, \tag{19}$$

as required. $\square$

We now state and prove a simple fact that will be required soon.

Fact 10. (i) For every $\eta > 0$ and $\varepsilon > 0$, there is $\delta > 0$ such that, for every $1 - \delta \leq \theta \leq 1$ and every $\alpha \geq 1 + \eta$, we have

$$\genfrac\{\}{0pt}{}{\theta\alpha}{2} \geq \left(1 - \frac{\varepsilon}{2}\right) \genfrac\{\}{0pt}{}{\alpha}{2}. \tag{20}$$

(ii) For every $\eta > 0$ and $\varepsilon > 0$, there is $\delta > 0$ such that, for every $\alpha \geq 1 + \eta$, we have

$$\genfrac\{\}{0pt}{}{(1+\delta)\alpha}{2} \leq \left(1 + \frac{\varepsilon}{2}\right) \genfrac\{\}{0pt}{}{\alpha}{2}. \tag{21}$$

Proof. Let us start with the proof of (i). Let η and ε be given. Let us first recall that $\genfrac\{\}{0pt}{1}{\alpha}{2} \geq \binom{\alpha}{2}$ for every α and that $\genfrac\{\}{0pt}{1}{\alpha}{2} \sim \binom{\alpha}{2}$ as $\alpha \to \infty$. Therefore, if $\alpha \geq \alpha_0(\varepsilon)$ and $0 < \delta \leq \delta_0(\varepsilon)$, then, for every $1 - \delta \leq \theta \leq 1$, we have

$$\genfrac\{\}{0pt}{}{\theta\alpha}{2} \geq \binom{\theta\alpha}{2} \geq \left(1 - \frac{\varepsilon}{4}\right) \binom{\alpha}{2} \geq \left(1 - \frac{\varepsilon}{2}\right) \genfrac\{\}{0pt}{}{\alpha}{2}, \tag{22}$$

and (20) holds. We now suppose $\alpha < \alpha_0(\varepsilon)$. Let $\delta_1 = \delta_1(\eta, \varepsilon) = \eta\varepsilon/4\alpha_0(\varepsilon)$ and suppose $1 - \delta_1 \leq \theta \leq 1$. Then

$$\alpha - \delta_1\alpha \leq \theta\alpha \leq \alpha. \tag{23}$$

Recall that $\genfrac\{\}{0pt}{1}{\alpha}{2}$ is a piecewise linear function. Moreover, the derivative of $\genfrac\{\}{0pt}{1}{\alpha}{2}$ at α is $\lfloor \alpha \rfloor$ and, more generally, it is m in the interval $(m, m+1)$ for every integer m. Therefore, in view of (23),

$$\genfrac\{\}{0pt}{}{\theta\alpha}{2} \geq \genfrac\{\}{0pt}{}{\alpha}{2} - \delta_1\alpha\lfloor\alpha\rfloor = \genfrac\{\}{0pt}{}{\alpha}{2}\left(1 - \delta_1\alpha\lfloor\alpha\rfloor\genfrac\{\}{0pt}{}{\alpha}{2}^{-1}\right). \tag{24}$$

However,

$$\begin{aligned} \delta_1\alpha\lfloor\alpha\rfloor\genfrac\{\}{0pt}{}{\alpha}{2}^{-1} &= \frac{\delta_1\alpha\lfloor\alpha\rfloor}{\lfloor\alpha\rfloor(\lfloor\alpha\rfloor - 1)/2 + \lfloor\alpha\rfloor(\alpha - \lfloor\alpha\rfloor)} = \frac{2\delta_1\alpha}{2\alpha - \lfloor\alpha\rfloor - 1} \\ &= \frac{2\delta_1\alpha}{\alpha + (\alpha - \lfloor\alpha\rfloor) - 1} \leq \frac{2\delta_1\alpha}{\alpha - 1} \leq \frac{2\delta_1\alpha}{\eta} \leq \frac{\varepsilon}{2}. \end{aligned} \tag{25}$$

Inequalities (24) and (25) therefore imply (20) in this case. The discussion above tells us that $\delta = \min\{\delta_0(\varepsilon), \delta_1(\eta, \varepsilon)\}$ will do in the statement of (i).

Let us now turn to (ii) of our fact, whose proof is very similar. Let η and ε be given. One can again check that if $\alpha \geq \alpha_1(\varepsilon)$ and $0 < \delta \leq \delta_2(\varepsilon)$, then (21) holds. We now suppose $\alpha < \alpha_1(\varepsilon)$. Let $\delta_3 = \delta_3(\eta, \varepsilon) = \eta\varepsilon/8\alpha_1(\varepsilon)$.

Recalling that $\left\{\begin{smallmatrix}\alpha\\2\end{smallmatrix}\right\}$ has derivative $\lfloor\alpha\rfloor$ in the interval $(\lfloor\alpha\rfloor, \lfloor\alpha\rfloor+1)$, we see that, for any $0 < \delta \le \delta_3$, we have

$$\left\{\begin{matrix}(1+\delta)\alpha\\2\end{matrix}\right\} \le \left\{\begin{matrix}\alpha\\2\end{matrix}\right\} + \delta_3\alpha(\lfloor\alpha\rfloor+1) = \left\{\begin{matrix}\alpha\\2\end{matrix}\right\}\left(1+2\delta_3\alpha\lfloor\alpha\rfloor\left\{\begin{matrix}\alpha\\2\end{matrix}\right\}^{-1}\right). \tag{26}$$

Calculations very similar to those in (25) show that (26) implies (21) in this case. It now suffices to take $\delta = \min\{\delta_2(\varepsilon), \delta_3(\eta,\varepsilon)\}$ for the statement of (ii). $\square$

We are now ready to prove the lower bounds in Theorem 1.

Proof of the lower bounds in (4) *and* (5). Let $\eta > 0$, $k = k(n)$, $t = t(n)$ and $\alpha = \alpha(n)$ be as in the statement of Theorem 1. We first show that, for any $\varepsilon > 0$, if n is large enough, then

$$\mu(n,k,t,\alpha) \ge (2-\varepsilon)\left\{\begin{matrix}\alpha\\2\end{matrix}\right\}\binom{n-t}{k-t}\binom{n-k-t}{k-t}. \tag{27}$$

Since we suppose that $\alpha \ge 1+\eta$, Fact 10(i) tells us that there is $\delta = \delta(\eta,\varepsilon) > 0$ so that, for every $1-\delta \le \theta \le 1$, we have

$$\frac{2}{\theta}\left\{\begin{matrix}\theta\alpha\\2\end{matrix}\right\} \ge 2\left\{\begin{matrix}\theta\alpha\\2\end{matrix}\right\} \ge (2-\varepsilon)\left\{\begin{matrix}\alpha\\2\end{matrix}\right\}. \tag{28}$$

Since we suppose that (k,t) is an η-admissible pair, Lemma 6 tells us that almost complete (k,t)-systems exist. Let n_0 be so that, for every $n \ge n_0$, a (θ,k,t)-system exists for some $\theta \ge 1-\delta$.

It now suffices to notice that, because of (28), the right-hand side of (8) is at least as large of the right-hand of (27). We remark that the extra hypotheses on k and α specified in the statement of Theorem 1(ii) are not required in the derivation of (27). (They are required in the proof of the upper bound in (5).)

We now turn to the lower bound in (4). In this case, we suppose $k \ll n^{1/2}$ and wish to show that

$$\mu(n,k,t,\alpha) \gtrsim 2\left\{\begin{matrix}\alpha\\2\end{matrix}\right\}\binom{n-t}{k-t}^2. \tag{29}$$

It suffices to notice that (29) follows from (27), because $\binom{n-k-t}{k-t} \sim \binom{n-t}{k-t}$ if $k \ll n^{1/2}$. $\square$

2.2. **Proof of the upper bounds.** We now prove the upper bounds in (4) and (5). Let $\eta > 0$, $k = k(n)$, $t = t(n)$, $\alpha = \alpha(n)$ be as in the statement of Theorem 1.

Let us start with the following observation. Suppose $|\mathscr{F}| = \left\lfloor \alpha\binom{n-t}{k-t}\right\rfloor$ and $\alpha \to \infty$. Recalling Remark 2(a), we see that the trivial bound $|\mathscr{F}^{(2)}_{<t}| \le |\mathscr{F}|^2 \sim \alpha^2\binom{n-t}{k-t}^2$ implies the upper bound in (4). In what follows, we suppose that

$$\alpha \ll n/k \tag{30}$$

and prove that

$$\mu(n,k,t,\alpha) \lesssim 2\left\{\begin{matrix}\alpha\\2\end{matrix}\right\}\binom{n-t}{k-t}\binom{n-k-t}{k-t} \le 2\left\{\begin{matrix}\alpha\\2\end{matrix}\right\}\binom{n-t}{k-t}^2. \tag{31}$$

Clearly, this will complete the proof of the upper bound in (4) and will establish the upper bound in (5). Let us turn to the proof of (31), assuming (30).

Let $\varepsilon > 0$ be given. Recall that we assume that $\alpha \ge 1+\eta$. Fact 10(ii) tells us that there is $\delta > 0$ so that if $\beta = (1+\delta)\alpha$, then

$$\left\{\begin{matrix}\beta\\2\end{matrix}\right\} \le \left(1+\frac{\varepsilon}{2}\right)\left\{\begin{matrix}\alpha\\2\end{matrix}\right\}. \tag{32}$$

In what follows, whenever necessary, we tacitly assume that n is large enough for our inequalities to hold. We shall construct a family $\mathscr{F} \subset \binom{[n]}{k}$ with $|\mathscr{F}| \gtrsim \beta\binom{n-t}{k-t} \ge \alpha\binom{n-t}{k-t}$ such that

$$\begin{aligned}|\mathscr{F}^{(2)}_{<t}| \lesssim 2\left\{\begin{matrix}\beta\\2\end{matrix}\right\}\binom{n-t}{k-t}\binom{n-k-t}{k-t} &\le 2\left(1+\frac{\varepsilon}{2}\right)\left\{\begin{matrix}\alpha\\2\end{matrix}\right\}\binom{n-t}{k-t}\binom{n-k-t}{k-t}\\ &= (2+\varepsilon)\left\{\begin{matrix}\alpha\\2\end{matrix}\right\}\binom{n-t}{k-t}\binom{n-k-t}{k-t}.\end{aligned} \tag{33}$$

Since $\mathscr{F}$ will be constructed for arbitrary $\varepsilon > 0$, this will establish (31). Let us proceed with the construction of $\mathscr{F}$.

Given a set $T \subset [n]$ with $|T| = t$, let

$$\mathscr{I}(T) = \left\{F \in \binom{[n]}{k} : T \subset F\right\}. \tag{34}$$

Clearly, $|\mathscr{I}(T)| = \binom{n-t}{k-t}$ for any such T. Our $\mathscr{F}$ will be, roughly speaking, a union of certain $\mathscr{I}(T)$ for a collection of β sets T (the fact that β is not necessarily an integer will be dealt with in a certain natural way). For $1 \le b \le \lceil\beta\rceil$, let

$$T_b = [t-1] \cup \{b+t-1\}. \tag{35}$$

We may now define $\mathscr{F}$ in the case β is an integer.

Definition 11 ($\mathscr{F}$ for integer β)**.** Let $\mathscr{F}_b = \mathscr{I}(T_b)$ for all $1 \le b \le \beta$ and let $\mathscr{F} = \bigcup_{1\le b\le\beta}\mathscr{F}_b$.

We shall define $\mathscr{F}$ for non-integer β in a short while. The following claim will help us estimate the cardinality of $\mathscr{F}$ (both for integer β and non-integer β).

Claim 12. *We have*

$$\left|\bigcup_{1\le b\le\beta}\mathscr{I}(T_b)\right| \sim \lfloor\beta\rfloor\binom{n-t}{k-t}. \tag{36}$$

Furthermore, we have

$$\Big|\mathscr{I}(T_{\lceil\beta\rceil})\cap\bigcup_{1\le j<\lceil\beta\rceil}\mathscr{I}(T_j)\Big|\ll\binom{n-t}{k-t}. \tag{37}$$

Proof. By Bonferroni's inequalities, we have

$$\lfloor\beta\rfloor\binom{n-t}{k-t}-\binom{\lfloor\beta\rfloor}{2}\binom{n-t-1}{k-t-1}\le\Big|\bigcup_{1\le b\le\beta}\mathscr{I}(T_b)\Big|\le\lfloor\beta\rfloor\binom{n-t}{k-t}. \tag{38}$$

In view of (30), a quick calculation shows that (38) implies that (36) holds. To see that (37) holds, note that the left-hand side of (37) is at most $(\lceil\beta\rceil-1)\binom{n-t-1}{k-t-1}$, and hence (37) follows from (30) (recall that $\beta=(1+\delta)\alpha$). ☐

We need to introduce a piece of notation. Given $\mathscr{J}$ and $\mathscr{J}'\subset\binom{[n]}{k}$, let $(\mathscr{J},\mathscr{J}')_{<t}$ be the set of pairs $(F,F')\in\mathscr{J}\times\mathscr{J}'$ with $|F\cap F'|<t$. Note that, for any $1\le b<b'\le\lceil\beta\rceil$, if $F\in\mathscr{I}(T_b)$, then

$$(\{F\},\mathscr{I}(T_{b'}))\le\binom{n-t}{k-t}\binom{n-k-1}{k-t}. \tag{39}$$

Recalling that $t\ll k^{1/2}\le n^{1/4}$, we see that

$$\big|\big(\mathscr{I}(T_b),\mathscr{I}(T_{b'})\big)_{<t}\big|\le\binom{n-t}{k-t}\binom{n-k-1}{k-t}\sim\binom{n-t}{k-t}\binom{n-k-t}{k-t}. \tag{40}$$

To define $\mathscr{F}$ in the case of non-integer β, we need one further observation, which we state in the claim below.

Claim 13. *There is* $\mathscr{F}'\subset\mathscr{I}(T_{\lceil\beta\rceil})$ *such that*

$$|\mathscr{F}'|=\left\lfloor(\beta-\lfloor\beta\rfloor)\binom{n-t}{k-t}\right\rfloor \tag{41}$$

and

$$\left|\left(\bigcup_{1\le b\le\beta}\mathscr{I}(T_b),\mathscr{F}'\right)_{<t}\right|\lesssim\lfloor\beta\rfloor(\beta-\lfloor\beta\rfloor)\binom{n-t}{k-t}\binom{n-k-t}{k-t} \tag{42}$$

Proof. Let $\mathscr{J}=\bigcup_{1\le b\le\beta}\mathscr{I}(T_b)$ and $\mathscr{J}'=\mathscr{I}(T_{\lceil\beta\rceil})$. Then $|\mathscr{J}|\le\lfloor\beta\rfloor\binom{n-t}{k-t}$ and $|\mathscr{J}'|=\binom{n-t}{k-t}$. Consider a bipartite graph Γ with vertex classes $\mathscr{J}$ and $\mathscr{J}'$, with $\{F,F'\}$ an edge if and only if $|F\cap F'|<t$. In view of (39), applied with $b'=\lceil\beta\rceil$, a simple averaging argument shows that, for any integer $M\ge 0$, one may select $\mathscr{G}\subset\mathscr{J}'$ with $|\mathscr{G}|=M$ so that the number of edges in our graph Γ induced by $\mathscr{J}\cup\mathscr{G}$ is at most

$$|\mathscr{J}|\binom{n-k-1}{k-t}\frac{M}{|\mathscr{J}'|}. \tag{43}$$

The claim now follows by setting M to be the right-hand side of (41) and by observing that $\binom{n-k-1}{k-t} \sim \binom{n-k-t}{k-t}$, as $t \ll k^{1/2} \le n^{1/4}$. $\square$

We are ready to define $\mathscr{F}$ in the case in which β is not an integer.

Definition 14 ($\mathscr{F}$ for non-integer β)**.** Let $\mathscr{F}_b = \mathscr{I}(T_b)$ for all $1 \le b \le \beta$ and let $\mathscr{F}_{\lceil\beta\rceil}$ be a family $\mathscr{F}'$ as in Claim 13. Let $\mathscr{F} = \bigcup_{1\le b\le\lceil\beta\rceil} \mathscr{F}_b$.

We shall now prove that the system $\mathscr{F}$ defined in Definitions 11 and 14 will do. Note first that Claim 12 implies that $|\mathscr{F}| \sim \beta\binom{n-t}{k-t}$ holds. Let us now estimate $|\mathscr{F}^{(2)}_{<t}|$. If β is an integer, then, using (40), we see that

$$\begin{aligned}
|\mathscr{F}^{(2)}_{<t}| &\le 2 \sum_{1\le b<b'\le\beta} \left|\left(\mathscr{F}_b, \mathscr{F}_{b'}\right)_{<t}\right| = 2 \sum_{1\le b<b'\le\beta} \left|\left(\mathscr{I}(T_b), \mathscr{I}(T_{b'})\right)_{<t}\right| \\
&\lesssim 2 \sum_{1\le b<b'\le\beta} \binom{n-t}{k-t}\binom{n-k-t}{k-t} = 2\binom{\beta}{2}\binom{n-t}{k-t}\binom{n-k-t}{k-t} \\
&= 2\left\{\begin{matrix}\beta\\2\end{matrix}\right\}\binom{n-t}{k-t}\binom{n-k-t}{k-t}, \qquad (44)
\end{aligned}$$

establishing the first inequality in (33) in the case in which β is an integer. If β is not an integer, then

$$\begin{aligned}
|\mathscr{F}^{(2)}_{<t}| &\le 2 \sum_{1\le b<b'\le\beta} \left|\left(\mathscr{F}_b, \mathscr{F}_{b'}\right)_{<t}\right| + 2\left|\left(\bigcup_{1\le b\le\beta} \mathscr{F}_b, \mathscr{F}_{\lceil\beta\rceil}\right)_{<t}\right| \\
&= 2 \sum_{1\le b<b'\le\beta} \left|\left(\mathscr{I}(T_b), \mathscr{I}(T_{b'})\right)_{<t}\right| + 2\left|\left(\bigcup_{1\le b\le\beta} \mathscr{I}(T_b), \mathscr{F}_{\lceil\beta\rceil}\right)_{<t}\right|. \qquad (45)
\end{aligned}$$

By (40) and by the choice of $\mathscr{F}_{\lceil\beta\rceil}$, the right-hand side of (45) is

$$\begin{aligned}
&\lesssim 2 \sum_{1\le b<b'\le\beta} \binom{n-t}{k-t}\binom{n-k-t}{k-t} + 2\lfloor\beta\rfloor(\beta-\lfloor\beta\rfloor)\binom{n-t}{k-t}\binom{n-k-t}{k-t} \\
&\le 2\binom{\lfloor\beta\rfloor}{2}\binom{n-t}{k-t}\binom{n-k-t}{k-t} + 2\lfloor\beta\rfloor(\beta-\lfloor\beta\rfloor)\binom{n-t}{k-t}\binom{n-k-t}{k-t} \\
&= 2\left\{\begin{matrix}\beta\\2\end{matrix}\right\}\binom{n-t}{k-t}\binom{n-k-t}{k-t}, \qquad (46)
\end{aligned}$$

again establishing the first inequality in (33). Thus $\mathscr{F}$ is as required. This concludes the proof of (31).

3. Concluding remarks

In this note, for simplicity, we have restricted ourselves to the case in which α is bounded away from 1. Our method does give results for $\alpha \to 1$, but we do not elaborate on those here. We have also omitted some results that one may obtain via standard eigenvalue methods in the case in which $t = 1$.

Let us now focus on the case $t > 1$. Our main tool, namely, Lemma 8, requires the existence of 'good' (k, t)-systems. As it turns out, this rules out the case $k \gg n^{1/2}$ (see [9]). We believe that it would be interesting to investigate the behaviour of $\mu(n, k, t, \alpha)$ for $t > 1$ and $k \gg n^{1/2}$. For instance, how much is $\mu(n, k, 2, \alpha)$ for, say, $k \sim n^{2/3}$ and $\alpha \sim \frac{1}{2}\binom{n}{k}\binom{n-2}{k-2}^{-1}$?

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