A structural result for 3-graphs

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Abstract. Suppose that $\mathcal{F} \subset \binom{[n]}{3}$ contains no three sets whose intersection is empty and their union has size at most 6. We prove a structure theorem for such families which easily implies the best possible bound, $|\mathcal{F}| \leq \binom{n-1}{2}$.

Let $[n] = \{1, 2, \ldots, n\}$. For an integer k, $0 \le k \le n$ let $\binom{[n]}{k}$ denote the collection of all k-element subsets of $[n]$. A family $\mathcal{F} \subset \binom{[n]}{k}$ is called a k-**graph**.

Definition 1. *The k-element sets A, B, C are called a **Katona-triple** if both $A \cap B \cap C = \emptyset$ and $|A \cup B \cup C| \le 2k$ hold.*

Let $m(n,k)$ denote the maximum of $|\mathcal{F}|$ for $\mathcal{F} \subset \binom{[n]}{k}$ over all $\mathcal{F}$ without a Katona-triple.

For $n < \frac{3}{2}k$, $A \cap B \cap C \neq \emptyset$ for all choices of A, B, $C \in \binom{[n]}{k}$. Consequently, $m(n,k) = \binom{n}{k}$ holds.

For $\frac{3}{2}k \le n \le 2k$ the second condition, $|A \cup B \cup C| \le 2k$ is satisfied automatically. Thus $m(n,k) = \binom{n-1}{k-1}$ follows from the following.

Theorem 1 (Frankl [Fra76], 1976)**.** *Let $r \ge 3$ be an integer, $n \ge \frac{r}{r-1}k$ and suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is r-wise intersecting. That is, $F_1 \cap \ldots F_r \neq \emptyset$ for all $F_1, \ldots, F_r \in \mathcal{F}$. Then*

$$|\mathcal{F}| \le \binom{n-1}{k-1}. \tag{1}$$

Moreover, in case of equality, for some fixed element $x \in [n]$ one has $\mathcal{F} = \left\{F \in \binom{[n]}{k} \,:\, x \in F\right\}$.

Remark 1. *The statement about uniqueness was not stated in [Fra76] but it is proved, e. g., in [Fra87].*

What happens for $n > 2k$? This was a problem asked by Katona and answered by Frankl and Füredi [FF83] who proved the following result.

Theorem 2 ([FF83], 1983)**.** *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$, $\mathcal{F}$ contains no Katona-triple and $n > k^2 + 3k$. Then (1) holds and the only optimal family is the star.*

Frankl and Füredi conjectured that the same holds true for all $n > 2k$ as well. They proved it in [FF83] for $k = 3$ and claim it for $k = 4, 5$ (without proof). Mubayi [Mub06] proved this conjecture for all k and $n > 2k$ via an entirely different proof. For four and more sets cf. [Mub07] and [MR09].

Our aim is to use a different approach and derive (1) in the first non-trivial case, $k = 3$ from a structure theorem.

Definition 2. *Let $\mathcal{F} \subset 2^{[n]}$ be a family, $F \in \mathcal{F}$. The subset $G \subset F$ is called* ***unique*** *if there is no different $F' \in \mathcal{F}$ with $G \subset F'$.*

Theorem 3 (Bollobás [Bol65], 1963)**.** *Suppose that for every member H of the family $\mathcal{H} \subset 2^{[n]}$, $G(H) \subset H$ is a unique subset. Then*

$$\sum_{H\in\mathcal{H}} \frac{1}{\binom{n-|H-G(H)|}{|G(H)|}} \leq 1 \tag{2}$$

holds.

Since for an antichain $\mathcal{H}$ and for all $H \in \mathcal{H}$ the choice $G(H) = H$ provides a unique subset, (2) generalizes the famous LYM-inequality.

Corollary 1. *Suppose that $\mathcal{H} \subset \binom{[n]}{k}$ and every $H \in \mathcal{H}$ has a unique $(k-1)$-subset $G(H) \subset H$. Then*

$$|\mathcal{H}| \leq \binom{n-1}{k-1} \tag{3}$$

holds.

Indeed, for such H, $G(H)$ each term in (2) is exactly $\frac{1}{\binom{n-1}{k-1}}$. □

One can show that equality is possible only for the full star of a fixed vertex. Let us mention that (3) can be proved without using (2). One proof is using linear independence. Another one is using a weight function $w(H, G)$ for all pairs $G \subset H \in \mathcal{H}$, $|G| = k-1$. Assigning weights 1 to (H, G) for $G = G(H)$ and $\frac{1}{n-k+1}$ for $G \neq G(H)$ assures that for $G \in \binom{[n]}{k-1}$

$$\sum_{H\in\mathcal{H}} w(H, G) \leq 1$$

and (3) follows by simple calculation.

Let us state our main result.

Theorem 4. *Suppose that* $\mathcal{F} \subset \binom{[n]}{3}$ *contains no Katona-triple. Then* $\mathcal{F}$ *can be partitioned into two families* $\mathcal{H}$ *and* $\mathcal{B}$ *and the ground set* $[n]$ *into two disjoint subsets* Y *and* Z *such that*

- $\mathcal{H} \subset \binom{Y}{3}$ *and every* $H \in \mathcal{H}$ *contains a unique* 2*-element set,*
- $\mathcal{B} \subset \binom{Z}{3}$ *and* $\mathcal{B}$ *is the vertex-disjoint union of* $\frac{|Z|}{4}$ *complete* 3*-graphs on* 4 *vertices.*

Proof of the theorem:

Let us define

$$\mathcal{H} = \left\{ H \in \mathcal{F} \,:\, \exists G = G(H) \in \binom{[n]}{2} \text{ such that } G \text{ is unique} \right\}.$$

Set $\mathcal{B} = \mathcal{F} - \mathcal{H}$. Note that for all $B \in \mathcal{B}$ and every $b \in B$, there exists $F = F(B, b)$, $F \neq B$ such that $(B - \{b\})$ is contained in F. A priori there might be several choices for such an F. However we prove the following.

Lemma 1. *For every* $B \in \mathcal{B}$ *there exists an element* $c \in ([n] - B)$ *such that*

$$F(B, b) = (B - \{b\}) \cup \{c\}$$

holds for each $b \in B$.

Proof: Let $B = \{b_1, b_2, b_3\}$ and let c_1, c_2, c_3 be such that $F(B, b_i) = (B - b_i) \cup \{c_i\}$ holds. Note that

$$F(B, b_1) \cup F(B, b_2) \cup F(B, b_3) = \{b_1, b_2, b_3\} \cup \{c_1, c_2, c_3\},$$

i. e., it consist of at most six elements. Since it is not a Katona-triple, the intersection of these three sets is non-empty. Let c denote the common element. Then $c = c_i$, $i = 1, 2, 3$ and the uniqueness of the choice of c_i follows. $\square$

By the lemma every $B \in \mathcal{B}$ gives rise to a complete 3-graph on four vertices (with vertex set $B \cup \{c\}$). Since every 2-subset of a complete 3-graph on four vertices is contained in two edges, $(B - \{b_i\}) \cup \{c\}$ is in $\mathcal{B}$ for all $i = 1, 2, 3$.

The following lemma was essentially proved already in [FF83].

Lemma 2. *If F, $F' \in \mathcal{F}$ satisfy $F \cap F' = \{y\}$ for some $y \in [n]$ then $(F' - \{y\})$ is a unique subset.*

Proof: Suppose the contrary and let $F'' \in \mathcal{F}$ satisfy $F' \neq F''$ and $(F' - \{y\}) \subset F''$. Then $y \notin F''$ implies that $F \cap F' \cap F'' = \emptyset$. Since $F \cup F' \cup F'' = F \cup F' \cup (F'' - F')$, the size of the union is at most six. That is F, F', F'' form a Katona-triple, a contradiction. □

Let $D \in \binom{[n]}{4}$ satisfy that $\binom{D}{3} \subset \mathcal{F}$. Let us prove the following.

Proposition 1. *For every $F \in \mathcal{F}$ either $F \subset D$ or $F \cap D = \emptyset$ holds.*

Proof: Suppose $F \not\subset D$. If $|F \cap D| = 1$ or 2 then we have exactly two choices for $B \in \binom{D}{3}$ satisfying $|F \cap B| = 1$. Setting $F' = B$ and using $B \in \mathcal{B}$ this contradicts Lemma 2. The only remaining possibility is $F \cap D = \emptyset$. □

To conclude the proof of the theorem just let Z be the union of all $D \in \binom{[n]}{4}$ with $\binom{D}{3} \subset \mathcal{F}$, and set $Y = ([n] - Z)$. □□

In view of the Bollobás Theorem,

$$|\mathcal{F}| \leq \binom{|Y| - 1}{2} + 4 \cdot \left\lfloor \frac{|Z|}{4} \right\rfloor \leq \binom{|Y| - 1}{2} + n - |Y|.$$

For $n \geq 5$ the right hand side is maximized for $|Y| = n$ and its maximal value is $\binom{n-1}{2}$.

It would be interesting to find a structure theorem for k-graphs with $k \geq 4$ without Katona-triples that implies the $\binom{n-1}{k-1}$ upper bound, i. e., $m(n, k) = \binom{n-1}{k-1}$.

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