MULTIPLY UNION FAMILIES IN $\mathbb{N}^n$

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Abstract. Let $A \subset \mathbb{N}^n$ be an r-wise s-union family, that is, a family of sequences with n components of non-negative integers such that for any r sequences in A the total sum of the maximum of each component in those sequences is at most s. We determine the maximum size of A and its unique extremal configuration provided (i) n is sufficiently large for fixed r and s, or (ii) $n = r + 1$.

1. Introduction

Let $\mathbb{N} := \{0, 1, 2, \ldots\}$ denote the set of non-negative integers, and let $[n] := \{1, 2, \ldots, n\}$. Intersecting families in $2^{[n]}$ or $\{0, 1\}^n$ are one of the main objects in extremal set theory. The equivalent dual form of an intersecting family is a union family, which is the subject of this paper. In [5] Frankl and Tokushige proposed to consider such problems not only in $\{0, 1\}^n$ but also in $[q]^n$. They determined the maximum size of 2-wise s-union families (i) in $[q]^n$ for $n > n_0(q, s)$, and (ii) in $\mathbb{N}^3$ for all s (the definitions will be given shortly). In this paper we extend their results and determine the maximum size and structure of r-wise s-union families in $\mathbb{N}^n$ for the following two cases: (i) $n \geq n_0(r, s)$, and (ii) $n = r+1$. Much research has been done for the case of families in $\{0, 1\}^n$, and there are many challenging open problems. The interested reader is referred to [2, 3, 4, 8, 9].

For a vector $\mathbf{x} \in \mathbb{R}^n$, we write x_i or $(\mathbf{x})_i$ for the ith component, so $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Define the *weight* of $\mathbf{a} \in \mathbb{N}^n$ by

$$|\mathbf{a}| := \sum_{i=1}^{n} a_i.$$

For a finite number of vectors $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{z} \in \mathbb{N}^n$ define the join $\mathbf{a} \vee \mathbf{b} \vee \cdots \vee \mathbf{z}$ by

$$(\mathbf{a} \vee \mathbf{b} \vee \cdots \vee \mathbf{z})_i := \max\{a_i, b_i, \ldots, z_i\},$$

and we say that $A \subset \mathbb{N}^n$ is r-wise *s-union* if

$$|\mathbf{a}_1 \vee \mathbf{a}_2 \vee \cdots \vee \mathbf{a}_r| \leq s \text{ for all } \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r \in A.$$

In this paper we address the following problem.

Problem. *For given n, r and s, determine the maximum size $|A|$ of r-wise s-union families $A \subset \mathbb{N}^n$.*

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To describe candidates A that give the maximum size to the above problem, we need some more definitions. Let us introduce a partial order $\prec$ in $\mathbb{R}^n$. For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ we let $\mathbf{a} \prec \mathbf{b}$ iff $a_i \le b_i$ for all $1 \le i \le n$. Then we define a *down set* for $\mathbf{a} \in \mathbb{N}^n$ by

$$\mathcal{D}(\mathbf{a}) := \{\mathbf{c} \in \mathbb{N}^n : \mathbf{c} \prec \mathbf{a}\},$$

and for $A \subset \mathbb{N}^n$ let

$$\mathcal{D}(A) := \bigcup_{\mathbf{a} \in A} \mathcal{D}(\mathbf{a}).$$

We also introduce $\mathcal{S}(\mathbf{a}, d)$, which can be viewed as a part of sphere centered at $\mathbf{a} \in \mathbb{N}^n$ with radius $d \in \mathbb{N}$, defined by

$$\mathcal{S}(\mathbf{a}, d) := \{\mathbf{a} + \boldsymbol{\epsilon} \in \mathbb{N}^n : \boldsymbol{\epsilon} \in \mathbb{N}^n, |\boldsymbol{\epsilon}| = d\}.$$

We say that $\mathbf{a} \in \mathbb{N}^n$ is a *balanced partition*, if all a_i's are as close to each other as possible, more precisely, $|a_i - a_j| \le 1$ for all i, j. Let $\mathbf{1} := (1, 1, \dots, 1) \in \mathbb{N}^n$.

For $r, s, n, d \in \mathbb{N}$ with $0 \le d \le \lfloor \frac{s}{r} \rfloor$ and $\mathbf{a} \in \mathbb{N}^n$ with $|\mathbf{a}| = s - rd$ let us define a family K by

$$K = K(r, n, \mathbf{a}, d) := \bigcup_{i=0}^{\lfloor \frac{d}{u} \rfloor} \mathcal{D}(\mathcal{S}(\mathbf{a} + i\mathbf{1}, d - ui)), \tag{1}$$

where $u = n - r + 1$. This is the candidate family. Intuitively K is a union of balls, and the corresponding centers and radii are chosen so that K is r-wise s-union as we will see in Claim 3 in the next section.

Conjecture. *Let $r \ge 2$ and s be positive integers. If $A \subset \mathbb{N}^n$ is r-wise s-union, then*

$$|A| \le \max_{0 \le d \le \lfloor \frac{s}{r} \rfloor} |K(r, n, \mathbf{a}, d)|,$$

where $\mathbf{a} \in \mathbb{N}^n$ is a balanced partition with $|\mathbf{a}| = s - rd$. Moreover if equality holds, then $A = K(r, n, \mathbf{a}, d)$ for some $0 \le d \le \lfloor \frac{s}{r} \rfloor$.

We first verify the conjecture when n is sufficiently large for fixed r, s. Let $\mathbf{e}_i$ be the i-th standard base of $\mathbb{R}^n$, that is, $(\mathbf{e}_i)_j = \delta_{ij}$. Let $\tilde{\mathbf{e}}_0 = \mathbf{0}$, and $\tilde{\mathbf{e}}_i = \sum_{j=1}^{i} \mathbf{e}_j$ for $1 \le i \le n$, e.g., $\tilde{\mathbf{e}}_n = \mathbf{1}$.

Theorem 1. *Let $r \ge 2$ and s be fixed positive integers. Write $s = dr + p$ where d and p are non-negative integers with $0 \le p < r$. Then there exists an $n_0(r, s)$ such that if $n > n_0(r, s)$ and $A \subset \mathbb{N}^n$ is r-wise s-union, then*

$$|A| \le |\mathcal{D}(\mathcal{S}(\tilde{\mathbf{e}}_p, d))|.$$

Moreover if equality holds, then A is isomorphic to $\mathcal{D}(\mathcal{S}(\tilde{\mathbf{e}}_p, d)) = K(r, n, \tilde{\mathbf{e}}_p, d)$.

We mention that the case $A \subset \{0, 1\}^n$ of Conjecture is posed in [2] and partially solved in [2, 3], and the case $r = 2$ of Theorem 1 is proved in [5] in a slightly stronger form. We also notice that if $A \subset \{0, 1\}^n$ is 2-wise $(2d + p)$-union, then the Katona's t-intersection theorem [7] states that $|A| \le |\mathcal{D}(\mathcal{S}(\tilde{\mathbf{e}}_p, d) \cap \{0, 1\}^n)|$ for all $n \ge s$.

Next we show that the conjecture is true if $n = r + 1$. We also verify the conjecture on general n if A satisfies some additional properties described below.

Let $A \subset \mathbb{N}^n$ be r-wise s-union. For $1 \le i \le n$ let

$$m_i := \max\{x_i : \mathbf{x} \in A\}. \tag{2}$$

If $n-r$ divides $|\mathbf{m}|-s$, then we define

$$d := \frac{|\mathbf{m}|-s}{n-r} \ge 0, \tag{3}$$

and for $1 \le i \le n$ let

$$a_i := m_i - d, \tag{4}$$

and we assume that $a_i \ge 0$. In this case we have $|\mathbf{a}| = s - rd$. Since $|\mathbf{a}| \ge 0$ it follows that $d \le \lfloor \frac{s}{r} \rfloor$. For $1 \le i \le n$ define $P_i \in \mathbb{N}^n$ by

$$P_i := \mathbf{a} + d\mathbf{e}_i, \tag{5}$$

where $\mathbf{e}_i$ denotes the ith standard base, for example, $P_2 = (a_1, a_2 + d, a_3, \dots, a_n)$.

Theorem 2. *Let $A \subset \mathbb{N}^n$ be r-wise s-union. Assume that the sequences P_i are well-defined and*

$$\{P_1, \dots, P_n\} \subset A. \tag{6}$$

Then it follows that

$$|A| \le \max_{0 \le d' \le \lfloor \frac{s}{r} \rfloor} |K(r, n, \mathbf{a}', d')|,$$

where $\mathbf{a}' \in \mathbb{N}^n$ is a balanced partition with $|\mathbf{a}'| = s - rd'$. Moreover if equality holds, then $A = K(r, n, \mathbf{a}', d')$ for some $0 \le d' \le \lfloor \frac{s}{r} \rfloor$.

We will show that the assumption (6) is satisfied when $n = r+1$, see Corollary 3 in the last section.

Notation: For $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$ we define $\mathbf{a} \setminus \mathbf{b} \in \mathbb{N}^n$ by $(\mathbf{a} \vee \mathbf{b}) - \mathbf{b}$, in other words, $(\mathbf{a} \setminus \mathbf{b})_i := \max\{a_i - b_i, 0\}$. The support of $\mathbf{a}$ is defined by $\operatorname{supp}(\mathbf{a}) := \{j : a_j > 0\}$.

2. Proof of Theorem 1 — the case when n is large

Let r, s be given, and let $s = dr + p$, $0 \le p < r$. We consider the situation $n \to \infty$ for fixed r, s, d, and p.

Claim 1. $|\mathcal{D}(\mathcal{S}(\tilde{\mathbf{e}}_p, d))| = \sum_{j=0}^{p} \binom{p}{j}\binom{n-j+d}{d} = (2^p/d!)n^d + O(n^{d-1})$.

Proof. By definition we have

$$\mathcal{D}(\mathcal{S}(\tilde{\mathbf{e}}_p, d)) = \{\mathbf{x} + \mathbf{y} \in \mathbb{N}^n : |\mathbf{x}| \le d, \, \mathbf{y} \prec \tilde{\mathbf{e}}_p\}.$$

We rewrite the RHS by classifying vectors according to their supports. For $I \subset [p]$ let $\tilde{\mathbf{e}}_p|_I$ be the restriction of $\tilde{\mathbf{e}}_p$ to I, that is, $(\tilde{\mathbf{e}}_p|_I)_i$ is 1 if $i \in I$ and 0 otherwise, and let

$$R(I) := \{\tilde{\mathbf{e}}_p|_I + \mathbf{z} : \operatorname{supp}(\mathbf{z}) \subset I \sqcup ([n] \setminus [p]), \, |\mathbf{z}| \le d\}.$$

Then we have $\mathcal{D}(\mathcal{S}(\tilde{\mathbf{e}}_p, d)) = \bigsqcup_{I \subset [p]} R(I)$. For each $I \in \binom{[p]}{i}$ the number of $\mathbf{z}$ in $R(I)$ equals the number of nonnegative integer solutions of $z_1 + z_2 + \cdots + z_{i+(n-p)} \le d$. Thus it follows that $|R(I)| = \binom{n-(p-i)+d}{d}$, and

$$|\mathcal{D}(\mathcal{S}(\tilde{\mathbf{e}}_p, d))| = \sum_{i=0}^{p} \binom{p}{i}\binom{n-(p-i)+d}{d} = \sum_{j=0}^{p} \binom{p}{j}\binom{n-j+d}{d}.$$

The RHS is further rewritten using $\binom{n-j+d}{d} = n^d/d! + O(n^{d-1})$ and $\sum_{j=0}^{p}\binom{p}{j} = 2^p$, as needed. $\square$

Let $A \subset \mathbb{N}^n$ be r-wise s-union with maximal size. So A is a down set. We will show that $|A| \le |\mathcal{D}(\mathcal{S}(\tilde{\mathbf{e}}_p, d))|$.

First suppose that there is a t with $2 \le t \le r$ such that A is t-wise $(dt+p)$-union, but not $(t-1)$-wise $(d(t-1)+p)$-union. In this case, by the latter condition, there are $\mathbf{b}_1, \dots, \mathbf{b}_{t-1} \in A$ such that $|\mathbf{b}| \ge d(t-1)+p+1$, where $\mathbf{b} = \mathbf{b}_1 \vee \cdots \vee \mathbf{b}_{t-1}$. Then, by the former condition, for every $\mathbf{a} \in A$ it follows that $|\mathbf{a} \vee \mathbf{b}| \le dt+p$, so $|\mathbf{a} \setminus \mathbf{b}| \le d-1$. This gives us

$$A \subset \{\mathbf{x}+\mathbf{y} \in \mathbb{N}^n : |\mathbf{x}| \le d-1,\, \mathbf{y} \prec \mathbf{b}\}.$$

There are $\binom{n+(d-1)}{d-1}$ choices for $\mathbf{x}$ satisfying $|\mathbf{x}| \le d-1$. On the other hand, the number of $\mathbf{y}$ with $\mathbf{y} \prec \mathbf{b}$ is independent of n (so it is a constant depending on r and s only). In fact $|\mathbf{b}| \le (t-1)s < rs$, and there are less than 2^{rs} choices for $\mathbf{y}$. Thus we get $|A| < \binom{n+(d-1)}{d-1}2^{rs} = O(n^{d-1})$ and we are done.

Next we suppose that

$$A \text{ is } t\text{-wise } (dt+p)\text{-union for all } 1 \le t \le r. \tag{7}$$

The case $t=1$ gives us $|\mathbf{a}| \le d+p$ for every $\mathbf{a} \in A$. If $p=0$, then this means that $A \subset \mathcal{D}(\mathcal{S}(\mathbf{0}, d))$, which finishes the proof for this case. So, from now on, we assume that $1 \le p < r$. We will see that there is a u with $u \ge 1$ such that there exist $\mathbf{b}_1, \dots, \mathbf{b}_u \in A$ satisfying

$$|\mathbf{b}| = u(d+1), \tag{8}$$

where $\mathbf{b} := \mathbf{b}_1 \vee \cdots \vee \mathbf{b}_u$. In fact we have (8) for $u=1$, if otherwise $A \subset \mathcal{D}(\mathcal{S}(\mathbf{0}, d))$. On the other hand, setting $t = p+1 \le r$ in (7), we see that A is $(p+1)$-wise $((p+1)(d+1)-1)$-union, and (8) fails if $u = p+1$. So we choose maximal u with $1 \le u \le p$ satisfying (8), and fix $\mathbf{b} = \mathbf{b}_1 \vee \cdots \vee \mathbf{b}_u$. By this maximality, for every $\mathbf{a} \in A$, it follows that $|\mathbf{a} \vee \mathbf{b}| \le (u+1)(d+1)-1$, and

$$|\mathbf{a} \setminus \mathbf{b}| = |\mathbf{a} \vee \mathbf{b}| - |\mathbf{b}| \le d. \tag{9}$$

Using (9) we have $A \subset \bigcup_{i=0}^{d} A_i$, where

$$A_i := \{\mathbf{x}+\mathbf{y} \in A : |\mathbf{x}| = i,\, \mathbf{y} \prec \mathbf{b}\}.$$

Then we have $|A_i| \le \binom{n+i}{i}2^{|\mathbf{b}|}$. Noting that $|\mathbf{b}| \le u(d+1) < r(d+1) = O(1)$ it follows $\sum_{i=0}^{d-1}|A_i| = O(n^{d-1})$. So the size of A_d is essential.

We naturally identify $\mathbf{a} \in A$ with a subset of $[n] \times \{1, \dots, d+p\}$. Formally let

$$\phi(\mathbf{a}) := \{(i,j) : 1 \le i \le n,\, 1 \le j \le a_i\},$$

for example, if $\mathbf{a} = (1,0,2)$, then $\phi(\mathbf{a}) = \{(1,1),(3,1),(3,2)\}$. Define $m = m(d)$ to be $r+1$ if $d=1$ and dr if $d \ge 2$. We say that $\mathbf{b}' \prec \mathbf{b}$ is *rich* if there exist m vectors $\mathbf{c}_1, \dots, \mathbf{c}_m$ of weight d such that $\mathbf{b}' \vee \mathbf{c}_j \in A$ for every j, and the $m+1$ subsets $\phi(\mathbf{c}_1), \dots, \phi(\mathbf{c}_m), \phi(\mathbf{b})$ are pairwise disjoint. In this case $\mathbf{b}'' \vee \mathbf{c}_j \in A$ for all $\mathbf{b}'' \prec \mathbf{b}'$ because A is a down set. This means that richness is hereditary, namely, if $\mathbf{b}'$ is rich and $\mathbf{b}'' \prec \mathbf{b}'$, then $\mathbf{b}''$ is rich as well. Informally, $\mathbf{b}'$ is rich if it can be extended to a $(|\mathbf{b}'|+d)$-element subset of A in m ways disjointly outside $\mathbf{b}$. We are comparing our family A with the reference family $\mathcal{D}(\mathcal{S}(\tilde{\mathbf{e}}_p),d)$, and we define $\tilde{\mathbf{b}}$ which plays the role of $\tilde{\mathbf{e}}_p$ in our family, namely, let us define

$$\tilde{\mathbf{b}} := \bigvee\{\mathbf{b}' \prec \mathbf{b} : \mathbf{b}' \text{ is rich}\}.$$

Claim 2. $|\tilde{\mathbf{b}}| \le p$.

Proof. Suppose the contrary. Then $|\tilde{\mathbf{b}}| > p$ and we can find rich $\mathbf{b}'_1, \mathbf{b}'_2, \dots, \mathbf{b}'_{p+1}$ (with repetition if necessary) such that $|\mathbf{b}'_1 \vee \cdots \vee \mathbf{b}'_{p+1}| \ge p+1$. Since richness is hereditary we may assume that $|\mathbf{b}'_1 \vee \cdots \vee \mathbf{b}'_{p+1}| = p+1$. Let $\mathbf{c}^{(i)}_1, \dots, \mathbf{c}^{(i)}_m$ support the richness of $\mathbf{b}'_i$. By definition $\phi(\mathbf{c}^{(i)}_1), \dots, \phi(\mathbf{c}^{(i)}_m)$ and $\phi(\mathbf{b})$ are pairwise disjoint. Let $\mathbf{a}_1 := \mathbf{b}'_1 \vee \mathbf{c}^{(1)}_{j_1} \in A$, say, $j_1 = 1$. Then choose $\mathbf{a}_2 := \mathbf{b}'_2 \vee \mathbf{c}^{(2)}_{j_2}$ so that $\phi(\mathbf{c}^{(1)}_{j_1})$ and $\phi(\mathbf{c}^{(2)}_{j_2})$ are disjoint. If $i \le p$, then having $\mathbf{a}_1, \dots, \mathbf{a}_i$ chosen, we only used id elements as $\bigcup_{l=1}^{i} \phi(\mathbf{c}^{(l)}_{j_l})$, which intersect at most id of $\mathbf{c}^{(i+1)}_1, \dots, \mathbf{c}^{(i+1)}_m$. Then, since $id \le pd < rd \le m$, we still have some $\mathbf{c}^{(i+1)}_{j_{i+1}}$, which is disjoint from any already chosen vectors. So we can continue this procedure until we get $\mathbf{a}_{p+1} := \mathbf{b}'_{p+1} \vee \mathbf{c}^{(p+1)}_{j_{p+1}} \in A$ such that all $\phi(\mathbf{c}^{(1)}_{j_1}), \dots, \phi(\mathbf{c}^{(p+1)}_{j_{p+1}})$ and $\phi(\mathbf{b})$ are disjoint. However, these vectors yield that

$$\begin{aligned} |\mathbf{a}_1 \vee \cdots \vee \mathbf{a}_{p+1}| &= |\mathbf{b}'_1 \vee \cdots \vee \mathbf{b}'_{p+1}| + |\mathbf{c}^{(1)}_{j_1}| + \cdots + |\mathbf{c}^{(p+1)}_{j_{p+1}}| \\ &= (p+1) + (p+1)d = (p+1)(d+1), \end{aligned}$$

which contradicts (7) at $t = p+1$. □

If $\mathbf{y} \prec \mathbf{b}$ is not rich, then

$$\{\phi(\mathbf{x}) : \mathbf{x} + \mathbf{y} \in A_d, |\mathbf{x}| = d\}$$

is a family of d-element subsets on $(d+p)n$ vertices, which has no m pairwise disjoint subsets (so the matching number is $m-1$ or less). Thus, by the Erdős matching theorem [1], the size of this family is $O(n^{d-1})$. There are at most $2^{|\mathbf{b}|} = O(1)$ choices for non-rich $\mathbf{y} \prec \mathbf{b}$, and we can conclude that the number of vectors in A_d coming from non-rich $\mathbf{y}$ is $O(n^{d-1})$. Then the remaining vectors in A_d come from rich $\mathbf{y} \prec \tilde{\mathbf{b}}$, and the number of such vectors is at most $2^{|\tilde{\mathbf{b}}|}\binom{n+d}{d}$. Note also that $\sum_{i=0}^{d-1} |A_i| = O(n^{d-1})$. Consequently we get

$$|A| \le 2^{|\tilde{\mathbf{b}}|}\binom{n+d}{d} + O(n^{d-1}) = (2^{|\tilde{\mathbf{b}}|}/d!)\, n^d + O(n^{d-1}).$$

Recall that the reference family is of size $(2^p/d!)n^d + O(n^{d-1})$, and $|\tilde{\mathbf{b}}| \le p$ from Claim 2. So we only need to deal with the case when $|\tilde{\mathbf{b}}| = p$ and there are exactly 2^p rich sets. In other words, $\tilde{\mathbf{b}} = \tilde{\mathbf{e}}_p$ (by renaming coordinates if necessary) and every $\mathbf{b}' \prec \tilde{\mathbf{e}}_p$ is rich. We show that $A \subset \mathcal{D}(\mathcal{S}(\tilde{\mathbf{e}}_p, d))$. Suppose the contrary, then there is an $\mathbf{a} \in A$ such that $|\mathbf{a}'| \ge d+1$, where $\mathbf{a}' = \mathbf{a} \setminus \tilde{\mathbf{e}}_p$. Since A is a down set we may assume that $|\mathbf{a}'| = d+1$. Now $\tilde{\mathbf{e}}_p$ is rich and let $\mathbf{c}_1, \dots, \mathbf{c}_m$ be vectors assured by the richness. We remark that $m - (d+1) \ge r - 1$. In fact if $d = 1$ then

$m-(d+1)=r-1$, and if $d\geq 2$ then $m-(d+1)=(r-1)(d-1)+r-2\geq r-1$. So we may assume that $\phi(\mathbf{c}_1),\dots,\phi(\mathbf{c}_{r-1})$ are pairwise disjoint and disjoint to $\phi(\mathbf{a})$ as well. Let $\mathbf{a}_i:=\tilde{\mathbf{e}}_p\vee\mathbf{c}_i\in A$ for $1\leq i\leq r-1$. Then we get

$$\begin{aligned}|\mathbf{a}\vee\mathbf{a}_1\vee\cdots\vee\mathbf{a}_{r-1}|&=|\tilde{\mathbf{e}}_p\vee\mathbf{a}'|+|\mathbf{c}_1|+\cdots+|\mathbf{c}_{r-1}|\\&=(p+d+1)+(r-1)d=dr+p+1=s+1,\end{aligned}$$

which contradicts that A is r-wise s-union. This completes the proof of Theorem 1.

3. The polytope P and proof of Theorem 2

Let $\mathbf{a}=(a_1,\dots,a_n)\in\mathbb{N}^n$ with $|\mathbf{a}|=s-rd$ for some $d\in\mathbb{N}$. We introduce a convex polytope $\mathbf{P}\subset\mathbb{R}^n$, which will play a key role in our proof. This polytope is defined by the following $n+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n-r+1}$ inequalities:

$$x_i\geq 0 \qquad \text{if } 1\leq i\leq n, \tag{10}$$

$$\sum_{i\in I}x_i\leq\sum_{i\in I}a_i+d \qquad \text{if } 1\leq |I|\leq n-r+1,\ I\subset[n]. \tag{11}$$

Namely,

$$\mathbf{P}:=\{\mathbf{x}\in\mathbb{R}^n:\mathbf{x}\text{ satisfies (10) and (11)}\}.$$

Let L denote the integer lattice points in $\mathbf{P}$:

$$L=L(r,n,\mathbf{a},d):=\{\mathbf{x}\in\mathbb{N}^n:\mathbf{x}\in\mathbf{P}\}.$$

Lemma 1. *The two sets K (defined by* (1)*) and L are the same, and r-wise s-union.*

Proof. This lemma is a consequence of the following three claims.

Claim 3. *The set K is r-wise s-union.*

Proof. Let $\mathbf{x}_1,\mathbf{x}_2,\dots,\mathbf{x}_r\in K$. We show that $|\mathbf{x}_1\vee\mathbf{x}_2\vee\cdots\vee\mathbf{x}_r|\leq s$. We may assume that $\mathbf{x}_j\in\mathcal{S}(\mathbf{a}+i_j\mathbf{1},d-ui_j)$, where $u=n-r+1$. We may also assume that $i_1\geq i_2\geq\cdots\geq i_r$. Let $\mathbf{b}:=\mathbf{a}+i_1\mathbf{1}$. Then, informally, $|\mathbf{x}\setminus\mathbf{b}|:=|(\mathbf{x}\vee\mathbf{b})-\mathbf{b}|$ counts the excess of $\mathbf{x}$ above $\mathbf{b}$, more precisely, it is $\sum_{j\in[n]}\max\{0,x_j-b_j\}$. Thus we have

$$\begin{aligned}|\mathbf{x}_1\vee\mathbf{x}_2\vee\cdots\vee\mathbf{x}_r|&\leq|\mathbf{b}|+\sum_{j=1}^r|\mathbf{x}_j\setminus\mathbf{b}|\\&\leq|\mathbf{a}|+ni_1+\sum_{j=1}^r\big((d-ui_j)-(i_1-i_j)\big)\\&=|\mathbf{a}|+dr+(n-r)i_1-\sum_{j=1}^r(u-1)i_j\\&=s-(n-r)\sum_{j=2}^r i_j\leq s,\end{aligned}$$

as required. □

Claim 4. *$K\subset L$.*

Proof. Let $\mathbf{x} \in K$. We show that $\mathbf{x} \in L$, that is, $\mathbf{x}$ satisfies (10) and (11). Since (10) is clear by definition of K, we show that (11). To this end we may assume that $\mathbf{x} \in \mathcal{S}(\mathbf{a}+i\mathbf{1}, d-ui)$, where $u = n-r+1$ and $i \leq \lfloor \frac{d}{u} \rfloor$. Let $I \subset [n]$ with $1 \leq |I| \leq u$. Then $i|I| \leq ui$. Thus it follows

$$\sum_{j\in I} x_j \leq \sum_{j\in I} a_j + i|I| + (d-ui) \leq \sum_{j\in I} a_j + d,$$

which confirms (11). □

Claim 5. $K \supset L$.

Proof. Let $\mathbf{x} \in L$. We show that $\mathbf{x} \in K$, that is, there exists some i' such that $0 \leq i' \leq \lfloor \frac{d}{n-r+1} \rfloor$ and

$$|\mathbf{x} \setminus (\mathbf{a} + i'\mathbf{1})| \leq d - (n-r+1)i'.$$

We write $\mathbf{x}$ as

$$\mathbf{x} = (a_1 + i_1, a_2 + i_2, \ldots, a_n + i_n),$$

where we may assume that $d \geq i_1 \geq i_2 \geq \cdots \geq i_n$. We notice that some i_j can be negative. Since $\mathbf{x} \in L$ it follows from (11) (a part of the definition of L) that if $1 \leq |I| \leq n-r+1$ and $I \subset [n]$, then

$$\sum_{j\in I} i_j \leq d.$$

Let $J := \{j : x_j \geq a_j\}$ and we argue separately by the size of $|J|$.

If $|J| \leq n-r+1$, then we may choose $i' = 0$. In fact,

$$\begin{aligned}|\mathbf{x} \setminus \mathbf{a}| &= \max\{0, i_1\} + \max\{0, i_2\} + \cdots + \max\{0, i_{n-r+1}\} \\ &= \max\left\{\sum_{j\in I} i_j : I \subset [n-r+1]\right\} \leq d.\end{aligned}$$

If $|J| \geq n-r+2$, then we may choose $i' = i_{n-r+2}$. In fact, by letting $i' := i_{n-r+2}$, we have

$$\begin{aligned}|\mathbf{x} \setminus (\mathbf{a} + i'\mathbf{1})| &= (i_1 - i') + (i_2 - i') + \cdots + (i_{n-r+1} - i') \\ &\leq d - (n-r+1)i'.\end{aligned}$$

We need to check $0 \leq i' \leq \lfloor \frac{d}{n-r+1} \rfloor$. It follows from $|J| \geq n-r+2$ that $i' \geq 0$. Also $d \geq i_1 \geq i_2 \geq \cdots \geq i_{n-r+2}$ and $i_1 + i_2 + \cdots + i_{n-r+1} \leq d$ yield $i' \leq \lfloor \frac{d}{n-r+1} \rfloor$. □

This completes the proof of Lemma 1. □

Let

$$\sigma_k(\mathbf{a}) := \sum_{K\in\binom{[n]}{k}} \prod_{i\in K} a_i$$

be the kth elementary symmetric polynomial of $a_1, \ldots, a_n$.

Lemma 2. *The size of $K(r,n,\mathbf{a},d)$ is given by*

$$|K(r,n,\mathbf{a},d)| = \sum_{j=0}^{n}\binom{d+j}{j}\sigma_{n-j}(\mathbf{a})$$
$$+\sum_{i=1}^{\lfloor \frac{d}{u}\rfloor}\sum_{j=u+1}^{n}\left(\binom{d-ui+j}{j}-\binom{d-ui+u}{j}\right)\sigma_{n-j}(\mathbf{a}+i\mathbf{1}),$$

where $u=n-r+1$. Moreover, for fixed n,r,d and $|\mathbf{a}|$, this size is maximized if and only if $\mathbf{a}$ is a balanced partition.

Proof. For $J\subset[n]$ let $\mathbf{x}|_J$ be the restriction of $\mathbf{x}$ to J, that is, $(\mathbf{x}|_J)_i$ is x_i if $i\in J$ and 0 otherwise.

First we count the vectors in the base layer $\mathcal{D}(\mathcal{S}(\mathbf{a},d))$. To this end we partition this set into $\bigsqcup_{J\subset[n]}A_0(J)$, where

$$A_0(J)=\{\mathbf{a}|_J+\mathbf{e}+\mathbf{b} : \mathrm{supp}(\mathbf{e})\subset J,\ |\mathbf{e}|\leq d,\ \mathrm{supp}(\mathbf{b})\subset[n]\setminus J,\ b_i<a_i \text{ for } i\notin J\}.$$

The number of vectors $\mathbf{e}$ with the above property is equal to the number of non-negative integer solutions of the inequality $x_1+x_2+\cdots+x_{|J|}\leq d$, which is $\binom{d+|J|}{|J|}$. The number of vectors $\mathbf{b}$ is clearly $\prod_{l\in[n]\setminus J}a_l$. Thus we get

$$\sum_{J\in\binom{[n]}{j}}|A_0(J)|=\sum_{J\in\binom{[n]}{j}}\binom{d+|J|}{|J|}\prod_{l\in[n]\setminus J}a_l=\binom{d+j}{j}\sigma_{n-j}(\mathbf{a}),$$

and $|\mathcal{D}(\mathcal{S}(\mathbf{a},d))|=\sum_{j=0}^{n}\binom{d+j}{j}\sigma_{n-j}(\mathbf{a})$.

Next we count the vectors in the ith layer:

$$\mathcal{D}(\mathcal{S}(\mathbf{a}+i\mathbf{1},d-ui))\setminus\left(\bigcup_{j=0}^{i-1}\mathcal{D}(\mathcal{S}(\mathbf{a}+j\mathbf{1},d-uj))\right).$$

For this we partition the above set into $\bigsqcup_{J\subset[n]}A_i(J)$, where

$$A_i(J)=\{(\mathbf{a}+i\mathbf{1})|_J+\mathbf{e}+\mathbf{b} : \mathrm{supp}(\mathbf{e})\subset J,\ d-u(i-1)-|J|<|\mathbf{e}|\leq d-ui,$$
$$\mathrm{supp}(\mathbf{b})\subset[n]\setminus J,\ b_l<a_l+i \text{ for } l\notin J\}.$$

In this case we need $d-u(i-1)<|J|+|\mathbf{e}|$ because the vectors satisfying the opposite inequality are already counted in the lower layers $\bigcup_{j<i}A_j(J)$. We also notice that $d-u(i-1)-|J|<d-ui$ implies that $|J|>u$. So $A_i(J)=\emptyset$ for $|J|\leq u$. Now we count the number of vectors $\mathbf{e}$ in $A_i(J)$, or equivalently, the number of non-negative integer solutions of

$$d-u(i-1)-|J|<x_1+x_2+\cdots+x_{|J|}\leq d-ui.$$

This number is $\binom{d-ui+j}{j}-\binom{d-ui+u}{j}$, where $j=|J|$. On the other hand, the number of vectors $\mathbf{b}$ in $A_i(J)$ is $\prod_{l\in[n]\setminus J}(a_l+i)$. Consequently we get

$$\sum_{J\subset[n]}|A_i(J)|=\sum_{j=u+1}^{n}\left(\binom{d-ui+j}{j}-\binom{d-ui+u}{j}\right)\sigma_{n-j}(\mathbf{a}+i\mathbf{1}).$$

Summing this term over $1 \le i \le \lfloor \frac{d}{u} \rfloor$ we finally obtain the second term of the RHS of $|K|$ in the statement of this lemma. Then, for fixed $|\mathbf{a}|$, the size of K is maximized when $\sigma_{n-j}(\mathbf{a})$ and $\sigma_{n-j}(\mathbf{a}+i\mathbf{1})$ are maximized. By the property of symmetric polynomials, this happens if and only if $\mathbf{a}$ is a balanced partition, see e.g., Theorem 52 in section 2.22 of [6]. ☐

Proof of Theorem 2. Let $A \subset \mathbb{N}^n$ be an r-wise s-union with (6). For $I \subset [n]$ let

$$m_I := \max\left\{\sum_{i\in I} x_i : \mathbf{x} \in A\right\}.$$

Claim 6. *If $I \subset [n]$ and $1 \le |I| \le n-r+1$, then*

$$m_I = \sum_{i\in I} a_i + d.$$

Proof. Choose $j \in I$. By (6) we have $P_j \in A$ and

$$m_I \ge \sum_{i\in I} (P_j)_i = \sum_{i\in I} a_i + d. \tag{12}$$

We need to show that this inequality is actually an equality. Let $[n] = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_r$ be a partition of $[n]$. Then it follows that

$$s \ge m_{I_1} + m_{I_2} + \cdots + m_{I_r} \ge \sum_{i\in[n]} a_i + rd = s,$$

where the first inequality follows from the r-wise s-union property of A, and the second inequality follows from (12). Since the left-most and the right-most sides are the same s, we see that all inequalities are equalities. This means that (12) is equality, as needed. ☐

By this claim if $\mathbf{x} \in A$ and $1 \le |I| \le n-r+1$, then we have

$$\sum_{i\in I} x_i \le m_I = \sum_{i\in I} a_i + d.$$

This means that $A \subset L$. Finally the theorem follows from Lemmas 1 and 2. ☐

Corollary 3. *If $n = r+1$, then Conjecture is true.*

Proof. Let $n = r+1$ and let $A \subset \mathbb{N}^{r+1}$ be r-wise s-union with maximum size. Define $\mathbf{m}$ by (2). Since $n - r = 1$ we can define d by (3). Then define $\mathbf{a}$ by (4). We need to verify $a_i \ge 0$ for all i. To this end we may assume that $m_1 \ge m_2 \ge \cdots \ge m_{r+1}$. Then $a_i \ge a_{r+1} = m_{r+1} - d$, so it suffices to show $m_{r+1} \ge d$. Since A is r-wise s-union it follows that $m_1 + m_2 + \cdots + m_r \le s$. This together with the definition of d implies $d = |\mathbf{m}| - s \le m_{r+1}$, as needed. So we can properly define P_i by (5).

Next we check that $\mathbf{x} \in A$ satisfies (10) and (11). By definition we have $x_i \le m_i = a_i + d$, so we have (10). Since A is r-wise s-union, we have

$$(x_1 + x_2) + m_3 + \cdots + m_{r+1} \le s,$$

or equivalently,

$$(x_1 + x_2) + (a_3 + d) + \cdots + (a_{r+1} + d) \le s = |\mathbf{a}| + rd.$$

Rearranging we get $x_1 + x_2 \le a_1 + a_2 + d$, and we get the other cases similarly, so we obtain (11). Thus $A \subset L$ and the result follows from Lemmas 1 and 2. $\square$

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