INTERSECTION PROBLEMS IN THE q-ARY CUBE

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ABSTRACT. We propose new intersection problems in the q-ary n-dimensional hypercube. The answers to the problems include the Katona's t-intersection theorem and the Erdős–Ko–Rado theorem as special cases. We solve some of the basic cases of our problems, and for example we get an Erdős–Ko–Rado type result for t-intersecting k-uniform families of multisets with bounded repetitions. Another case is obtained by counting the number of lattice points in a polytope having an intersection property.

1. INTRODUCTION

1.1. The problem and conjecture.

Intersection problems in extremal set theory typically deal with a family of subsets in the n-element set, or equivalently, a family of n-dimensional binary sequences. Two of the most important results are perhaps the Katona's t-intersection theorem for non-uniform families [13], and the Erdős–Ko–Rado theorem [7, 8, 18, 2] for uniform families. In this paper we extend such problems by working in the space of n-dimensional q-ary sequences so that the above two results naturally appear as special cases in our new setting. We present conjectures concerning the extremal configurations of our problems, where a part of ball-like or sphere-like structures appears. We then solve some of the basic cases of our problems both in non-uniform and uniform settings.

Let $\mathbb{N}$ denote the set of nonnegative integers, and let $n, q, s \in \mathbb{N}$ with $s \le (q-1)n$. Let

$$X_q := \{0, 1, \ldots, q-1\}$$

be the q-ary base set, and we will consider problems in the n-dimensional q-ary cube X_q^n. We will sometimes drop q and write X for X_q if there is no confusion. For $\mathbf{a} \in \mathbb{R}^n$, let $a_i \in X$ denote the i-th entry of $\mathbf{a}$, that is, $\mathbf{a} = (a_1, \ldots, a_n)$. Define the *weight* of $\mathbf{a}$ by

$$|\mathbf{a}| := \sum_{i=1}^{n} a_i.$$

Let $k \in \mathbb{N}$ and let $X_q^{n,k}$ be the collection of weight k sequences in X_q^n, that is,

$$X_q^{n,k} := \{\mathbf{a} \in X_q^n : |\mathbf{a}| = k\},$$

which we refer to as the k-uniform part of X_q^n. We remark that $k > n$ is possible.

For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ define the *join* $\mathbf{a} \vee \mathbf{b}$ by

$$(\mathbf{a} \vee \mathbf{b})_i := \max\{a_i, b_i\},$$

and we say that $A \subset \mathbb{R}^n$ is *s-union* if

$$|\mathbf{a} \vee \mathbf{b}| \leq s \text{ for all } \mathbf{a}, \mathbf{b} \in A.$$

The *width* of $A \subset X^n$ is defined to be the maximum s such that A is s-union.

In this paper we address the following problems concerning the maximum size of s-union sets.

Problem 1. *Determine*

$$w_q^n(s) := \max\{|A| : A \subset X_q^n \text{ is } s\text{-union}\},$$
$$w_q^{n,k}(s) := \max\{|A| : A \subset X_q^{n,k} \text{ is } s\text{-union}\}.$$

It is easy to see that

$$w_q^{n,k}(s) = \begin{cases} |X_q^{n,k}| & \text{if } s \geq 2k, \\ 1 & \text{if } s = k, \\ 0 & \text{if } s < k. \end{cases}$$

So when we consider $w_q^{n,k}(s)$ we always assume that $k < s < 2k$.

To describe candidates A for the w functions in Problem 1, we need some more definitions. Let us introduce a partial order $\prec$ in $\mathbb{R}^n$. For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ we let $\mathbf{a} \prec \mathbf{b}$ iff $a_i \leq b_i$ for all $1 \leq i \leq n$. Then we define a *down set* for $\mathbf{a} \in X^n$ by

$$\mathcal{D}(\mathbf{a}) := \{\mathbf{c} \in X^n : \mathbf{c} \prec \mathbf{a}\},$$

and for $A \subset X^n$ let

$$\mathcal{D}(A) := \bigcup_{\mathbf{a} \in A} \mathcal{D}(\mathbf{a}).$$

We remark that if $A \subset X^n$ has width s, then $\mathcal{D}(A)$ has the same width. So if $A \subset X^n$ is an extremal configuration for the problem, then A is a down set, namely, $A = \mathcal{D}(B)$ for some $B \subset X^n$.

Conversely we define an *up set* for $A \subset X_q^n$ by

$$\mathcal{U}_q(A) := \{\mathbf{c} \in X_q^n : \mathbf{a} \prec \mathbf{c} \text{ for some } \mathbf{a} \in A\}.$$

We also need an important structure $\mathcal{S}_q(\mathbf{a}, d)$, which can be viewed as a sphere centered at $\mathbf{a}$ with radius d. Formally, for $\mathbf{a} \in X_q^n$ and $d \in \mathbb{N}$ with $|\mathbf{a}| + 2d \leq (q-1)n$, we define

$$\mathcal{S}_q(\mathbf{a}, d) = \mathcal{S}(\mathbf{a}, d) := \{\mathbf{a} + \boldsymbol{\epsilon} \in X_q^n : \boldsymbol{\epsilon} \in X_q^n,\ |\boldsymbol{\epsilon}| = d\}.$$

Notice that $\mathcal{S}(\mathbf{a}, d)$ is $(|\mathbf{a}| + d)$-uniform and has width $|\mathbf{a}| + 2d$.

For given s and n we say that $\mathbf{a} \in X^n$ is an equitable (s, n)*-partition*, or simply, *equitable partition*, if all a_i's are as close to s/n as possible, more precisely,

$$s = a_1 + a_2 + \cdots + a_n, \text{ and } |a_i - a_j| \leq 1 \text{ for all } i, j.$$

Let $\mathbf{1} := (1, 1, \ldots, 1) \in X^n$.

Before stating a construction of a large s-union set, let us begin with a small concrete example: what is $w_5^3(10)$? It is sometimes helpful to visualize a sequence $\mathbf{x} \in X_q^n$ by a picture of a $(q-1) \times n$ box with $|\mathbf{x}|$ dots from the bottom. For example, Figure 1 shows a picture corresponding to $\mathbf{x} = (4, 3, 3)$. Since $|\mathbf{x}| = 10$, $\mathcal{D}(\mathbf{x})$ is 10-

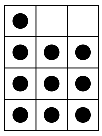

FIGURE 1. A picture for $\mathbf{x} = (4, 3, 3)$

union, and $|\mathcal{D}(\mathbf{x})| = 5 \cdot 4^2 = 80$. This shows that $w_5^3(10) \geq 80$. Can we do better than this? Actually we should start with the following 4 sequences (see Figure 2):

$$\mathbf{p}_1 = (4, 2, 2),\ \mathbf{p}_2 = (2, 4, 2),\ \mathbf{p}_3 = (2, 2, 4),\ \mathbf{q} = (3, 3, 3).$$

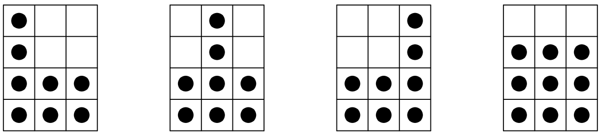

FIGURE 2. Pictures for $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ and $\mathbf{q}$

Then it is easy to see that $A := \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{q}\}$ is 10-union, and so is $\mathcal{D}(A)$. Let us count $|\mathcal{D}(A)|$. We have $|\mathcal{D}(\mathbf{q})| = 4^3 = 64$. A sequence in $D := \mathcal{D}(\mathbf{p}_1) \setminus \mathcal{D}(\mathbf{q})$ has a form of $(4, y, z)$ where $0 \leq y, z \leq 2$, and $|D| = 3^2 = 9$. By symmetry we get $|\mathcal{D}(A) \setminus \mathcal{D}(\mathbf{q})| = 3 \times 9 = 27$. Consequently we have $|\mathcal{D}(A)| = 64 + 27 = 91$. This yields $w_5^3(10) \geq 91$, and this is the best we can do as we will see in the next section. We also notice that, letting $\mathbf{a} = (2, 2, 2)$, it follows that $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} \subset \mathcal{S}_5(\mathbf{a}, 2)$ and $\mathbf{q} \in \mathcal{S}_5(\mathbf{a} + \mathbf{1}, 0)$. Thus the set of 91 sequences coincides with

$$\mathcal{D}(A) = \mathcal{D}(\mathcal{S}_5(\mathbf{a}, 2) \sqcup \mathcal{S}_5(\mathbf{a} + \mathbf{1}, 0)).$$

Figure 3 shows how these 91 integer lattice points corresponding to $\mathcal{D}(A)$ look like in $\mathbb{R}^3$, where the hidden corner is the origin. One may recognize 64 points for $\mathcal{D}(\mathbf{q})$ and 9 points for $\mathcal{D}(\mathbf{p}_1) \setminus \mathcal{D}(\mathbf{q})$, etc.

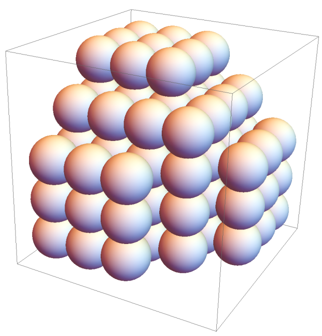

FIGURE 3. The 3D view of 91 sequences

We are ready to present an important construction of a large s-union set.

Example 1. *Let n, q, s be given. For an integer d with $0 \leq d \leq s/2$ choose an equitable partition $\mathbf{a} \in X_q^n$ of weight $s - 2d$. For $i \in \mathbb{N}$ with $d - (n-1)i \geq 0$ let*

$$U_i(d) := \mathcal{S}_q\big(\mathbf{a} + i\mathbf{1}, d - (n-1)i\big),$$

and let

$$A_q^n(d) := \mathcal{D}\Bigg(\bigcup_{i=0}^{\lfloor \frac{d}{n-1} \rfloor} U_i(d) \Bigg).$$

Then $A_q^n(d)$ is s-union.

Proof. Let $0 \leq i \leq j \leq \lfloor \frac{d}{n-1} \rfloor$, and let $\mathbf{b} \in U_i(d)$ and $\mathbf{c} \in U_j(d)$. Then we have

$$|\mathbf{c}| = |\mathbf{a} + j\mathbf{1}| + d - (n-1)j = |\mathbf{a}| + d + j,$$

and

$$|\mathbf{b} \setminus \mathbf{c}| := \sum_{1 \leq l \leq n} \max\{b_l - c_l, 0\} \leq d - (n-1)i - (j-i) = d - (n-2)i - j.$$

Thus it follows

$$|\mathbf{b} \vee \mathbf{c}| = |\mathbf{c}| + |\mathbf{b} \setminus \mathbf{c}| \leq |\mathbf{a}| + 2d - (n-2)i \leq |\mathbf{a}| + 2d = s.$$

This means that $A_q^n(d)$ is s-union. □

We mention that $A_q^n(d)$ has the following disjoint union decomposition, which we will show in the next section:

$$A_q^n(d) = \mathcal{D}(U_0(d)) \sqcup \Bigg(\bigsqcup_{i \geq 1} U_i(d) \Bigg).$$

In particular, noting that $U_0(d)$ is $(s-d)$-uniform, if $k \leq s - d$, then the k-uniform part of $A_q^n(d)$ is in $\mathcal{D}(U_0(d))$, namely $A_q^n(d) \cap X_q^{n,k} = \mathcal{D}(U_0(d)) \cap X_q^{n,k}$.

Now we state a general conjecture, which would give an answer to Problem 1.

Conjecture 1. *Let n, q, s be given, and let $A_q^n(d) \subset X_q^n$ be an s-union set defined in Example 1. Then it follows that*

$$w_q^n(s) = \max_{0 \leq d \leq s/2} |A_q^n(d)|.$$

If moreover $k < s < 2k$ then

$$w_q^{n,k}(s) = \max_{0 \leq d \leq s/2} |A_q^n(d) \cap X_q^{n,k}| = \max_{0 \leq d \leq s-k} |\mathcal{D}(U_0(d)) \cap X_q^{n,k}|.$$

It is sometimes convenient to consider the equivalent dual version of Problem 1. To this purpose, for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, define the *meet* $\mathbf{a} \wedge \mathbf{b} \in \mathbb{R}^n$ by

$$(\mathbf{a} \wedge \mathbf{b})_i := \min\{a_i, b_i\},$$

and we say that $A \subset \mathbb{R}^n$ is *t-intersecting* if

$$|\mathbf{a} \wedge \mathbf{b}| \geq t \text{ for all } a, b \in A.$$

Then we define

$$m_q^n(t) := \max\{|A| : A \subset X_q^n \text{ is } t\text{-intersecting}\},$$
$$m_q^{n,k}(t) := \max\{|A| : A \subset X_q^{n,k} \text{ is } t\text{-intersecting}\}.$$

We can relate functions w and m as we will see below. For $\mathbf{a} \in X_q^n$ define the complement $\bar{\mathbf{a}} \in X_q^n$ by

$$\bar{a}_i := (q-1) - a_i,$$

and for $A \subset X_q^n$ let $\bar{A} := \{\bar{\mathbf{a}} : \mathbf{a} \in A\}$. Clearly $|A| = |\bar{A}|$. Notice that

$$|\mathbf{a}| + |\bar{\mathbf{a}}| = (q-1)n$$

for every $\mathbf{a} \in X_q^n$, and $|\mathbf{a} \vee \mathbf{b}| \leq s$ is equivalent to $|\bar{\mathbf{a}} \wedge \bar{\mathbf{b}}| \geq (q-1)n - s$. (We may assume that $(q-1)n \geq s$ whenever we consider s-union family in X_q^n.) Thus $A \subset X_q^n$ is s-union iff $\bar{A}$ is $((q-1)n - s)$-intersecting, and

$$w_q^n(s) = m_q^n(t) \text{ where } t = (q-1)n - s. \tag{1}$$

On the other hand, if $\mathbf{a}, \mathbf{b} \in X_q^{n,k}$, then

$$2k = |\mathbf{a}| + |\mathbf{b}| = |\mathbf{a} \vee \mathbf{b}| + |\mathbf{a} \wedge \mathbf{b}|,$$

and $|\mathbf{a} \vee \mathbf{b}| \leq 2k - t$ is equivalent to $|\mathbf{a} \wedge \mathbf{b}| \geq t$ for $0 < t < k$. Thus, for $k < s < 2k$, $A \subset X_q^{n,k}$ is s-union iff it is $(2k - s)$-intersecting, namely,

$$w_q^{n,k}(s) = m_q^{n,k}(t) \text{ where } k < s < 2k \text{ and } t = 2k - s. \tag{2}$$

We need some more notation. For $\mathbf{a}, \mathbf{b} \in X^n$ let

$$(\mathbf{a} \setminus \mathbf{b})_i := \max\{a_i - b_i, 0\} = |\mathbf{a} \vee \mathbf{b}| - |\mathbf{b}|,$$

and the support of $\mathbf{a}$ be denoted by

$$\operatorname{supp}(\mathbf{a}) := \{i : a_i \neq 0\}.$$

Let us define $\mathbf{0}, \mathbf{e}_i, \tilde{\mathbf{e}}_t \in X^n$. Let $\mathbf{0} = (0, \ldots, 0)$ be the zero sequence, $\mathbf{e}_i$ be the i-th standard base, e.g., $\mathbf{e}_1 = (1, 0, \ldots, 0)$, and let $\tilde{\mathbf{e}}_t = (1, \ldots, 1, 0, \ldots, 0)$ be the basic sequence of weight t, that is,

$$\tilde{\mathbf{e}}_t := \mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_t.$$

1.2. **Easy cases, known results, and new results.** We list some easy cases and known results.

(i) $w_q^n(1) = |\mathcal{D}(\mathcal{S}_q(\mathbf{e}_1, 0))| = |\{\mathbf{0}, \mathbf{e}_1\}| = 2$.

(ii) $w_q^n(2) = |\mathcal{D}(\mathcal{S}_q(\mathbf{0}, 1))| = |\{\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_n\}| = n + 1$.

(iii) $w_q^1(s) = |\mathcal{D}(\mathcal{S}_q((s), 0))| = |\{(0), (1), \ldots, (s)\}| = s + 1$.

(iv) $w_q^2(2d) = |\mathcal{D}(\mathcal{S}_q(\mathbf{0}, d))| = |\{(i, j) : i, j \in X_{d+1}\}| = (d+1)^2$.

(v) $w_q^2(2d+1) = |\mathcal{D}(\mathcal{S}_q(\mathbf{e}_1, d))| = |\{(i, j) : i \in X_{d+2}, j \in X_{d+1}\}| = (d+2)(d+1)$.

(vi) $w_2^n(2d) = |\mathcal{D}(\mathcal{S}_2(\mathbf{0}, d))| = |\{\mathbf{a} \in X_2^n : |\mathbf{a}| \leq d\}| = \sum_{i=0}^{d} \binom{n}{i}$.

(vii) $w_2^n(2d+1) = |\mathcal{D}(\mathcal{S}_2(\mathbf{e}_1, d))|$
$= |\{\mathbf{a} \in X_2^n : |\mathbf{a}| \leq d \text{ or } (a_1 = 1 \text{ and } |\mathbf{a}| = d + 1)\}| = \sum_{i=0}^{d} \binom{n}{i} + \binom{n-1}{d}$.

(viii) $m_2^{n,k}(t)$ for $k > t \geq 1$, $n \geq 2k - t$ is determined by Ahlswede and Khachatrian, see Theorem 7.

(ix) $m_q^{n,k}(t)$ for $k > t \ge 1$, $n \ge 2k - t$, and $q \ge k - t + 2$ is determined by Füredi, Gerbner and Vizer, see Theorem 9.

We remark that the (vi) and (vii) are equivalent to the Katona's t-intersection theorem [13], which states that

$$m_2^n(t) = \begin{cases} \sum_{i=l}^n \binom{n}{i} & \text{if } n+t=2l, \\ \sum_{i=l}^n \binom{n}{i} + \binom{n-1}{l-1} & \text{if } n+t=2l-1. \end{cases}$$

Letting $s = n - t$ and $d = n - l$ we can rewrite the above formula using (2) as

$$w_2^n(s) = \begin{cases} \sum_{i=n-d}^n \binom{n}{i} = \sum_{i=0}^d \binom{n}{i} & \text{if } s=2d, \\ \sum_{i=n-d}^n \binom{n}{i} + \binom{n-1}{n-d-1} = \sum_{i=0}^d \binom{n}{i} + \binom{n-1}{d} & \text{if } s=2d+1. \end{cases}$$

In this paper we determine the functions $w_q^n(s)$ and $w_q^{n,k}(s)$ (or the dual functions $m_q^n(t)$ and $w_q^{n,k}(t)$) to verify Conjecture 1 in the following special cases:

(I) $w_q^n(s)$ for $n = 3$ and $q \ge q_0(s)$ in Theorem 1.
(II) $w_q^n(s)$ for $n > n_0(s, q)$ in Theorem 2.
(III) $w_q^{n,k}(s)$ for $k < s < 2k$ and $n > n_0(k, s, q)$ in Theorem 3.
(IV) $m_q^n(t)$ for $t = 1$ in Theorem 4.
(V) $m_q^{n,k}(t)$ for $t = 1$ and $n \ge \max\{2k - q + 2, k + 1\}$ in Theorem 5.
(VI) $m_q^{n,k}(t)$ for $k > t \ge 1$, $n \ge 2k - t$, and $q \ge k - t + 1$ in Theorem 6.

As in (iii) and (iv) it is easy to determine $w_q^n(s)$ for $n = 1, 2$, and they have simple formulas. So it is somewhat surprising that the case $n = 3$ is not so easy already, and the formula for $w_q^3(s)$ is rather involved. In Section 2 we discuss how to estimate $w_q^n(s)$ for $q > q_0(n, s)$, and we verify that $w_q^3(s)$ is given by Conjecture 1 as follows:

Theorem 1. *If $q - 1 \ge 4s/5$, then*

$$w_q^3(s) = \max\{|A_q^3(d)| : 0 \le d \le 2/s\},$$

and equality is attained only by $A_q^3(d)$ up to isomorphism (of renaming the coordinates), where $A_q^3(d)$ is defined in Example 1.

We actually prove in the next section that the same holds for general n provided additional assumptions are satisfied, see Proposition 1. (These assumptions hold automatically for $n = 3$ case.) For the proof we will find the maximal s-union polytope in $\mathbb{R}^n$, and then we count the number of integer lattice points contained in the polytope. This approach works not only to deal with pairwise s-union sets, but also 'r-wise' s-union sets, see [10].

If n is large enough compared with the other parameters, then the situation become rather simple, and we can verify Conjecture 1. For non-uniform case we show the following in Section 3.

Theorem 2. *Let s and q be fixed. If $n > n_0(s, q)$ then*

$$w_q^n(s) = \begin{cases} |\mathcal{D}(\mathcal{S}_q(\mathbf{0}, d))| & \text{if } s = 2d, \\ |\mathcal{D}(\mathcal{S}_q(\mathbf{e}_1, d))| & \text{if } s = 2d+1. \end{cases}$$

Moreover equality is attained only by $\mathcal{D}(\mathcal{S}_q(\mathbf{0}, d))$ *if* $s = 2d$ *and* $\mathcal{D}(\mathcal{S}_q(\mathbf{e}_1, d))$ *if* $s = 2d + 1$ *up to isomorphism.*

Similarly, for the k-uniform case, we show the following in Section 4 by proving the equivalent dual form Theorem 8.

Theorem 3. *Let* k, s *and* q *be fixed with* $k < s < 2k$*. If* $n \geq n_0(k, s, q)$ *then*

$$w_q^{n,k}(s) = |\mathcal{S}_q(\tilde{\mathbf{e}}_{2k-s}, s - k)|.$$

Moreover equality is attained only by $\mathcal{S}_q(\tilde{\mathbf{e}}_{2k-s}, s - k)$ *up to isomorphism.*

Both proofs of the above two results are based on the so-called kernel method introduced by Erdős, Ko, and Rado [7].

Another easy situation is 1-intersecting case, and we show the following two results – one for non-uniform case and the other for k-uniform case in Sections 3 and 4, respectively.

Theorem 4. *For* 1*-intersecting families, we have*

$$m_q^n(1) = \begin{cases} |\mathcal{U}_q(\mathcal{S}_2(\mathbf{0}, d+1))| & \text{if } n = 2d+1, \\ |\mathcal{U}_q(\mathcal{S}_2(\mathbf{0}, d+1) \cap X_2^{n-1})| & \text{if } n = 2d+2. \end{cases}$$

Theorem 5. *If* $n \geq \max\{2k - q + 2, k + 1\}$*, then*

$$m_q^{n,k}(1) = |\mathcal{S}_q(\mathbf{e}_1, k - 1)|.$$

As we mentioned in (viii) Ahlswede and Khachatrian [2] completely determined $m_2^n(t)$. Recently Füredi, Gerbner and Vizer [12] observed that $m_q^n(t)$ for the case $q \geq k - t + 2$ is represented by using $m_2^n(t)$. We slightly extend this result as follows, which will be proved in Section 4.

Theorem 6. *Let* $k > t \geq 1$*,* $n \geq 2k - t$*, and* $q \geq k - t + 1$*. Then*

$$m_q^{n,k}(t) = \max\{|\mathcal{D}(\mathcal{S}_q(\tilde{\mathbf{e}}_{t+2i}, k - t - i)) \cap X_q^{n,k}| : i = 0, 1, \ldots, (k - t)/2\}.$$

As in [12] one can view this result as an intersection result for multisets with bounded repetitions, or an intersection result for weighted subsets.

Finally we record the following simple fact on the number of nonnegative integer solutions, which will be used several times in the proofs, for a proof see e.g., p. 117 in [16].

Lemma 1. *Let* t *and* d *be positive integers. Then the number of nonnegative integer solutions of* $x_1 + x_2 + \cdots + x_t \leq d$ *is* $\binom{t+d}{d}$*, and the number of nonnegative integer solutions of* $x_1 + x_2 + \cdots + x_t = d$ *is* $\binom{t+d}{d} - \binom{t+d-1}{d-1} = \binom{t+d-1}{d}$*.*

2. s-UNION FAMILIES FOR $q > q_0(n, s)$

In this section we will determine $w_q^n(s)$ for the case $n \geq 3$ and $q > q_0(n, s)$ under two additional assumptions (see Proposition 1), and prove Theorem 1 with some more detailed information of $w_q^3(s)$ as a function of s.

2.1. **Counting lattice points in a polytope.** In this section let $n \in \mathbb{N}$ with $n \geq 3$. We recall that for $\mathbf{x} \in \mathbb{R}^n$ we write x_i for the i-th component, so

$$\mathbf{x} = (x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i \mathbf{e}_i,$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the standard basis of $\mathbb{R}^n$.

Let n, s, q be fixed positive integers with $q > q_0(n, s)$. We write X^n for X_q^n. Let $A \subset X^n$ be s-union with $|A| = w_q^n(s)$. For $1 \leq i \leq n$ let

$$m_i := \max\{x_i : \mathbf{x} \in A\}.$$

Let $\mathbf{m} = (m_1, \ldots, m_n) \in X^n$ and let $m := |\mathbf{m}|/n$ be the average.

If $nm < s$ then we can increase $|A|$ without violating s-union property. So we may assume that $nm \geq s$.

We will make three assumptions. The first one is the following.

Supposition 1. $n - 2$ *divides* $nm - s$.

We remark that the above supposition is automatically satisfied if $n = 3$. Let

$$d := \frac{nm - s}{n - 2} \in \mathbb{N},$$

which can be rewrite as $s - 2d = n(m - d)$. Then define $\mathbf{a} = (a_1, \ldots, a_n)$, which will play a role of a 'center' of A, by

$$a_i := m_i - d. \tag{3}$$

If $n = 3$, then $m_1 + m_2 + m_3 = 3m = d + s$, which implies $m_i \geq d$ for all i. In fact if $m_1 < d$ then $m_2 + m_3 > s$ and this contradicts the s-union property of A. Our second assumption is the following.

Supposition 2. $a_i \geq 0$ *for all* $i = 1, 2, \ldots, n$.

We have

$$|\mathbf{a}| = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} (m_i - d) = n(m - d) = s - 2d \geq 0,$$

and

$$s = |\mathbf{a}| + 2d = nm - (n - 2)d, \quad 2d \leq s \leq nm.$$

Then we define n integer lattice points $P_1, \ldots, P_n \in \mathcal{S}(\mathbf{a}, d)$ by

$$P_i := \mathbf{a} + d\mathbf{e}_i,$$

so, for example, $P_2 = (a_1, m_2, a_3, \ldots, a_n)$. These n points are crucial for the argument below.

Let $\mathbf{x} \in A$. Then, for each $i = 1, 2, \ldots, n$, we have

$$x_i \geq 0, \tag{4}$$

$$x_i \leq m_i, \tag{5}$$

$$\left(\sum_{j=1}^{n} x_j \right) - x_i \leq s - m_i, \tag{6}$$

where (5) follows from the definition of m_i, and (6) is the consequence of the s-union property of A. These $3n$ inequalities define a convex polytope $\mathbf{P}_0 \subset \mathbb{R}^n$ containing A, namely, $\mathbf{P}_0 := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \text{ satisfies (4),(5),(6)}\} \supset A$.

If $n = 3$, then the polyhedron $\mathbf{P}_0$ is also s-union, that is, if $\mathbf{x}, \mathbf{y} \in \mathbf{P}_0$, then $|\mathbf{x} \vee \mathbf{y}| \leq s$. (In fact we may assume that two of $\max\{x_1, y_1\}, \max\{x_2, y_2\}, \max\{x_2, y_2\}$ come from $\mathbf{x}$, say, $x_1 \geq y_1$, $x_2 \geq y_2$, and $|\mathbf{x} \vee \mathbf{y}| \leq x_1 + x_2 + m_3$. Then $|\mathbf{x} \vee \mathbf{y}| \leq s$ follows from (6).) So A is obtained by taking all integer lattice points in $\mathbf{P}_0$:

$$A = \{\mathbf{x} \in \mathbb{N}^3 : \mathbf{x} \in \mathbf{P}_0\}.$$

In particular, if $n = 3$, then $P_1, P_2, P_3 \in A$.

On the other hand, if $n \geq 4$, then the polytope $\mathbf{P}_0$ is not necessarily s-union in general. It has $P_1, \ldots, P_n$ as (a part of) vertices, for example, P_1 comes from (5) for $i = 1$ and (6) for $i = 2, \ldots, n$. Our last assumption is the following.

Supposition 3. *All $P_1, \ldots, P_n$ are in A.*

As we have already noticed, this supposition is satisfied when $n = 3$.

Claim 1. *$\{P_1, \ldots, P_n\}$ is $(s - d)$-uniform and has width s.*

Proof. Recall $\mathcal{S}(\mathbf{a}, d)$ is $(|\mathbf{a}| + d)$-uniform and has width $|\mathbf{a}| + 2d$. Then this claim follows from the fact that $|\mathbf{a}| = s - 2d$ and $P_i \in \mathcal{S}(\mathbf{a}, d)$. □

Another important vertex $Q \in \mathbb{R}^n$ (not necessarily in $\mathbb{N}^n$) of the polytope $\mathbf{P}_0$ is obtained by solving (6) for $i = 1, \ldots, n$, that is,

$$Q := \mathbf{m} - \frac{nm - s}{n - 1}\mathbf{1} = \mathbf{a} + \frac{d}{n - 1}\mathbf{1},$$

see Figure 4. Let $\mathbf{T}$ be an n-dimensional simplex spanned by the $n + 1$ vertices $P_1, \ldots, P_n$ and Q. From the above claim we see that $P_1, \ldots, P_n$ form an $(n - 1)$-dimensional *regular* simplex $\mathbf{F}$ in the hyperplane $x_1 + \cdots + x_n = s - d$. Moreover, the distance from Q to each P_i does not depend on i.

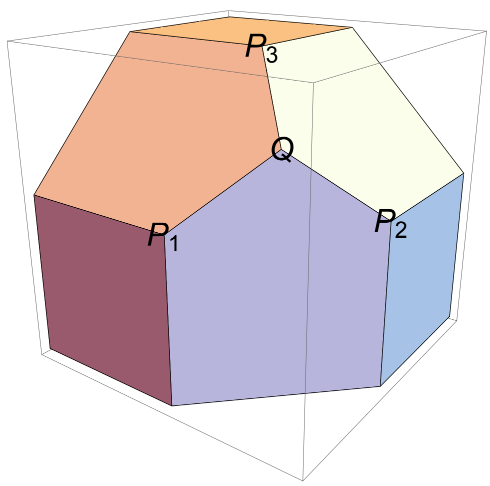

FIGURE 4. The polyhedron $\mathbf{P}_0$ for $n = 3$

We are going to construct an s-union convex polytope $\mathbf{P} \subset \mathbf{P}_0$ which defines A in the following way:

$$A = \{\mathbf{x} \in \mathbb{N}^n : \mathbf{x} \in \mathbf{P}\}.$$

Informally, $\mathbf{P}$ will be obtained as the union of $\mathbf{T}$ and the down set of $\mathbf{F}$.

For the defining inequalities of $\mathbf{P}$ we extend the definition of m_i. For an index set $I \subset [n]$ with $1 \leq |I| \leq n-1$, let

$$m_I := \max\left\{ \sum_{i \in I} x_i : \mathbf{x} \in A \right\},$$

so $m_i = m_{\{i\}}$. By definition we have $\sum_{i \in I} x_i \leq m_I$ for $\mathbf{x} \in A$. The next claim shows that all m_I's are actually completely determined by $m_1, \ldots, m_n$.

Claim 2. *For $I \subset [n]$, $1 \leq |I| \leq n-1$, it follows*

$$m_I = \sum_{i \in I} m_i - (|I| - 1)d.$$

Proof. Let $j \in I$. By Supposition 3 we have P_j in A, and this yields

$$m_I \geq m_j + \sum_{l \in I \setminus \{j\}} a_l = \sum_{i \in I} m_i - (|I| - 1)d. \tag{7}$$

Similarly we have

$$m_{[n]\setminus I} \geq \sum_{i \in [n]\setminus I} m_i - (n - |I| - 1)d.$$

Then it follows

$$s \geq m_I + m_{[n]\setminus I} \geq \sum_{i \in [i]} m_i - (n-2)d = nm - (n-2)d = s. \tag{8}$$

This means that all inequalities in (8) are equalities, and thus the inequality in (7) is also equality, which shows the claim. □

By the above claim and the definition of m_I we get the following inequalities.

Claim 3. *Let $I \subset [n]$, $1 \leq |I| \leq n-1$. If $\mathbf{x} \in A$, then*

$$\sum_{i \in I} x_i \leq \sum_{i \in I} m_i - (|I| - 1)d. \tag{9}$$

We notice that (5) and (6) are special cases of (9) corresponding to the cases $|I| = 1$ and $|I| = n-1$, respectively. Now we can define the convex polytope $\mathbf{P}$ formally by (4) and (9):

$$\mathbf{P} := \left\{ \mathbf{x} \in \mathbb{R}^n : \begin{array}{c} x_i \geq 0 \text{ for } i = 1, \ldots, n, \text{ and} \\ \sum_{i \in I} x_i \leq \sum_{i \in I} m_i - (|I| - 1)d \text{ for } \emptyset \neq I \subsetneq [n] \end{array} \right\}.$$

So $\mathbf{P}$ is defined by $n + (2^n - 2)$ inequalities. By the construction it follows

$$A \subset \mathbf{P}.$$

Claim 4. *The polytope $\mathbf{P}$ is s-union.*

Proof. Recall from (8) that $m_I + m_{[n]\setminus I} = s$ for all $\emptyset \neq I \subsetneq [n]$. If $\mathbf{x}, \mathbf{y} \in \mathbf{P}$ and letting $I = \{i : x_i \geq y_i\}$, then

$$|\mathbf{x} \vee \mathbf{y}| = \sum_{i \in I} x_i + \sum_{j \in [n]\setminus I} y_j \leq m_I + m_{[n]\setminus I} = s.$$

Also, noting that the width of $\mathbf{P}$ in the $\mathbf{1}$ direction is given by $\mathbf{0}$ and Q, we have

$$|\mathbf{x}| \leq |Q| = s - \frac{mn - s}{n-1} < s$$

for all $\mathbf{x} \in \mathbf{P}$. This completes the proof of the s-union property of $\mathbf{P}$. □

Since $A \subset \mathbf{P}$ and A is size maximal, we infer that A is obtained by taking all integer lattice points contained in $\mathbf{P}$. In other words, we have the following.

Claim 5. $A = \{\mathbf{x} \in \mathbb{N}^n : \mathbf{x} \in \mathbf{P}\}$.

In the rest of this subsection we shall show that A coincides with one of $A_q^n(d)$ in Example 1. For this let $U_i = \mathcal{S}_q\big(\mathbf{a} + i\mathbf{1}, d - (n-1)i\big)$ and define

$$K := \mathcal{D}\Bigg(\bigcup_{i=0}^{\lfloor \frac{d}{n-1} \rfloor} U_i\Bigg),$$

which is almost the same as $A_q^n(d)$ but here the $\mathbf{a}$ defined by (3) is not necessarily equitable. Then K is s-union. (Recall that in the proof that $A_q^n(d)$ is s-union in Example 1 we did not use the property that $\mathbf{a}$ is equitable.)

Claim 6. $A \subset K$.

Proof. Let $\mathbf{x} \in A$. To show that $\mathbf{x} \in K$ we need to find some i' with $0 \leq i' \leq \lfloor \frac{d}{n-1} \rfloor$ such that

$$|\mathbf{x} \setminus (\mathbf{a} + i'\mathbf{1})| \leq d - (n-1)i'. \tag{10}$$

We write $\mathbf{x}$ as $\mathbf{x} = (a_1 + i_1, a_2 + i_2, \dots, a_n + i_n)$, where we may assume that $i_1 \geq i_2 \geq \cdots \geq i_n$. For $J \subset [n]$ it follows from $a_j = m_j - d$ that

$$\sum_{j \in J} x_j = \sum_{j \in J} (a_j + i_j) = \sum_{j \in J} m_j - |J|d + \sum_{j \in J} i_j.$$

This together with (9) yields that if $1 \leq |J| \leq n-1$ then

$$\sum_{j \in J} i_j \leq d.$$

Now we verify (10). If $i_n \geq 0$ then we set $i' = i_n$ and $J = [n-1]$. Then,

$$|\mathbf{x} \setminus (\mathbf{a} + i'\mathbf{1})| = \sum_{j=1}^{n} (i_j - i_n) = \sum_{j \in J} i_j - (n-1)i_n \leq d - (n-1)i'.$$

If $i_n < 0$ then let $i' = 0$ and $J = \{j : i_j \geq 0\}$. It follows that

$$|\mathbf{x} \setminus \mathbf{a}| = \sum_{j=1}^{n} \max\{0, i_j\} = \sum_{j \in J} i_j.$$

The RHS is $\leq d$ if $J \neq \emptyset$. If $J = \emptyset$, then we have $\mathbf{x} \prec \mathbf{a}$ and $|\mathbf{x} \setminus \mathbf{a}| = 0$. So in both cases we have $|\mathbf{x} \setminus \mathbf{a}| \leq d$, and we are done. □

Since both K and A are s-union and A has the maximum size it follows that $|A| \geq |K|$. Thus by Claim 6 we have

$$A = K.$$

Finally we compute the size of K and show that $\mathbf{a}$ needs to be equitable to maximize the size. Let

$$\sigma_j(\mathbf{a}) := \sum_{J \in \binom{[n]}{j}} \prod_{i \in J} a_i$$

be the j-th elementary symmetric function of $a_1, \ldots, a_n$.

Claim 7.

$$|A| = |K| = \sum_{j=0}^{n} \binom{d+j}{j} \sigma_{n-j}(\mathbf{a}) + \sum_{i=1}^{\lfloor \frac{d}{n-1} \rfloor} \binom{d-(n-1)(i-1)}{n-1}. \tag{11}$$

Proof. By the definition of U_i we see that $\mathcal{D}(U_{i+1}) \setminus U_{i+1} \subset U_i$ for $i \geq 0$. Noting also that U_i is $(|\mathbf{a}| + d + i)$-uniform we have the following disjoint union decomposition of $K = \mathcal{D}(U_0) \sqcup S$, where $S := \bigsqcup_{i \geq 1} U_i$.

If $\mathbf{x} \in \mathcal{D}(U_0)$ then $|\mathbf{x} \setminus \mathbf{a}| \leq d$. Moreover, we have $\sum_{j \in J_{\mathbf{x}}} (x_j - a_j) \leq d$, where $J_{\mathbf{x}} = \{j : x_j \geq a_j\}$, and $x_i < a_i$ for $i \notin J_{\mathbf{x}}$. So, for each $J \subset [n]$, by letting

$$K_0(J) := \{\mathbf{x} \in \mathcal{D}(U_0) : J_{\mathbf{x}} = J\},$$

we have the decomposition $\mathcal{D}(U_0) = \bigsqcup_{J \subset [n]} K_0(J)$. If $\mathbf{x} \in K_0(J)$ then the number of solutions of $\sum_{j \in J} (x_j - a_j) \leq d$ is equal to the number of nonnegative solutions of $\sum_{j \in J} y_j \leq d$, which is $\binom{d+|J|}{|J|}$ by Lemma 1. So it follows that

$$|K_0(J)| = \binom{d+|J|}{|J|} \prod_{i \in [n] \setminus J} a_i,$$

and

$$|\mathcal{D}(U_0)| = \sum_{J \subset [n]} |K_0(J)| = \sum_{j=0}^{n} \binom{d+j}{j} \sigma_{n-j}(\mathbf{a}).$$

If $\mathbf{x} \in U_i$ then we can write $\mathbf{x} = \mathbf{a} + i\mathbf{1} + \mathbf{y}$, where $|\mathbf{y}| = d - (n-1)i$. By Lemma 1 the number of such vectors $\mathbf{y}$ is $\binom{n+d-(n-1)i-1}{d-(n-1)i} = \binom{d-(n-1)(i-1)}{n-1}$. Thus we have

$$|S| = \sum_{i=1}^{\lfloor \frac{d}{n-1} \rfloor} |U_i| = \sum_{i=1}^{\lfloor \frac{d}{n-1} \rfloor} \binom{d-(n-1)(i-1)}{n-1},$$

which completes the proof. □

Recall that n is fixed, and d and $\mathbf{a}$ are determined by A. We have assumed that $|A|$ is maximal. In order to maximize the RHS of (11), $\mathbf{a}$ needs to be an equitable partition for each d, because $\sigma_i(\mathbf{a})$ is maximized when $\mathbf{a}$ is equitable. Then d is chosen so that (11) (with equitable $\mathbf{a}$) is maximized. We summarize what we have shown and state the main result in this subsection:

Proposition 1. *Let n and s be given, and let $q > q_0(n,s)$. If $n = 3$, then Conjecture 1 is true. If $n \geq 4$ and all Suppositions 1, 2, and 3 are satisfied, then Conjecture 1 is true.*

An optimistic conjecture is the following.

Conjecture 2. *All Suppositions 1, 2, and 3 are satisfied for any maximum s-union set in X_q^n with $q > q_0(n,s)$.*

We have the exact size of A by (11) which gives $w_q^n(s)$ provided q is large and all suppositions are satisfied. We also have an upper bound for $|A|$ under these conditions, which is easier to compute. Namely, using (11) and

$$\sigma_i(\mathbf{a}) \leq \binom{n}{i}\left(\frac{|\mathbf{a}|}{n}\right)^i = \binom{n}{i}\left(\frac{s-2d}{n}\right)^i,$$

we have

$$|A| \leq \sum_{j=0}^{n} \binom{d+j}{j}\binom{n}{j}\left(\frac{s-2d}{n}\right)^{n-j} + \sum_{i=1}^{\lfloor \frac{d}{n-1} \rfloor} \binom{d-(n-1)(i-1)}{n-1}. \tag{12}$$

2.2. **The case $n = 3$.** Let s be fixed. From (12) with $d = 3m - s$ we get

$$|A| \leq \frac{1}{8}\left(62m^3 - 3m^2(30s+13) + 6m\left(7s^2+5s-1\right) - 6s^3 - 3s^2 + 10s + 7 + \epsilon\right)$$

where $\epsilon = 0$ if d is odd and $\epsilon = 1$ if d is even. Let $f(m)$ be the RHS of the above inequality with $\epsilon = 1$. Since $s \geq 2d = 2(3m - s) \geq 0$ we have $3s \geq 6m \geq 2s$. So letting

$$m = ps$$

we have $1/3 \leq p \leq 1/2$. In this domain $f(ps)$ attains the maximum at $p = p_0$ where p_0 is the smaller root of

$$\frac{4}{3s}\frac{d}{dp}f(ps) = 31s^2p^2 + \left(-30s^2 - 13s\right)p + 7s^2 + 5s - 1, \tag{13}$$

more concretely,

$$\begin{aligned} p_0 &= \frac{30s - \sqrt{32s(s+5)+293} + 13}{62s} \\ &= \frac{15}{31} - \frac{2\sqrt{2}}{31} + \left(\frac{13}{62} - \frac{5\sqrt{2}}{31}\right)\frac{1}{s} + O(s^{-2}). \end{aligned} \tag{14}$$

Namely, $f(m)$ attains its maximum at around $m \approx 0.3926s$. In this case, by taking the polynomial remainder of $f(ps)$ divided by the RHS of (13), we have

$$\begin{aligned} f(p_0 s) &= \frac{1}{248}\big(-s(32s^2+160s+293)p_0+24s^3+148s^2+345s+235\big) \\ &= \frac{33+8\sqrt{2}}{961}s^3+\frac{15\,(33+8\sqrt{2})}{1922}s^2+\frac{(5260+1479\sqrt{2})}{7688}s \\ &\quad +\frac{10761+3395\sqrt{2}}{15376}+\frac{27}{512\sqrt{2}s}+O(s^{-2}). \end{aligned}$$

We can also get the exact formula. For given s and d let $\mathbf{a}$ be an equitable partition with $|\mathbf{a}| = s-2d$. Let $g(d)$ be the RHS of (11). Noting that $d = 3m-s = (3p-1)s$ let $d^+ = \lceil (3p_0-1)s \rceil$ and $d^- = \lfloor (3p_0-1)s \rfloor$. Then for $d \in \mathbb{N}$ we have

$$g(d) \le \max\{g(d^+), g(d^-)\}.$$

This shows that $w_q^3(s) = \max\{g(d^+), g(d^-)\}$, and $A_q^3(d^+)$ and $A_q^3(d^-)$ are the only possible extremal configurations (up to isomorphism) whose size attains $w_q^3(s)$. In some cases, including $s = 2, 4, 7, 9, 16, 37, 44, 65, \ldots$, all of them attain the maximal size. For example, if $s = 16$, then $m_q^3(16) = 291 = g(3) = g(2)$ and there are two different extremal configurations $A_q^3(3)$ and $A_q^3(2)$ (see Figure 5), both have the same size 291:

$$\begin{aligned} A_q^3(3) &= \mathcal{D}\big(\mathcal{S}_q((3,3,4),3) \cup \mathcal{S}_q((4,4,5),1)\big), \\ A_q^3(2) &= \mathcal{D}\big(\mathcal{S}_q((4,4,4),2) \cup \mathcal{S}_q((5,5,5),0)\big). \end{aligned}$$

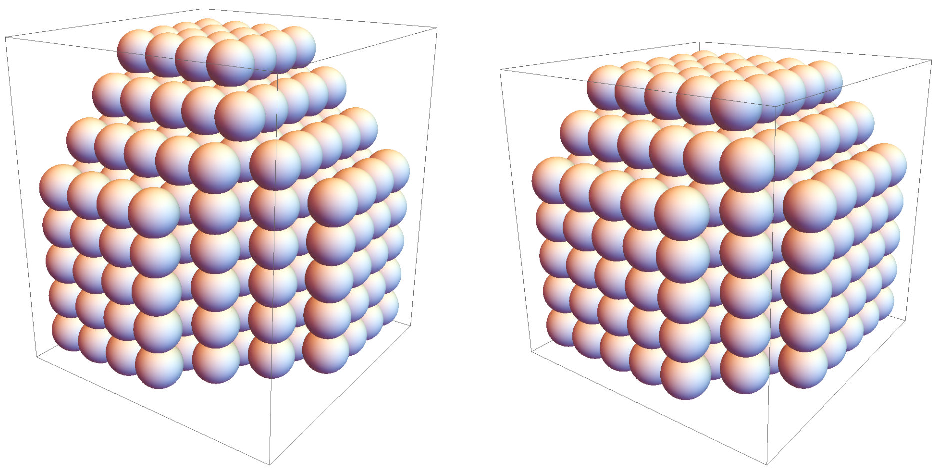

FIGURE 5. $A_q^3(3)$ and $A_q^3(2)$ that attain $w_q^3(16)$

For the lower bound for q, it suffices that $q-1 \ge m$, that is, $q \ge \lceil 2p_0 s \rceil + 1$. Consequently we get the following.

Theorem 1 (slightly stronger version). *If $q \geq \lceil 2p_0 s \rceil + 1$, where p_0 is given in* (14), *then*

$$\begin{aligned} w_q^3(s) &= \max\{g(d^+), g(d^-)\} \\ &\leq \lfloor f(p_0 s) \rfloor = \frac{33 + 8\sqrt{2}}{961} s^3 + O(s^2). \end{aligned}$$

Moreover, the only extremal configuration that attains $w_q^3(s)$ is one of (or possibly both of) $A_q^3(d^+)$ and $A_q^3(d^-)$ (up to isomorphism).

From (14) one can show that $0.8s > \lceil 2p_0 s \rceil + 1$ for all $s \geq 1$. So the above upper bound for $w_q^n(s)$ is valid for $q \geq 4s/5$. We also have $2p_0 s < \frac{2(15-2\sqrt{2})}{31} s \approx 0.785s$. Here is numeric data of $w_q^n(s)$ and its upper bound $f(p_0 s)$ for $1 \leq s \leq 30$.

s	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$w_q^3(s)$	2	4	8	12	20	28	39	54	69	91	113	140	173	206	248	291
$\lfloor f(p_0 s) \rfloor$	2	4	8	13	20	29	40	54	71	91	114	141	173	208	248	293

17	18	19	20	21	22	23	24	25	26	27	28	29	30
341	399	457	526	598	677	767	857	959	1068	1182	1311	1440	1582
343	399	460	527	600	680	767	860	961	1070	1186	1311	1444	1585

3. s-union families for $n > n_0(s, q)$

3.1. **A general bound for s-union families for n large enough.** Since we know that Conjecture 1 is true for the cases (i)–(ix) in Section 1, from now on, we will consider $w_q^n(s)$ for

$$n \geq 3, s \geq 3 \text{ and } q \geq 3.$$

We restate the result we are going to prove.

Theorem 2. *Let s and q be fixed. If $n > n_0(s, q)$ then*

$$w_q^n(s) = \begin{cases} |\mathcal{D}(\mathcal{S}_q(\mathbf{0}, d))| & \text{if } s = 2d, \\ |\mathcal{D}(\mathcal{S}_q(\mathbf{e}_1, d))| & \text{if } s = 2d + 1. \end{cases}$$

Moreover equality is attained only by $\mathcal{D}(\mathcal{S}_q(\mathbf{0}, d))$ if $s = 2d$ and $\mathcal{D}(\mathcal{S}_q(\mathbf{e}_1, d))$ if $s = 2d + 1$ up to isomorphism.

Proof. We remark that for every $\mathbf{x} \in X_q^n$ we have

$$|\mathcal{D}(\mathcal{S}_q(\mathbf{x}, d))| \geq |\mathcal{S}_q(\mathbf{x}, d)|.$$

Then $|\mathcal{S}_q(\mathbf{x}, d)|$ is minimized when $q = 2$, and in this case $|\mathcal{S}_2(\mathbf{x}, d)| = \binom{n-|\mathbf{x}|}{d}$. In particular, both $\mathcal{D}(\mathcal{S}_2(\mathbf{0}, d))$ and $\mathcal{D}(\mathcal{S}_2(\mathbf{e}_1, d))$ have size $\Omega(n^d)$.

Let $A \subset X^n$ be s-union with $|A| = w_q^n(s)$.

First let $s = 2d$. We show that

$$A = \mathcal{D}(\mathcal{S}_q(\mathbf{0}, d)) = \{\mathbf{a} \in X^n : |\mathbf{a}| \leq d\}.$$

To the contrary, suppose that there is an $\mathbf{a} \in A$ with $|\mathbf{a}| = d + r$ for some $1 \le r \le d$. We may assume that $|\mathbf{a}| \ge |\mathbf{c}|$ for all $\mathbf{c} \in A$ and $\mathrm{supp}(\mathbf{a}) = [t] := \{1, 2, \dots, t\}$ for some $1 \le t \le d + r$. Then for every $\mathbf{c} \in A$ we have

$$c_1 + c_2 + \cdots + c_t \le |\mathbf{c}| \le |\mathbf{a}| = d + r.$$

On the other hand, since $\mathbf{a} \vee \mathbf{c} \le s = 2d$ we get

$$c_{t+1} + c_{t+2} + \cdots + c_n \le 2d - (a_1 + \cdots + a_t) = 2d - |\mathbf{a}| = d - r.$$

Then counting the number of nonnegative integer solutions of

$$x_1 + x_2 + \cdots + x_t \le d + r$$

and

$$x_{t+1} + x_{t+2} + \cdots + x_n \le d - r$$

with Lemma 1 we get

$$|A| \le \binom{t + (d + r)}{d + r}\binom{(n - t) + (d - r)}{d - r} = O(n^{d-r}) \le O(n^{d-1}),$$

a contradiction.

Next let $s = 2d + 1$. We show that

$$A \cong \mathcal{D}(\mathcal{S}_q(\mathbf{e}_1, d)) = \{\mathbf{a} \in X^n : |\mathbf{a}| \le d\} \cup \{\mathbf{a} \in X^n : 1 \in \mathrm{supp}(\mathbf{a}),\ |\mathbf{a}| = d + 1\}.$$

Notice that both of the two subfamilies on the RHS have size $\Omega(n^d)$.

To the contrary, if there is an $\mathbf{a} \in A$ with $|\mathbf{a}| \ge d + 2$, then as in the previous case we get $|A| = O(n^{d-1})$. So we may assume that $|\mathbf{c}| \le d + 1$ for all $\mathbf{c} \in A$. We may further assume that there is an $\mathbf{a} \in A$ with $|\mathbf{a}| = d + 1$ and $\mathrm{supp}(\mathbf{a}) = [t]$ for some $1 \le t \le d + 1$. We focus on

$$A_{d+1} := \{\mathbf{c} \in A : |\mathbf{c}| = d + 1\}.$$

Since A_{d+1} is $(2d+1)$-union and $(d+1)$-uniform, it is 1-intersecting, that is, $\mathrm{supp}(\mathbf{c}) \cap \mathrm{supp}(\mathbf{c}') \ne \emptyset$ for all $\mathbf{c}, \mathbf{c}' \in A$. For $1 \le i \le t$ we define $B_i \subset A_{d+1}$ by

$$B_i := \{\mathbf{b} \in A_{d+1} : \mathrm{supp}(\mathbf{b}) \cap [t] = \{i\}\}.$$

Note that if $\mathbf{b} \in B_i$ then $|\,\mathrm{supp}(\mathbf{b}) \cap [t + 1, n]| \le d$. We partition A_{d+1} as

$$A_{d+1} = B_1 \cup B_2 \cup \cdots \cup B_t \cup F,$$

where

$$F := \{\mathbf{f} \in A_{d+1} : |\,\mathrm{supp}(\mathbf{f}) \cap [t]| \ge 2\}.$$

Note that if $\mathbf{f} \in F$ then it follows

$$f_1 + \cdots + f_t = i,$$
$$f_{t+1} + \cdots + f_n = d + 1 - i$$

for some $2 \le i \le d+1$. Thus

$$\begin{aligned}|F| &\le \sum_{i=2}^{d+1} \binom{t+i-1}{i}\binom{(n-t)+(d+1-i)-1}{d+1-i} \\ &\le d\binom{2d+1}{d+1}\binom{n-t+d-2}{d-1} = O(n^{d-1}).\end{aligned}$$

Let $I := \{i : B_i \neq \emptyset\}$.

Claim 8. *If $|I| \ge 2$, then $\sum_{i=1}^{t} |B_i| = O(n^{d-1})$.*

Proof. Let $i, j \in I$, $i \neq j$, and let $\mathbf{b} \in B_i$, $\mathbf{c} \in B_j$. Since $\mathbf{b}$ and $\mathbf{c}$ intersect, that is, $\operatorname{supp}(\mathbf{b}) \cap \operatorname{supp}(\mathbf{c}) \cap [t+1, n] \neq \emptyset$, and $\sum_{j\in[t+1,n]} c_j = d$ we have that

$$\sum \{c_j : j \in [t+1, n] \setminus \operatorname{supp}(\mathbf{b})\} \le d - 1.$$

Then, by Lemma 1, we have

$$|B_j| \le \binom{(n-t-|\operatorname{supp}(\mathbf{b})|)+(d-1)}{d-1}\binom{|\operatorname{supp}(\mathbf{b})|+d}{d} = O(n^{d-1}).$$

Thus $\sum_{i=1}^{t} |B_i| \le t \max |B_i| \le (d+1)O(n^{d-1}) = O(n^{d-1})$. ☐

By the claim above, if $|I| \ge 2$, then

$$|A_{d+1}| = \sum_{i=1}^{d} |B_i| + |F| = O(n^{d-1}),$$

which is a contradiction.

So we may assume that $|I| = 1$, say $I = \{1\}$. If there is an $\mathbf{f} \in F$ such that $1 \notin \operatorname{supp}(\mathbf{f})$, then as in the above claim we have $|B_1| = O(n^{d-1})$ and $|A_{d+1}| = O(n^{d-1})$, a contradiction. Consequently we need $1 \in \operatorname{supp}(\mathbf{f})$ for all $\mathbf{f} \in F$, and thus $1 \in \operatorname{supp}(\mathbf{c})$ for all $\mathbf{c} \in A_{d+1}$. This means $A_{d+1} \subset \{\mathbf{a} \in X^n : 1 \in \operatorname{supp}(\mathbf{a}), |\mathbf{a}| = d+1\}$. ☐

3.2. **Towards a sharp lower bound of n for the case when $q > q_0(s)$.**

Corollary 1. *Let s be fixed. Let $q-1 \ge d$ if $s = 2d$ and let $q-1 \ge d+1$ if $s = 2d+1$. If $n > n_0(s)$ then*

$$w_q^n(s) = \begin{cases} \dbinom{n+d}{d} & \text{if } s = 2d, \\[2ex] \dbinom{n+d}{d} + \dbinom{n+d-1}{d} & \text{if } s = 2d+1. \end{cases}$$

Proof. By Theorem 2 it suffices to show that if $q-1 \ge d$ then $|\mathcal{D}(\mathcal{S}_q(\mathbf{0}, d))| = \binom{n+d}{d}$ and if $q-1 \ge d+1$ then $|\mathcal{D}(\mathcal{S}_q(\mathbf{e}_1, d))| = \binom{n+d}{d} + \binom{n+d-1}{d}$. These identities follow from the next claim. ☐

Claim 9. *For $q-1 \geq d \geq 0$ we have*

$$|\mathcal{D}(\mathcal{S}_q(\mathbf{0}, d))| = \binom{n+d}{d}.$$

For $q-2 \geq d \geq t \geq 1$ we have

$$|\mathcal{D}(\mathcal{S}_q(\tilde{\mathbf{e}}_t, d))| = \sum_{j=0}^{t} \binom{t}{j}\binom{n+d-j}{d}.$$

Proof. By counting the number of nonnegative integer solutions of the inequality $x_1 + x_2 + \cdots + x_n \leq d$, we get from Lemma 1 that

$$|\mathcal{D}(\mathcal{S}_q(\mathbf{0}, d))| = \binom{n+d}{d}.$$

Let $t \geq 1$. Let J be a j-element subset in $[t]$, and let

$$A_J := \left\{ \sum_{i \in [t] \setminus J} \mathbf{e}_i + \mathbf{x} : x_i = 0 \text{ for } j \in J, \sum_{l \in [n] \setminus J} x_l \leq d \right\}.$$

In other words, if $\mathbf{a} \in A_J$, then $a_j = 0$ for $j \in J$, $a_i \geq 1$ for $i \in [t] \setminus J$, and $|\mathbf{a}| \leq d + (t-j)$. Then we have a partition

$$\mathcal{D}(\mathcal{S}_q(\tilde{\mathbf{e}}_t, d)) = \bigcup_{J \subset [t]} A_J. \tag{15}$$

Now we compute $|A_J|$. This is the number of nonnegative solutions of the inequality

$$\sum_{l \in [n] \setminus J} x_l \leq d,$$

and it follows from Lemma 1 that

$$|A_J| = \binom{(n-j)+d}{d}. \tag{16}$$

By (15) and (16) we get the desired identity. □

Now we try to find a sharp lower bound for n that guarantees the formula for $w_q^n(s)$ in Corollary 1.

Claim 10. *Let $q-1 \geq d \geq 3$. If*

$$n > n_0 := \frac{(1+\sqrt{5})d}{2} + \frac{3}{2},$$

then

$$|\mathcal{D}(\mathcal{S}_q(\tilde{\mathbf{e}}_2, d-1))| < |\mathcal{D}(\mathcal{S}_q(\mathbf{0}, d))|. \tag{17}$$

Proof. By direct computation using Claim 9 we see that (17) holds iff

$$n^2 - (d+3)n - d^2 + 2d + 2 \geq 0. \tag{18}$$

Solving the above inequality we get

$$n \geq n_0 - \frac{1}{2\sqrt{5}} + O(1/d).$$

In particular, if $n = n_0$, then the LHS of (18) is equal to $(2d-1)/4$, which is positive. On the other hand, if $n = n_0 - \frac{1}{2\sqrt{5}}$, then the LHS of (18) is equal to $-\frac{1}{5}$. So the minimum integer n satisfying (18) is in the interval $[n_0 - \frac{1}{2\sqrt{5}}, n_0]$. □

Claim 11. *Let $q-1 \geq d \geq 4$. If*

$$n > n_0 := \frac{(1+\sqrt{5})d}{2} + 2$$

then

$$|\mathcal{D}(\mathcal{S}_q(\tilde{\mathbf{e}}_3, d-1))| < |\mathcal{D}(\mathcal{S}_q(\mathbf{e}_1, d))|. \tag{19}$$

Proof. By Claim 9 the LHS of (19) is

$$\binom{n+d-4}{d-1} + 3\binom{n+d-3}{d-1} + 3\binom{n+d-2}{d-1} + \binom{n+d-1}{d-1}.$$

Then we see that (19) holds iff

$$2n^3 - (d+12)n^2 - (3d^2 - 6d - 22)n - d^3 + 6d^2 - 7d - 12 \geq 0. \tag{20}$$

Solving the above inequality we get

$$n \geq n_0 - \frac{3}{\sqrt{5}} + 1 + O(1/d).$$

In particular, if $n = n_0$, then the LHS of (20) is equal to $d(\sqrt{5}(d-1) + d)$, which is positive. On the other hand, if $n = n_0 - \frac{3}{\sqrt{5}} + 1$, then the LHS of (20) is equal to $\frac{1}{25}\left(5\left(19\sqrt{5} - 52\right)d - 114\sqrt{5} + 270\right)$, which is negative. So the minimum integer n satisfying (20) is in the interval $[n_0 - \frac{3}{\sqrt{5}} + 1, n_0]$. □

The previous two claims suggest the following lower bound for n, which, if true, would be almost sharp.

Conjecture 3. *We can replace $n_0(s)$ in Corollary 1 with*

$$\begin{cases} \dfrac{(1+\sqrt{5})d}{2} + \dfrac{3}{2} & \text{if } s = 2d, \\[2ex] \dfrac{(1+\sqrt{5})d}{2} + 2 & \text{if } s = 2d+1. \end{cases}$$

3.3. **The 1-intersecting case.** Recall that

$$\mathcal{U}_q(A) := \{\mathbf{c} \in X_q^n : \mathbf{a} \prec \mathbf{c} \text{ for some } \mathbf{a} \in A\}$$

for $A \subset X_q^n$. We restate the result that we are going to prove.

Theorem 4. *For 1-intersecting families, we have*

$$m_q^n(1) = \begin{cases} |\mathcal{U}_q(\mathcal{S}_2(\mathbf{0}, d+1))| & \text{if } n = 2d+1, \\ |\mathcal{U}_q(\mathcal{S}_2(\mathbf{0}, d+1) \cap X_2^{n-1})| & \text{if } n = 2d+2. \end{cases}$$

The following equivalent dual form via (1) verifies the Conjecture 1 in this case.

$$w_q^n\big((q-1)n-1\big) = \begin{cases} \big|\mathcal{D}\big(\mathcal{S}_q((q-2)\tilde{\mathbf{e}}_n, d)\big)\big| & \text{if } n = 2d+1, \\ \big|\mathcal{D}\big(\mathcal{S}_q((q-2)\tilde{\mathbf{e}}_n + \mathbf{e}_n, d)\big)\big| & \text{if } n = 2d+2. \end{cases}$$

For $\mathbf{b} \in 2^{[n]}$ we define its W_q-weight by

$$W_q(\mathbf{b}) := (q-1)^{|\mathbf{b}|},$$

and for $B \subset 2^{[n]}$ let

$$W_q(B) := \sum_{\mathbf{b}\in B} W_q(\mathbf{b}).$$

We mention that the product measure μ_p, where $p := 1 - \frac{1}{q} \in [1/2, 1)$, is obtained by normalizing the W_q-weight, that is,

$$\mu_p(\mathbf{b}) := W_q(\mathbf{b})/q^n = p^{|\mathbf{b}|}(1-p)^{n-|\mathbf{b}|}.$$

Proof of Theorem 4. Let $A \subset X_q^n$ be 1-intersecting. Then the base set

$$B_A := \{\text{supp}(\mathbf{a}) : \mathbf{a} \in A\}$$

is also 1-intersecting (in the usual sense, that is, any two members in B_A have non-empty intersection), and $|A| \le W_q(B_A)$. Thus we have that

$$|A| \le \max\{W_q(B) : B \subset 2^{[n]} \text{ is 1-intersecting}\}.$$

If $q = 2$, then the RHS is 2^{n-1}, and equality holds if

$$B = \begin{cases} B_0 := \mathcal{U}_2(\mathcal{S}_2(\mathbf{0}, d+1)) & \text{for } n = 2d+1, \\ B_1 := \mathcal{U}_2(\mathcal{S}_2(\mathbf{0}, d+1) \cap X_2^{n-1}) & \text{for } n = 2d+2. \end{cases}$$

For $q \ge 3$ we use the following Bey–Engel version of the comparison lemma (Theorem 7 in [4]) originally due to Ahlswede and Khachatrian (Lemma 7 in [3]).

Lemma 2 (Comparison lemma). *Let P be a set of points in $\mathbb{R}_{\ge 0}^{n+1-t}$ whose coordinates are indexed by $t, t+1, \dots, n$. Let $\mathbf{v} \in \mathbb{R}_{\ge 0}^{n+1-t}$ be a given positive weight vector. Suppose that there is some $\mathbf{f}^* \in P$ such that the standard inner product of $\mathbf{v}$ and $\mathbf{f}^*$ satisfies*

$$\mathbf{v} \cdot \mathbf{f}^* = \max\{\mathbf{v} \cdot \mathbf{f} : \mathbf{f} \in P\},$$

and for some $u \in [t, n]$

$$f_i^* = 0 \text{ if } t \le i < u,$$
$$f_i \le f_i^* \text{ if } u \le i \le n \text{ and } \mathbf{f} \in P.$$

Let $\mathbf{v}' \in \mathbb{R}_{\ge 0}^{n+1-t}$ be another positive weight vector with

$$\frac{v_i}{v_{i+1}} \ge \frac{v'_i}{v'_{i+1}} \text{ for } i = t, \dots, n.$$

Then also

$$\mathbf{v}' \cdot \mathbf{f}^* = \max\{\mathbf{v}' \cdot \mathbf{f} : \mathbf{f} \in P\}.$$

To apply the lemma, let $t = 1$, let P be the set of profile vectors of 1-intersecting families in $2^{[n]}$, let $\mathbf{f}^*$ be the profile vector of $B = B_0$ or B_1 according to the parity of n. (So, for example, if $n = 2d + 1$, then $u = d + 1$ and $f_i^* = \binom{n}{i}$ for $u \le i \le n$.) We choose $\mathbf{v}$ and $\mathbf{v}'$ corresponding to W_2 and W_q, respectively, namely, let $\mathbf{v} = \mathbf{1}$ and define $\mathbf{v}'$ by $v_i' = (q-1)^i$. Then

$$\frac{v_i}{v_{i+1}} = 1 > \frac{1}{q-1} = \frac{v_i'}{v_{i+1}'}.$$

Thus, by the lemma, it follows that the same B ($= B_0$ or B_1) gives the maximum W_q-weight for $q \ge 3$ as well. Consequently we have $|A| \le W_q(B)$ and the RHS coincides with the RHS of the formula in the theorem. Moreover both families in the formula ($\mathcal{U}_q(\mathcal{S}_2(\mathbf{0}, d+1))$ for $n = 2d+1$ and $\mathcal{U}_q(\mathcal{S}_2(\mathbf{0}, d+1) \cap X_2^{n-1})$ for $n = 2d+2$) are 1-intersecting, which completes the proof. □

4. k-uniform t-intersecting families

4.1. **The case when $t = 1$ or n is large.** In this section we assume that $k < s < 2k$. We rewrite Conjecture 1 in terms of $m_q^{n,k}(t)$ using (2). Consider the situation that $\mathcal{S}_q(\mathbf{a}, d)$ is s-union. By solving

$$2k - t = s = |\mathbf{a}| + 2d$$

we get

$$d = k - \frac{t + |\mathbf{a}|}{2}.$$

If $\mathbf{b} \in \mathcal{S}_q(\mathbf{a}, d)$ then

$$|\mathbf{b}| = |\mathbf{a}| + d = k - \frac{t}{2} + \frac{|\mathbf{a}|}{2},$$

and $|\mathbf{b}| \ge k$ iff $|\mathbf{a}| \ge t$. So to ensure $\mathcal{D}(\mathcal{S}_q(\mathbf{a}, d)) \cap X_q^{n,k}$ is nonempty, we need $|\mathbf{a}| \ge t$. Consequently Conjecture 1 is equivalent to the following.

Conjecture 4. *Let $0 < t < k$, and let $d = k - \frac{t+|\mathbf{a}|}{2}$ be nonnegative integer. Then*

$$m_q^{n,k}(t) = \max_{0 \le d \le k-t} \{|\mathcal{D}(\mathcal{S}_q(\mathbf{a}, d)) \cap X_q^{n,k}| : \mathbf{a} \in X_q^{n,\ge t} \text{ is an equitable partition}\},$$

where $X_q^{n,\ge t} = \bigcup_{j=t}^{n} X_q^{n,j}$.

Conjecture 1 is true if $t = 1$ as follows.

Theorem 5. *If $n \ge \max\{2k - q + 2, k + 1\}$, then*

$$m_q^{n,k}(1) = |\mathcal{S}_q(\mathbf{e}_1, k-1)|.$$

Proof. Let $n \ge \max\{2k - q + 2, k + 1\}$. Since $m_q^{n,k}(1) = m_{k+1}^{n,k}(1)$ for $q \ge k + 1$ we may assume that $q \le k + 1$. Then $2k - q + 2 \ge k + 1$ and we may assume that $n \ge 2k - q + 2$. Let $A \subset X_q^{n,k}$ be 1-intersecting with $|A| = m_q^{n,k}(1)$. Let

$$Y_i := \{\mathbf{x} \in X_q^{n,k} : |\operatorname{supp}(\mathbf{x})| = i\}.$$

Then $X_q^{n,k} = \bigsqcup_i Y_i$, where $\lceil \frac{k}{q-1} \rceil \leq i \leq k$, is a partition. Since $A_i := A \cap Y_i$ is 1-intersecting,

$$\mathrm{supp}(A_i) := \{\mathrm{supp}(\mathbf{x}) : \mathbf{x} \in A_i\} \subset \binom{[n]}{i}$$

is 1-intersecting, too.

First suppose that $n \geq 2i$. Then, applying the Erdős–Ko–Rado theorem to $\mathrm{supp}(A_i)$, we see that $|\mathrm{supp}(A_i)|$ is maximized (and thus $|A_i|$ is maximized) when $\bigcap_{\mathbf{x}\in A_i} \mathrm{supp}(\mathbf{x}) \neq \emptyset$, say, the intersection is $\{1\}$, and this yields

$$|A_i| \leq |\mathcal{S}_q(\mathbf{e}_1, k-1) \cap Y_i|. \tag{21}$$

In particular, if $n \geq 2k$, then we have $n \geq 2i$ and

$$|A| = \sum_i |A_i| \leq \sum_i |\mathcal{S}_q(\mathbf{e}_1, k-1) \cap Y_i| = |\mathcal{S}_q(\mathbf{e}_1, k-1)|,$$

as needed. So we may assume that $n < 2k$.

Next suppose that $n < 2i$. We further partition Y_i into

$$Y_i = Y_i^1 \cup \cdots \cup Y_i^{N_i},$$

so that all distinct $\binom{n}{i}$ supports appear exactly once in each Y_i^l. Thus $|Y_i^l| = \binom{n}{i}$ and $\mathrm{supp}(Y_i^l) = \binom{[n]}{i}$. Also N_i is the number of nonnegative integer solutions of

$$x_1 + \cdots + x_i = k - i,$$

and $N_i = \binom{k-1}{i-1}$ by Lemma 1. Let $j := n - i < i$ and partition Y_j similarly. We have $N_j = \binom{k-1}{j-1}$, and here we need $k - j \leq (q-1) - 1$. For this inequality we use our assumption $n \geq 2k - q + 2$, in fact,

$$k - j = k - (n - i) \leq k - (n - k) \leq (q-1) - 1.$$

Let $B^{l,l'}$ be a bipartite graph on $Y_i^l \cup Y_j^{l'}$ such that two vertices are adjacent if they have empty intersection. Then $B^{l,l'}$ is a perfect matching. Let $A_i^l := A \cap Y_i^l$. Then $A_i^l \cup A_j^{l'}$ is an independent set in $B^{l,l'}$. Thus we have $|A_i^l| + |A_j^{l'}| \leq \binom{n}{i}$ for all l, l'. Summing up this inequality over all l and l', and dividing both sides by $N_i N_j$ we get

$$|A_i|/N_i + |A_j|/N_j \leq \binom{n}{i}. \tag{22}$$

In the same way we have $|\mathcal{S}_q(\mathbf{e}_1, k-1) \cap Y_i^l| + |\mathcal{S}_q(\mathbf{e}_1, k-1) \cap Y_j^{l'}| = \binom{n}{i}$ for all l, l', and

$$|\mathcal{S}_q(\mathbf{e}_1, k-1) \cap Y_i|/N_i + |\mathcal{S}_q(\mathbf{e}_1, k-1) \cap Y_j|/N_j = \binom{n}{i}. \tag{23}$$

Since $n \geq 2j$ we can apply (21) to A_j to get

$$|A_j| \leq |\mathcal{S}_q(\mathbf{e}_1, k-1) \cap Y_j|. \tag{24}$$

We have $N_i \leq N_j$. In fact $N_i/N_j = \binom{k-1}{i-1}/\binom{k-1}{j-1} \leq 1$ is equivalent to $i - 1 \geq k - j$, which follows from our assumption $n \geq k + 1$. Thus (22) implies that $|A_i| + |A_j|$ is maximized when $|A_j|$ is maximized. In this case we have equality in (24).

Then comparing (22) and (23) we get (21) (under the assumption that $|A_i| + |A_j|$ is maximized). Consequently, if $n < 2i$ and $j = n - i$ then we always have

$$|A_i| + |A_j| \leq |\mathcal{S}_q(\mathbf{e}_1, k-1) \cap Y_i| + |\mathcal{S}_q(\mathbf{e}_1, k-1) \cap Y_j|. \tag{25}$$

Finally let $I_2 := \{i : n < 2i \leq 2k\}$ and $I_1 := \{i : \lceil \frac{k}{q-1} \rceil \leq i \leq k\} \setminus I_2$. Then, by (21) and (25), we get

$$|A| = \sum_{i \in I_1} |A_i| + \sum_{i \in I_2} \Big(|A_i| + |A_{n-i}| \Big) \leq |\mathcal{S}_q(\mathbf{e}_1, k-1)|,$$

which gives us the desired formula for $m_1^{n,k}(1)$. □

We mention that if $q \geq k+1$ in Theorem 5, then $|\mathcal{S}_q^{k-1}(\mathbf{e}_1)| = \binom{n+k-2}{k-1}$ and this case was already solved by Meagher and Purdy [17]. If $q \leq k$, then the lower bound for n in Theorem 5 is not sharp in general. For example, if $q = 3$, then, apparently one can replace the lower bound for n with $n \geq 3k/2$. Maybe the correct bound is obtained by comparing the sizes of $\mathcal{S}_3(\mathbf{e}_1, k-1)$ and $\mathcal{D}(\mathcal{S}_3(\tilde{\mathbf{e}}_3, k-2)) \cap X_3^{n,k}$.

Conjecture 4 is also true if $q = 2$. In fact this case is equivalent to the Ahlswede–Khachatrian version [2] of the Erdős–Ko–Rado [7] theorem as we will see below.

Theorem 7 ([2])**.** *Let $k > t \geq 1$ and $n \geq 2k - t$. Then*

$$m_2^{n,k}(t) = \max\{|\operatorname{AK}(n,k,t,i)| : i = 0, 1, \ldots, (k-t)/2\},$$

where

$$\operatorname{AK}(n,k,t,i) := \left\{ F \in \binom{[n]}{k} : |F \cap [t+2i]| \geq t + i \right\}.$$

Claim 12. *Conjecture 4 is true if $q = 2$. Namely, if $k > t \geq 1$ and $n \geq 2k - t$, then*

$$m_2^{n,k}(t) = \max\{|\mathcal{D}(\mathcal{S}_2(\tilde{\mathbf{e}}_{t+2i}, k-t-i)) \cap X_2^{n,k}| : i = 0, 1, \ldots, (k-t)/2\}. \tag{26}$$

Proof. We can identify $X_2^{n,k}$ with $\binom{[n]}{k}$ by sending $\mathbf{x} \in X_2^{n,k}$ to $\operatorname{supp}(\mathbf{x}) \in \binom{[n]}{k}$. So it suffices to show that $\mathcal{D}(\mathcal{S}_2(\mathbf{a}, d)) \cap X_2^{n,k}$ can be identified with $\operatorname{AK}(n,k,t,i)$, where $\mathbf{a} = \tilde{\mathbf{e}}_{t+2i}$ and $d = k - t - i$.

Let $\mathbf{c} \in \mathcal{D}(\mathcal{S}_2(\mathbf{a}, d)) \cap X_2^{n,k}$. Then there is some $\mathbf{b} \in \mathcal{S}_2(\mathbf{a}, d)$ with $\mathbf{c} \in \mathcal{D}(\mathbf{b}) \cap X_2^{n,k}$. Also, $|\mathbf{b}| = (t+2i)+(k-t-i) = k+i$, $|\mathbf{c}| = k$ and $|\mathbf{b} \setminus \mathbf{c}| = i$. Since $\operatorname{supp}(\mathbf{b}) \supset [t+2i]$ it follows $|\operatorname{supp}(\mathbf{c}) \cap [t+2i]| \geq (t+2i) - i = t + i$. Thus $\operatorname{supp}(\mathbf{c}) \in \operatorname{AK}(n,k,t,i)$.

Let $F \in \operatorname{AK}(n,k,t,i)$. Since $|F \cap [t+2i]| \geq t+i$, or equivalently, $|[t+2i] \setminus F| \leq i$, one can find $G \in \binom{[n]}{k+i}$ with $G \supset [t+2i]$ by adding i vertices to F. Let $\mathbf{b}, \mathbf{c} \in X_2^n$ be such that $\operatorname{supp}(\mathbf{b}) = G$, $\operatorname{supp}(\mathbf{c}) = F$. Then $\mathbf{b} \in \mathcal{S}_2(\mathbf{a}, d)$ and $\mathbf{c} \in \mathcal{D}(\mathbf{b}) \cap X_2^{n,k}$. This means that $\mathbf{c} \in \mathcal{D}(\mathcal{S}_2(\mathbf{a}, d)) \cap X_2^{n,k}$. □

If $n \geq (t+1)(k-t+1)$, then the RHS of (26) is attained by $i = 0$. In this case (26) reads

$$m_2^{n,k}(t) = |\mathcal{S}_2(\tilde{\mathbf{e}}_t, k-t)| = \binom{n-t}{k-t}.$$

This result is due to Frankl [8] and Wilson [18].

Now we will verify that if n is large enough, then the maximum of the RHS in Conjecture 4 is attained by $\mathbf{a} = \tilde{\mathbf{e}}_t$ (and thus $|\mathbf{a}| = t$ and $d = k - t$) as stated below.

Theorem 8. *Let k, t and q be fixed with $0 < t < k$. If $n \geq n_0(k, t, q)$ then*

$$m_q^{n,k}(t) = |\mathcal{S}_q(\tilde{\mathbf{e}}_t, k-t)|.$$

Moreover equality is attained only by $\mathcal{S}_q(\tilde{\mathbf{e}}_t, k-t)$ up to isomorphism.

We remark that Theorem 8 is an equivalent dual form of Theorem 3 via (2).

Proof of Theorem 8. The proof given here is very similar to a proof of the Erdős–Ko–Rado theorem for n sufficiently large, cf., [5] p. 48.

Let $A \subset X_q^{n,k}$ be t-intersecting with $|A| = m_q^{n,k}(t)$. Then there are $\mathbf{a}^1, \mathbf{a}^2 \in A$ such that

$$|\mathbf{a}^1 \wedge \mathbf{a}^2| = t.$$

Let $\mathbf{b} := \mathbf{a}^1 \wedge \mathbf{a}^2$. We may assume that $\mathbf{b} = (b_1, \ldots, b_n)$ with $|\mathbf{b}| = t$ and

$$q - 1 \geq b_1 \geq b_2 \geq \cdots \geq b_n \geq 0.$$

First suppose that $\mathbf{b} \prec \mathbf{a}$ for all $\mathbf{a} \in A$. Then $|A|$ is bounded from above by the number of nonnegative solutions of an equation

$$x_1 + \cdots + x_n = k, \text{ with } b_i \leq x_i < q \text{ for } 1 \leq i \leq n,$$

or equivalently,

$$y_1 + \cdots + y_n = k - t, \text{ with } 0 \leq y_i < q - b_i \text{ for } 1 \leq i \leq n. \tag{27}$$

Claim 13. *The number of solutions of (27) is maximized when $b_1 = \cdots = b_t = 1$, and it is at most $|\mathcal{S}_q(\tilde{\mathbf{e}}_t, k-t)|$.*

Proof. Let $\mathbf{b}$ and $\mathbf{b}'$ be weight t vectors with $\sum_{i=1}^j b_i \geq \sum_{i=1}^j b'_i$ for all j. Let $Y(\mathbf{b})$ be the set of nonnegative integer solutions of (27). We prove that $|Y(\mathbf{b})| \leq |Y(\mathbf{b}')|$ with equality holding iff $\mathbf{b} = \mathbf{b}'$. It suffices to consider the case $\mathbf{b}' = \mathbf{b} - \mathbf{e}_i + \mathbf{e}_j$ where $i < j$, $b'_i \geq b'_j$. Since $b'_i = b_i - 1$ and $b'_j = b_j + 1$ we have $b_i \geq b_j + 2$. Let

$$Y_l := \{(y_i, y_j) \in X_{q-b_i} \times X_{q-b_j} : y_i + y_j = l\}$$

and

$$Y'_l := \{(y'_i, y'_j) \in X_{q-b'_i} \times X_{q-b'_j} : y'_i + y'_j = l\}.$$

We show that $|Y_l| = |Y'_l|$ in most cases. There are two cases that $Y_l \neq Y'_l$, namely, if $l \geq q - 1 - b_j$ then

$$Y_l \setminus Y'_l = \{(l - (q - 1 - b_j), q - 1 - b_j)\},$$

and if $l \geq q - 1 - b'_i = q - b_i$ then

$$Y'_l \setminus Y_l = \{(q - 1 - b'_i, l - (q - 1 - b'_i))\}.$$

Even in these cases we have $|Y_l| = |Y'_l|$ for $l \geq q - 1 - b_j$, where we used $q - 1 - b_j > q - b_i$. But if $q - b_i \leq l < q - 1 - b_j$, then $|Y'_l| = |Y_l| + 1$. This proves $|Y(\mathbf{b})| < |Y(\mathbf{b}')| \leq Y(\tilde{\mathbf{e}}_t)|$. $\square$

Next suppose that there is an $\mathbf{a}^3 \in A$ such that $\mathbf{b} \not\prec \mathbf{a}^3$, that is,

$$|\mathbf{a}^3 \wedge \mathbf{b}| \leq t-1. \tag{28}$$

For $l = 0, 1, \ldots, t$ let

$$A_l := \{\mathbf{a} \in A : |\mathbf{a} \wedge \mathbf{b}| = t - l\}.$$

Then $|A| = |A_0| + \cdots + |A_t|$. We will prove $|A| = O(n^{k-t-1})$ by showing $|A_l| = O(n^{k-t-1})$ for all l. This will complete the proof of the theorem because

$$|\mathcal{S}_q(\tilde{\mathbf{e}}_t, k-t)| \geq |\mathcal{S}_2(\tilde{\mathbf{e}}_t, k-t)| = \binom{n-t}{k-t} = \Omega(n^{k-t}).$$

For the case $l = 0$, let $\mathbf{a} \in A_0$. Then $\mathbf{a} \succ \mathbf{b}$ implies that $a_i \geq b_i$ for all $1 \leq i \leq n$. Moreover $|\mathbf{a} \wedge \mathbf{a}^3| \geq t$ and (28) yield that there is some j such that $a_j \geq b_j + 1$ (informally, $|\mathbf{a} \cap (\mathbf{a}^3 \setminus \mathbf{b})| \geq 1$). Let N be the number of nonnegative solutions of equation

$$x_1 + \cdots + x_n = k - (|\mathbf{b}| + 1) = k - t - 1,$$

we get from Lemma 1 that $N = \binom{(n-1)+(k-t-1)}{k-t-1} = O(n^{k-t-1})$. Also, there are at most $|\mathbf{a}| = k$ choices for this j, so we have

$$|A_0| \leq kN = O(n^{k-t-1}).$$

Now let $l \geq 1$. Fix $\mathbf{b}' \prec \mathbf{b}$ with $|\mathbf{b}'| = t - l$, and we will count the number N' of $\mathbf{a} \in A_l$ such that $\mathbf{b}' = \mathbf{a} \wedge \mathbf{b}$. Clearly,

$$|\mathbf{a} \wedge \mathbf{b}| = t - l. \tag{29}$$

Since $|\mathbf{a} \wedge \mathbf{a}^1| \geq t$ we need

$$|(\mathbf{a} \wedge \mathbf{a}^1) \setminus \mathbf{b}| \geq l. \tag{30}$$

In the same reason we also have

$$|(\mathbf{a} \wedge \mathbf{a}^2) \setminus \mathbf{b}| \geq l. \tag{31}$$

So the number of $\mathbf{a} \in A_l$ satisfying (29), (30) and (31), is at most the number of nonnegative solutions of an equation

$$x_1 + \cdots + x_n = k$$

with $x_i \geq c_i$ for $1 \leq i \leq n$, where

$$c_1 + \cdots + c_n = (t - l) + l + l = t + l,$$

and the number is, by Lemma 1, at most $\binom{(n-1)+(k-(t+l))}{k-(t+l)} = O(n^{k-t-1})$. There are at most $\binom{t}{l}$ choices for $\mathbf{b}'$, and at most $\binom{k-t}{l}$ choices for the l positions in $\mathbf{a}^1$ for (30), and the same for (31), thus we get

$$|A_l| \leq \binom{t}{l}\binom{k-t}{l}^2\binom{(n-1)+(k-(t+l))}{k-(t+l)} = O(n^{k-t-1}).$$

This completes the proof of the theorem. □

4.2. **Intersecting families with weights for large q.** We remark that if q is large enough, then $|\mathcal{S}_q(\tilde{\mathbf{e}}_t, k-t)|$ does not depend on q. More precisely, if $k > t$ and $q \geq k-t+2$, then $|\mathcal{S}_q(\tilde{\mathbf{e}}_t, k-t)|$ is the number of the solutions of an equation

$$x_1 + \cdots + x_n = k - t,$$

so it follows from Lemma 1 that

$$|\mathcal{S}_q(\tilde{\mathbf{e}}_t, k-t)| = \binom{(n-1)+(k-t)}{k-t}.$$

If, moreover, $n \geq t(k-t)+2$, then direct computation shows that

$$|\mathcal{S}_q(\tilde{\mathbf{e}}_t, k-t)| \geq |\mathcal{D}(\mathcal{S}_q(\tilde{\mathbf{e}}_{t+2}, k-t-1)) \cap X_q^{n,k}|.$$

This suggests that if $k > t$, $q \geq k-t+2$, and $n \geq t(k-t)+2$, then

$$m_q^{n,k}(t) = |\mathcal{S}_q(\tilde{\mathbf{e}}_t, k-t)| = \binom{n+k-t-1}{k-t}.$$

Theorem 8 says that this is true if $n \geq n_0(k,t,q)$, and in fact Füredi, Gerbner and Vizer proved the following much stronger result, which confirms Conjecture 4 for the case when $q \geq k-t+2$.

Theorem 9 ([12])**.** *Let $k > t \geq 1$, $n \geq 2k-t$, and $q \geq k-t+2$. Then*

$$m_q^{n,k}(t) = m_2^{n+k-1,k}(t). \tag{32}$$

We notice that the value of the RHS is given by Theorem 7. We also mention that (32) is not necessarily true if $q < k-t+2$, see the comment after the proof of Theorem 6. We will discuss some possible extensions of Theorem 9 below by considering intersecting families with weights.

Let

$$x_q^{n,k} := |X_q^{n,k}|.$$

For $B \subset 2^{[n]}$ we define its (k,q)-weight by

$$W_q^k(B) := \sum_{\mathbf{b} \in B} W_q^k(\mathbf{b}),$$

where

$$W_q^k(\mathbf{b}) := \#\{\mathbf{x} \in X_q^{n,k} : \mathrm{supp}(\mathbf{x}) = \mathbf{b}\},$$

or equivalently,

$$W_q^k(\mathbf{b}) := \begin{cases} x_{q-1}^{|\mathbf{b}|,k-|\mathbf{b}|} & \text{if } |\mathbf{b}| \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3 ([12])**.** *Let $k > t \geq 1$, $n \geq 2k-t$, and q be arbitrary. Then*

$$m_q^{n,k}(t) = \max\{W_q^k(B) : B \subset 2^{[n]} \text{ is } t\text{-intersecting}\}.$$

The above lemma is proved in [12] by using a variant of shifting technique called "down compression," which requires $n \geq 2k-t$. On the other hand, the following lemma holds for $n \geq k-t+1$. These two lemmas clearly imply Theorem 9.

Lemma 4. *Let $k > t \geq 1$, $n \geq k - t + 1$, and $q \geq k - t + 2$. Then*

$$\max\{W_q^k(B) : B \subset 2^{[n]} \text{ is } t\text{-intersecting}\} = m_2^{n+k-1,k}(t). \tag{33}$$

Proof. If $q \geq k - t + 2$, then for $\mathbf{b} \subset [n]$ with $t \leq |\mathbf{b}| \leq k$ it follows

$$W_q^k(\mathbf{b}) = x_{q-1}^{|\mathbf{b}|,k-|\mathbf{b}|} = \binom{k-1}{k-|\mathbf{b}|}.$$

For $B \subset 2^{[n]}$ we construct $\tilde{B} \subset \binom{[n+k-1]}{k}$ by

$$\tilde{B} := \left\{\mathbf{b} \cup \mathbf{b}' : \mathbf{b} \in B,\ |\mathbf{b}| \leq k,\ \mathbf{b}' \in \binom{[n+1, n+k-1]}{k-|\mathbf{b}|}\right\}. \tag{34}$$

We are only interested in $W_q^k(B)$ and so we can neglect any subset in B of size larger than k. Then we have

$$W_q^k(B) = \sum_{\mathbf{b}\in B} W_q^k(\mathbf{b}) = \sum_{\mathbf{b}\in B} \binom{k-1}{k-|\mathbf{b}|} = |\tilde{B}|. \tag{35}$$

Now suppose that $B \subset 2^{[n]}$ is t-intersecting. Then $\tilde{B} \subset \binom{[n+k-1]}{k}$ is t-intersecting, too. Thus we have

$$|\tilde{B}| \leq m_2^{n+k-1,k}(t). \tag{36}$$

Moreover it follows from $n \geq k - t + 1$ that $n + k - 1 \geq 2k - t$, and we can apply Theorem 7 to get

$$m_2^{n+k-1,k}(t) = \max_i |\operatorname{AK}(n + k - 1, k, t, i)|. \tag{37}$$

Therefore, by (35), (36) and (37), we have

$$W_q^k(B) = |\tilde{B}| \leq \max_i |\operatorname{AK}(n + k - 1, k, t, i)|.$$

On the other hand, if

$$B = B(n, t, i) := \{\mathbf{b} \subset [n] : |\mathbf{b} \cap [t + 2i]| \geq t + i\}, \tag{38}$$

then B is t-intersecting and

$$\begin{aligned} W_q^k(B) = |\tilde{B}| &= \#\left\{\mathbf{a} \in \binom{[n+k-1]}{k} : |\mathbf{a} \cap [t + 2i]| \geq t + i\right\} \\ &= |\operatorname{AK}(n + k - 1, k, t, i)|. \end{aligned} \tag{39}$$

This gives us that

$$\max_B W_q^k(B) \geq \max_i |\operatorname{AK}(n + k - 1, k, t, i)|,$$

which completes the proof. □

We remark that the proof above shows that the LHS of (33) is attained by one of $B(n, t, i)$, namely, (33) reads

$$m_2^{n+k-1,k}(t) = \max_i W_q^k(B(n, k, i)). \tag{40}$$

Let us define a family in $X_q^{n,k}$ corresponding to (38). So let

$$\begin{aligned} A_q(n,k,t,i) &:= \mathcal{D}(\mathcal{S}_q(\tilde{\mathbf{e}}_{t+2i}, k-t-i)) \cap X_q^{n,k} \\ &= \{\mathbf{x} \in X_q^{n,k} : \operatorname{supp}(\mathbf{x}) \in B(n,t,i)\}. \end{aligned}$$

Then this family is t-intersecting with

$$|A_q(n,k,t,i)| = W_q^k(B(n,t,i)). \tag{41}$$

By (40) and (41) we see that (32) in Theorem 9 can be rewritten as

$$m_q^{n,k}(t) = \max\{|A_q(n,k,t,i)| : i = 0,1,\ldots,(k-t)/2\}, \tag{42}$$

and we can slightly extend Theorem 9 as follows.

Theorem 6. *Let $k > t \geq 1$, $n \geq 2k-t$, and $q \geq k-t+1$. Then (42) holds.*

Proof. Theorem 9 covers the cases $q \geq k-t+2$.

Let $q = k-t+1$. We use Lemma 3. So let $B \subset 2^{[n]}$ be t-intersecting with $W_q^k(B) = m_q^{n,k}(t)$. We may assume that B is shifted, that is, if $\mathbf{b} \in B$ and $\{i,j\} \cap \mathbf{b} = \{j\}$ for some $1 \leq i < j \leq n$, then $(\mathbf{b} \setminus \{j\}) \cup \{i\} \in B$. (For more details about shifting, see, e.g., [9]).

If $\bigcap B = [t]$, then we have $B = B(n,t,0)$ and

$$\begin{aligned} W_q^k(B) &= W_q^k(B(n,t,0)) \\ &= \#\big\{(x_1,\ldots,x_n) \in X_q^n : x_1 + \cdots + x_n = k-t\big\} \\ &= \binom{n-1+k-t}{k-t} - t, \end{aligned} \tag{43}$$

where the $-t$ in the last term comes from the vectors of type

$$\tilde{\mathbf{e}}_t + (k-t)\mathbf{e}_i \in X_{k-t+2}^{n,k} \setminus X_{k-t+1}^{n,k}$$

for $i = 1,\ldots,t$.

Now suppose that $\bigcap B \neq [t]$. Then $|\mathbf{b}| \geq t+1$ holds for all $\mathbf{b} \in B$. Define $\tilde{B}$ by (34). Since $|\mathbf{b}| \geq t+1$ it follows that $W_q^k(\mathbf{b}) = \binom{k-1}{k-|\mathbf{b}|}$ and

$$W_q^k(B) = |\tilde{B}|. \tag{44}$$

Since $\tilde{B} \subset \binom{[n+k-1]}{k}$ is non-trivially t-intersecting, we can apply a result of Ahlswede and Khachatrian from [1] with (39) to get

$$|\tilde{B}| \leq \max\{b, |C|\}, \tag{45}$$

where

$$b := \max\{|\operatorname{AK}(n+k-1,k,t,i)| = W_q^k(B(n,t,i)) : i \geq 1\},$$

and

$$C := \left\{\mathbf{c} \in \binom{[n+k-1]}{k} : [t] \subset \mathbf{c},\ \mathbf{c} \cap [t+1,k+1] \neq \emptyset\right\} \cup \big\{[k+1] \setminus \{i\} : 1 \leq i \leq t\big\}.$$

It is also known that $b \le |C|$ only if $n+k-1 > (t+1)(k-t+1)$ and $k > 2t+1$. But in this case direct computation shows that

$$\begin{aligned}|C| &= \binom{n+k-t-1}{k-t} - \binom{(n+k-1)-(k+1)}{k-t} + t \\ &< \binom{n+k-t-1}{k-t} - t = W_q^k(B(n,t,0)).\end{aligned}$$

This together with (44) and (45) yields

$$W_q^k(B) \le \max_{i \ge 0} W_q^k(B(n,t,i)). \tag{46}$$

Consequently using (41), (43) and (46) we get (42). □

As promised after stating Theorem 9 we give an example that does not satisfy (32). Let k and t be fixed, and let $q = k-t+1$. Suppose that n is large enough so that the RHS of (42) is attained by $A_q(n,k,t,0)$. Then as in the proof of Theorem 6 it follows that

$$m_q^{n,k}(t) = |A_q(n,k,t,0)| = \binom{n-1+k-t}{k-t} - t.$$

On the other hand, if $n+k-1 \ge (t+1)(k-t+1)$, then $m_2^{n+k-1,k}(t) = \binom{n-1+k-t}{k-t}$. Thus we have $m_2^{n+k-1,k}(t) = m_q^{n,k}(t) - t$ provided $q = k-t+1$ and n large enough. So, (32) fails in this situation.

5. Kruskal–Katona type problem

Recall that

$$\begin{aligned}X_q &= \{0,1,\dots,q-1\},\\ X_q^n &= \{(x_1,\dots,x_n) : x_i \in X_q\},\\ X_q^{n,k} &= \{\mathbf{x} \in X_q^n : |\mathbf{x}| = k\},\end{aligned}$$

where $|\mathbf{x}| = x_1 + \cdots + x_n$ for $\mathbf{x} \in X_q^n$. Let $[n] := \{1,2,\dots,n\}$. Let $[n]^k$ denote the set of non-decreasing sequences of length k, that is,

$$[n]^k := \{(\epsilon_1,\epsilon_2,\dots,\epsilon_k) : 1 \le \epsilon_1 \le \epsilon_2 \le \cdots \le \epsilon_k\}.$$

This can be viewed as a family of multisets, and let $[n]_q^k$ be the subfamily of $[n]^k$ with multiplicity (or repetition) at most q, that is, if $\epsilon = (\epsilon_1,\epsilon_2,\dots,\epsilon_k) \in [n]^k$, then $\epsilon \in [n]_q^k$ iff

$$\max\{j : \epsilon_i = \epsilon_{i+1} = \cdots = \epsilon_{i+j} \text{ for some } 1 \le i \le n\} \le q.$$

We identify $X_q^{n,k}$ and $[n]_q^k$ in the obvious way, for example,

$$(3,0,1,2) \in X_3^{4,6} \text{ and } (1,1,1,3,4,4) \in [4]_3^6$$

are corresponding. Clearly $[n]_q^k = [n]_k^k$ for all $q \ge k$.

We introduce a partial order (colex order) in $[n]_q^k$ as follows. For distinct two elements $\alpha = (\alpha_1,\dots,\alpha_k)$ and $\beta = (\beta_1,\dots,\beta_k)$ in $[n]_q^k$, we define $\alpha \prec \beta$ iff there is some i such that $\alpha_i < \beta_i$ and $\alpha_j = \beta_j$ for all $i < j \le k$. Let colex$(m, [n]_q^k)$ denote the first m elements in $[n]_q^k$ with respect to the colex order.

For $l < k$ and $A \subset X_q^{n,k}$ we define the l-th shadow $\Delta_l(A)$ of A by

$$\Delta_l(A) := \mathcal{D}(A) \cap X_q^{n,l}.$$

Conjecture 5. *For $l < k$ and $A \subset X_q^{n,k}$ with $|A| = m$, we have*

$$|\Delta_l(A)| \ge |\Delta_l(\mathrm{colex}(m, [n]_q^k))|.$$

This is known to be true when $q = 2$ by Kruskal [15] and Katona [14], and $q \ge k$ by Clements and Lindström [6]. The Kruskal–Katona theorem has played an important role for solving many intersection problems in $2^{[n]}$, see, e.g., [11]. The authors believe that understanding shadows in $[n]_q^k$ would also be very useful for attacking Problem 1 and other extremal problems in the q-ary cube.

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