A NOTE ON HUANG–ZHAO THEOREM ON INTERSECTING FAMILIES WITH LARGE MINIMUM DEGREE

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Abstract. Using the linear algebra method Huang and Zhao proved that if $n > 2k$ and $\mathcal{F}$ is an intersecting n-vertex k-uniform hypergraph with minimum degree at least $\binom{n-2}{k-2}$, then $\mathcal{F}$ is the star. In this note we present an elementary, combinatorial proof of this result for the case $n \geq 3k$. We also prove a vector space version of the Huang–Zhao result along the same line as their proof.

1. Introduction

For a positive integer n let $[n] = \{1, 2, \ldots, n\}$. By an n-vertex k-uniform family $\mathcal{F}$ we mean $\mathcal{F} \subset \binom{[n]}{k}$. Let $\deg_{\mathcal{F}}(i) := \#\{F \in \mathcal{F} : i \in F\}$ denote the degree of $i \in [n]$ in $\mathcal{F}$, and let $\delta(\mathcal{F}) := \min\{\deg_{\mathcal{F}}(i) : i \in [n]\}$ denote the minimum degree of $\mathcal{F}$. We say that $\mathcal{F}$ is intersecting if $F \cap F' \neq \emptyset$ for all $F, F' \in \mathcal{F}$. Define a star centered at i by

$$\mathcal{S}_k^n(i) := \{S \in \binom{[n]}{k} : i \in S\}.$$

Then $\mathcal{S}_k^n(i)$ is an intersecting family with $|\mathcal{S}_k^n(i)| = \binom{n-1}{k-1}$ and $\delta(\mathcal{S}_k^n(i)) = \binom{n-2}{k-2}$. The Erdős–Ko–Rado theorem states that if $n \geq 2k+1$ and $\mathcal{F}$ is an intersecting n-vertex k-uniform family, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality holding if and only if $\mathcal{F} = \mathcal{S}_k^n(i)$ for some $i \in [n]$. Recently Huang and Zhao proved the following.

Theorem 1 ([3])**.** *Let $n \geq 2k+1$ and let $\mathcal{F}$ be an intersecting n-vertex k-uniform family with $\delta(\mathcal{F}) \geq \binom{n-2}{k-2}$. Then $\mathcal{F} = \mathcal{S}_k^n(i)$ for some $i \in [n]$.*

Their beautiful proof is based on analysis of eigenvalues of the Kneser graph. They also use the following result from discrete geometry.

Lemma 1 ([3])**.** *Let $a, b \in \mathbb{R}$ with $a > 0$. Suppose that $u_1, \ldots, u_n \in \mathbb{R}^{n-1}$ satisfy*

$$\langle u_i, u_j \rangle = \begin{cases} a & \text{if } i = j, \\ b & \text{if } i \neq j, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product. Then for every $v \in \mathbb{R}^{n-1}$ there exists i such that

$$\langle v, u_i \rangle \leq \frac{-1}{n-1} \sqrt{\langle v, v \rangle} \sqrt{\langle u_i, u_i \rangle}.$$

In this note we present a completely different proof of Theorem 1 for the case $n \geq 3k$, which is elementary, and purely combinatorial. Our proof is based on a result concerning the size of intersecting families with maximum degree constraint, see

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Theorem 4 in the next section. We also present a vector space version of Theorem 1, whose proof is along the same lines as in [3]. To state our result we need some definitions. Let V_n denote an n-dimensional vector space over $\mathbb{F}_q$ (the q-element field). We say that a family of k-dimensional subspaces $H \subset \begin{bmatrix} V_n \\ k \end{bmatrix}$ is intersecting if $\dim(h \cap h') \geq 1$ for all $h, h' \in H$. For a fixed line $l \in \begin{bmatrix} V_n \\ 1 \end{bmatrix}$ we define a star centered at l by

$$S_k^n[l] := \{h \in \begin{bmatrix} V_n \\ k \end{bmatrix} : l \leq h\},$$

namely, $S_k^n[l]$ is the family of all k-dimensional subspaces containing l as a subspace.

Theorem 2. *Let $n \geq 2k$ and let $H \subset \begin{bmatrix} V_n \\ k \end{bmatrix}$ be intersecting. Suppose that every line in V_n is contained (as a subspace) in at least $\begin{bmatrix} n-2 \\ k-2 \end{bmatrix}$ members of H, that is,*

$$\#\{h \in H : x \leq h\} \geq \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} \tag{1}$$

for every $x \in \begin{bmatrix} V_n \\ 1 \end{bmatrix}$. Then $H = S_k^n[l]$ for some $l \in \begin{bmatrix} V_n \\ 1 \end{bmatrix}$.

We invite the readers to find a purely combinatorial proof for all $n > 2k$ in the case of sets and possibly for vector spaces.

2. Proof of Theorem 1 for $n \geq 3k$

In this section we give an elementary proof of the following slightly weaker version of Theorem 1.

Theorem 3. *Let $n \geq 3k$ and let $\mathcal{F}$ be an intersecting n-vertex k-uniform family with $\delta(\mathcal{F}) \geq \binom{n-2}{k-2}$. Then $\mathcal{F} = \mathcal{S}_k^n(i)$ for some $i \in [n]$.*

Proof. Suppose that $\mathcal{F}$ satisfies all the assumptions in Theorem 3. If $\mathcal{F}$ is trivial, that is, $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$, then $\mathcal{F} \subset \mathcal{S}_k^n(i)$ for some $i \in [n]$. Since $\delta(\mathcal{F}) \geq \binom{n-2}{k-2} = \delta(\mathcal{S}_k^n(i))$ we have $\mathcal{F} = \mathcal{S}_k^n(i)$ as needed.

So suppose that $\mathcal{F}$ is non-trivial, that is, $\bigcap_{F \in \mathcal{F}} F = \emptyset$. We will show that $\mathcal{F}$ cannot satisfy some of the assumptions. We note that

$$\Delta(\mathcal{F}) \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1},$$

where $\Delta(\mathcal{F}) := \max\{\deg_{\mathcal{F}}(i) : i \in [n]\}$ denotes the maximum degree of $\mathcal{F}$. In fact, since $\mathcal{F}$ is non-trivial, for every $i \in [n]$ there is some $F \in \mathcal{F}$ such that $i \notin F$. Then $\{\{i\} \cup G : G \in \binom{[n] \setminus (\{i\} \cup F)}{k-1}\} \cap \mathcal{F} = \emptyset$ because $\mathcal{F}$ is intersecting, and this means that $\deg_{\mathcal{F}}(i) \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1}$. Let us recall the following old result.

Theorem 4 ([1])**.** *Let $\mathcal{F} \subset \binom{[n]}{k}$ be an intersecting family. Suppose that*

$$\Delta(\mathcal{F}) \leq \binom{n-1}{k-1} - \binom{n-j-1}{k-1} \tag{2}$$

for some j, $2 \leq j \leq k$, then it follows that

$$|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-j-1}{k-1} + \binom{n-j-1}{k-j}. \tag{3}$$

First suppose that (2) holds for $j = 2$. (This clearly includes the case when (2) holds for $j = 0, 1$.) Then, using $\binom{n-1}{k-1} = \binom{n-3}{k-1} + 2\binom{n-3}{k-2} + \binom{n-3}{k-3}$, (3) reads

$$\begin{aligned} |\mathcal{F}| &\le \binom{n-1}{k-1} - \binom{n-3}{k-1} + \binom{n-3}{k-2} = 3\binom{n-3}{k-2} + \binom{n-3}{k-3} \\ &= \binom{n-2}{k-2} + 2\binom{n-3}{k-2} < 3\binom{n-2}{k-2}, \end{aligned} \tag{4}$$

where the last inequality holds for $n > k$. On the other hand it follows that

$$k|\mathcal{F}| = \sum_{x\in[n]} \deg_{\mathcal{F}}(x) \ge n\delta(\mathcal{F}) \ge n\binom{n-2}{k-2}. \tag{5}$$

By (4) and (5) we get

$$3k\binom{n-2}{k-2} > k|\mathcal{F}| \ge n\binom{n-2}{k-2},$$

which implies $3k > n$, a contradiction.

Next suppose that (2) holds for some j, $3 \le j \le k$. Let j be the smallest value of j such that (2) holds. (We may assume that $j \ge 3$.) Then $\binom{n-1}{k-1} - \binom{n-j}{k-1} < \Delta(\mathcal{F})$ implies $|\mathcal{F}| - \Delta(\mathcal{F}) < \binom{n-j-1}{k-2} + \binom{n-j-1}{k-j}$. Without loss of generality we may assume that $\deg_{\mathcal{F}}(n) = \Delta(\mathcal{F}) > |\mathcal{F}| - \binom{n-j-1}{k-j} - \binom{n-j-1}{k-2}$. This yields that

$$\begin{aligned} k|\mathcal{F}| &= \sum_{i\in[n]} \deg_{\mathcal{F}}(i) = \deg_{\mathcal{F}}(n) + \sum_{i\in[n-1]} \deg_{\mathcal{F}}(i) \ge \deg_{\mathcal{F}}(n) + (n-1)\delta(\mathcal{F}) \\ &> |\mathcal{F}| - \binom{n-j-1}{k-j} - \binom{n-j-1}{k-2} + (n-1)\binom{n-2}{k-2}. \end{aligned}$$

Using $(n-1)\binom{n-2}{k-2} = (k-1)\binom{n-1}{k-1}$ and rearranging we get

$$\binom{n-j-1}{k-j} + \binom{n-j-1}{k-2} > (k-1)\left(\binom{n-1}{k-1} - |\mathcal{F}|\right).$$

This together with (3) gives us

$$\binom{n-j-1}{k-j} + \binom{n-j-1}{k-2} > (k-1)\left(\binom{n-j-1}{k-1} - \binom{n-j-1}{k-j}\right),$$

that is,

$$k\binom{n-j-1}{k-j} > (k-1)\binom{n-j-1}{k-1} - \binom{n-j-1}{k-2} \ge (k-2)\binom{n-j-1}{k-1}, \tag{6}$$

where we need $n \ge 2k + j - 2$ in the last inequality (and this follows from our assumptions $n \ge 3k$ and $k \ge j$). Since $j \ge 3$ we have $k - 2 \ge k - j + 1$, and we deduce from (6) that

$$k\binom{n-j-1}{k-j} > (k-j+1)\binom{n-j-1}{k-1}.$$

Multiplying both sides by $(n-j)/(k(k-j+1))$, we finally get

$$\binom{n-j}{k-j+1} > \binom{n-j}{k},$$

or equivalently, $2k+1>n$, which contradicts our assumption. $\square$

3. Proof of Theorem 2

Proof. Let G be the q-Kneser graph defined by $V(G) = \begin{bmatrix} V_n \\ k \end{bmatrix}$ and $xy \in E(G)$ iff $x \cap y = \{\mathbf{0}\}$. Let A be q^{-k^2} times the adjacency matrix of G. Then it is known (e.g., [2]) that A has eigenvalues λ_s $(s = 0, 1, \ldots, k)$ with multiplicity $m_s := \begin{bmatrix} n \\ s \end{bmatrix} - \begin{bmatrix} n \\ s-1 \end{bmatrix}$, where

$$\lambda_s = (-1)^s q^{\binom{s}{2}-ks} \begin{bmatrix} n-k-s \\ k-s \end{bmatrix}.$$

Let E be the vector space of dimension $\begin{bmatrix} n \\ k \end{bmatrix}$ over $\mathbb{R}$ (with coordinates indexed by k-dimensional subspaces of V_n). Then E has an orthogonal decomposition $E = V_0 \oplus V_1 \oplus \cdots \oplus V_k$, where V_s is the eigenspace corresponding to λ_s.

For $s = 0$ we have $\lambda_0 = \begin{bmatrix} n-k \\ k \end{bmatrix}$, $m_0 = 1$, and the corresponding eigenspace V_0 is spanned by the unit length vector $v_1 := \mathbf{1}/\sqrt{\begin{bmatrix} n \\ k \end{bmatrix}}$, where $\mathbf{1} \in \mathbb{R}^{\begin{bmatrix} n \\ k \end{bmatrix}}$ denotes the all ones vector.

For $s = 1$ we have $\lambda_1 = -q^{-k}\begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}$, $m_1 = \begin{bmatrix} n \\ 1 \end{bmatrix} - 1$. Let $v_2, \ldots, v_{\begin{bmatrix} n \\ 1 \end{bmatrix}}$ be an orthonormal basis of V_1.

We remark that Hoffmans's ratio bound gives a sharp upper bound for the independence number of G, namely, if $n \geq 2k$, then

$$\alpha(G) \leq \frac{-\lambda_1}{\lambda_0 - \lambda_1} \begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

We label all lines in V_n, namely, let $\begin{bmatrix} V_n \\ 1 \end{bmatrix} = \{l_1, \ldots, l_{\begin{bmatrix} n \\ 1 \end{bmatrix}}\}$, and let g_i be the characteristic vector of the family of k-dimensional vector space (in V_n) containing l_i, in other words, g_i is corresponding to the star $S_k^n[l_i]$. Then g_i is contained in $V_0 \oplus V_1$ and one can write

$$g_i = \alpha_{i1} v_1 + \alpha_{i2} v_2 + \cdots + \alpha_{i\begin{bmatrix} n \\ 1 \end{bmatrix}} v_{\begin{bmatrix} n \\ 1 \end{bmatrix}}$$

for $i = 1, 2, \ldots, \begin{bmatrix} n \\ 1 \end{bmatrix}$. This yields that

$$\alpha_{i1} = \langle g_i, v_1 \rangle = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} / \sqrt{\begin{bmatrix} n \\ k \end{bmatrix}}. \tag{7}$$

We extend $v_1, \ldots, v_{\begin{bmatrix} n \\ 1 \end{bmatrix}}$ to get an orthonormal basis $v_1, \ldots, v_{\begin{bmatrix} n \\ k \end{bmatrix}}$ of E, where $v_{\begin{bmatrix} n \\ s-1 \end{bmatrix}+1}, \ldots, v_{\begin{bmatrix} n \\ s \end{bmatrix}}$ are the eigenvectors corresponding to λ_s. Let g_H be the characteristic vector of the family H, and we write

$$g_H = \sum_{j=1}^{\begin{bmatrix} n \\ k \end{bmatrix}} f_j v_j.$$

Let $e := |H|$ denote the number of edges in H. Then we have

$$f_1 = \langle g_H, v_1 \rangle = e/\sqrt{\left[{n \atop k}\right]}, \tag{8}$$

$$e = \langle g_H, g_H \rangle = \sum_{j=1}^{\left[{n \atop k}\right]} f_j^2.$$

By the assumption (1) it follows that $e\left[{k \atop 1}\right] \geq \left[{n \atop 1}\right]\left[{n-2 \atop k-2}\right]$, or equivalently,

$$e \geq \frac{q^n - 1}{q^k - 1}\begin{bmatrix} n-2 \\ k-2 \end{bmatrix} =: e_*.$$

Let $F := f_2^2 + \cdots + f_{\left[{n \atop 1}\right]}^2$. We will show the following two inequalities.

Claim 1. *We have that*

$$F \geq \frac{q^n - q}{(q-1)\left[{n \atop k}\right]}\, e\,(e - e_*)\,, \tag{9}$$

with equality holding only if $e - e^2/\left[{n \atop k}\right] - F = 0$, *and also that*

$$F \leq \frac{(q^k-1)(q^n-q)^2}{(q-1)^2(q^n-q^k)\left[{n \atop k}\right]}\,(e - e_*)^2\,. \tag{10}$$

By assuming the claim we can easily finish the proof of Theorem 2 as follows. By (9) and (10) we have either

$$e - e_* = 0 \tag{11}$$

or

$$e \leq (e - e_*)\,\frac{(q^k-1)(q^n-q)}{(q-1)(q^n-q^k)}. \tag{12}$$

In the case of (11) it follows from (9) and (10) that $F = 0$. Moreover equality holds in (9), and $e - e^2/\left[{n \atop k}\right] - F = 0$, that is, $e = \left[{n \atop k}\right]$. But (11) implies that $e = e_* = \frac{q^k-q}{q^n-q}\left[{n \atop k}\right] < \left[{n \atop k}\right]$, a contradiction. So only (12) can happen. In this case, after some computation, (12) yields $e \geq \left[{n-1 \atop k-1}\right]$. On the other hand it is known, e.g., [2], that if $n \geq 2k$, then the maximum size of intersecting families in $\left[{V_n \atop k}\right]$ is $\left[{n-1 \atop k-1}\right]$. Moreover if $n \geq 2k+1$ then the star $S_k^n[l]$ centered at some $l \in \left[{V_n \atop 1}\right]$ is the only optimal configuration. If $n = 2k$, then there are exactly two non-isomorphic optimal configurations; one is the star and the other is $\left[{Y \atop k}\right]$ for some $Y \in \left[{V_n \atop 2k-1}\right]$, see [4, 5, 6]. But the latter does not satisfy the assumption (1). Consequently the star is the only family that satisfies all the assumptions of Theorem 2.

Thus all we need to do is to prove Claim 1. Since H is intersecting, it follows that $\langle g_H, Ag_H\rangle = 0$. By expanding the LHS we get

$$\begin{aligned}
0 &= \langle \sum_{j=1}^{\left[{n \atop k}\right]} f_j v_j, \sum_{s=0}^{k} \lambda_s \sum_{j=\left[{n \atop s-1}\right]+1}^{\left[{n \atop s}\right]} f_j v_j \rangle \\
&= \sum_{s=0}^{k} \lambda_s \sum_{j=\left[{n \atop s-1}\right]+1}^{\left[{n \atop s}\right]} f_j^2 \\
&= \begin{bmatrix} n-k \\ k \end{bmatrix} f_1^2 - q^{-k} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} F + \sum_{s=2}^{k} \lambda_s \sum_{j=\left[{n \atop s-1}\right]+1}^{\left[{n \atop s}\right]} f_j^2 .
\end{aligned}$$

Using (8) we have that

$$0 = \begin{bmatrix} n-k \\ k \end{bmatrix} e^2 / \begin{bmatrix} n \\ k \end{bmatrix} - \frac{q^k - 1}{q^n - q^k} \begin{bmatrix} n-k \\ k \end{bmatrix} F + \sum_{s=2}^{k} \lambda_s \sum_{j=\left[{n \atop s-1}\right]+1}^{\left[{n \atop s}\right]} f_j^2 . \tag{13}$$

To estimate the last term of (13) we first note that

$$\sum_{s=2}^{k} \sum_{j=\left[{n \atop s-1}\right]+1}^{\left[{n \atop s}\right]} f_j^2 = \sum_{j=1}^{\left[{n \atop k}\right]} f_j^2 - f_1^2 - F = e - e^2/\left[{n \atop k}\right] - F. \tag{14}$$

We also have that if $s \geq 2$ then

$$\lambda_s \geq -q^{3-3k} \begin{bmatrix} n-k-3 \\ k-3 \end{bmatrix} > -\frac{q^k - q}{q^n - q^k} \begin{bmatrix} n-k \\ k \end{bmatrix}.$$

This together with (14) yields

$$\sum_{s=2}^{k} \lambda_s \sum_{j=\left[{n \atop s-1}\right]+1}^{\left[{n \atop s}\right]} f_j^2 \geq -\frac{q^k - q}{q^n - q^k} \begin{bmatrix} n-k \\ k \end{bmatrix} \left(e - e^2/\left[{n \atop k}\right] - F \right), \tag{15}$$

where equality holds only when $e - e^2/\left[{n \atop k}\right] - F = 0$. By (13) and (15) with some computation we get (9) with equality holding only if $e - e^2/\left[{n \atop k}\right] - F = 0$.

Our aim is to show that $g_H = g_i$ for some i (and $e = \left[{n-1 \atop k-1}\right]$). If this happens, then the LHS of (14) vanishes, and we get $F = e - e^2/\left[{n \atop k}\right] = \frac{q^n - q^k}{q^n - 1}\left[{n-1 \atop k-1}\right] =: F_*$. This value is useful to check the sharpness of (9) and (10). In fact if $e = \left[{n-1 \atop k-1}\right]$, then the RHS of both (9) and (10) coincides with F_*.

Next we prove (10). For $i = 1, 2, \ldots, \left[{n \atop 1}\right]$, let

$$u_i := (\alpha_{i2}, \alpha_{i3}, \ldots, \alpha_{i\left[{n \atop 1}\right]}) \in \mathbb{R}^{\left[{n \atop 1}\right]-1}.$$

We will verify that these vectors u_i satisfy the assumptions in Lemma 1, namely, we need to check that $\langle u_i, u_i\rangle$ is independent of i, and $\langle u_i, u_j\rangle$ $(i \neq j)$ is independent of the choice of i, j.

We have that

$$\begin{aligned}\langle u_i, u_i\rangle &= \alpha_{i2}^2 + \cdots + \alpha_{i{n \brack 1}}^2 \\ &= \sum_{j=1}^{{n \brack 1}} \alpha_{ij}^2 - \alpha_{i1}^2 \\ &= \langle g_i, g_i\rangle - \langle g_i, v_1\rangle^2 \\ &= \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^2 \Big/ \begin{bmatrix} n \\ k \end{bmatrix} \\ &= \frac{(q^k-1)(q^n-q^k)}{(q^n-1)^2} \begin{bmatrix} n \\ k \end{bmatrix}.\end{aligned}$$

Let $i \neq j$. Noting that

$$\langle g_i, g_j\rangle = \langle \sum \alpha_{il} v_l, \sum \alpha_{jl} v_l\rangle = \sum \alpha_{il}\alpha_{jl} = \alpha_{i1}\alpha_{j1} + \langle u_i, u_j\rangle$$

we have that

$$\langle u_i, u_j\rangle = \langle g_i, g_j\rangle - \alpha_{i1}\alpha_{j1} = \langle g_i, g_j\rangle - \langle g_i, v_1\rangle\langle g_j, v_1\rangle = \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} - \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^2 \Big/ \begin{bmatrix} n \\ k \end{bmatrix}.$$

Thus we can apply Lemma 1 to $v := (f_2, \dots, f_{{n \brack 1}})$, and there exists an i such that

$$\langle v, u_i\rangle \leq -\frac{1}{{n \brack 1} - 1}\sqrt{\langle u_i, u_i\rangle}\sqrt{\langle v, v\rangle}.$$

Recall that $\langle v, v\rangle = \sum_{j=2}^{{n \brack 1}} f_j^2 = F$. Then we can rewrite the inequality as

$$\sum_{j=2}^{{n \brack 1}} f_j \alpha_{ij} \leq -\frac{q-1}{q^n-q}\sqrt{\frac{(q^k-1)(q^n-q^k)}{(q^n-1)^2}\begin{bmatrix} n \\ k \end{bmatrix}}\sqrt{F}.$$

Note that the RHS is negative or zero. (So is the LHS.) Thus we obtain

$$F \leq \left(\sum_{j=2}^{{n \brack 1}} f_j \alpha_{ij}\right)^2 \Big/ \left(\frac{(q-1)^2(q^k-1)(q^n-q^k)}{(q^n-q)^2(q^n-1)^2}\begin{bmatrix} n \\ k \end{bmatrix}\right). \tag{16}$$

To estimate $\sum f_j \alpha_{ij}$ we first use the assumption (1) on the minimum degree. Then we have $\langle g_H, g_i\rangle \geq {n-2 \brack k-2}$. By expanding $\langle g_H, g_i\rangle = \langle \sum f_j v_j, \sum \alpha_{ij} v_j\rangle$ we get

$$\sum_{j=1}^{{n \brack k}} f_j \alpha_{ij} \geq \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}.$$

Next we use (7) and (8) to get $f_1\alpha_{i1} = e\left[{n-1 \atop k-1}\right] / \left[{n \atop k}\right] = \frac{q^k-1}{q^n-1}e$. Consequently we have

$$\sum_{j=2}^{\left[{n \atop 1}\right]} f_j\alpha_{ij} = \sum_{j=1}^{\left[{n \atop 1}\right]} f_j\alpha_{ij} - f_1\alpha_{i1} \geq \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} - \frac{q^k-1}{q^n-1}e = -\frac{q^k-1}{q^n-1}(e-e_*).$$

Substituting this into (16) we finally have

$$\begin{aligned} F &\leq \left(\frac{q^k-1}{q^n-1}(e-e_*)\right)^2 \Big/ \left(\frac{(q-1)^2(q^k-1)(q^n-q^k)}{(q^n-q)^2(q^n-1)^2}\begin{bmatrix} n \\ k \end{bmatrix}\right) \\ &= \frac{(q^k-1)(q^n-q)^2}{(q-1)^2(q^n-q^k)\left[{n \atop k}\right]}(e-e_*)^2, \end{aligned}$$

which proves (10). This completes the proof of Claim 1 and Theorem 2. □

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