A proof of the Hilton–Milner Theorem without computation

by Peter Frankl, Rényi Institute, Budapest, Hungary

Abstract

Let $n \geq 2k \geq 4$ be integers and $\mathcal{F}$ a family of k-subsets of $\{1, 2, \ldots, n\}$. It is called *intersecting* if $F \cap F' \neq \emptyset$ for all $F, F' \in \mathcal{F}$. It is called *non-trivial* if $\bigcap\limits_{F \in \mathcal{F}} F = \emptyset$. Strengthening the famous Erdős–Ko–Rado Theorem Hilton and Milner proved that $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$ if $\mathcal{F}$ is non-trivial and intersecting. We provide a proof by injection of this result.

1 Introduction

Let $[n] = \{1, \ldots, n\}$ be the standard n-element set and $2^{[n]}$ its power set. Subsets $\mathcal{F} \subset 2^{[n]}$ are called families. For $i \in [n]$ we use the standard notations $\mathcal{F}(i) = \{F \setminus \{i\} : i \in F \in \mathcal{F}\}$ and $\mathcal{F}(\bar{i}) = \{F : i \notin F \in \mathcal{F}\}$. Note that

$$|\mathcal{F}| = |\mathcal{F}(i)| + |\mathcal{F}(\bar{i})|.$$

For a positive integer t the family $\mathcal{F}$ is said to be t-*intersecting* if $|F \cap F'| \geq t$ for all $F, F' \in \mathcal{F}$. For $t = 1$ we use the term *intersecting*.

Let us recall the definition of the $S_{i,j}$ shift, an important operation on families, discovered by Erdős, Ko and Rado [EKR].

Definition 1.1. For $1 \leq i < j \leq n$ and a family $\mathcal{F} \subset 2^{[n]}$ one defines $S_{i,j}(\mathcal{F}) = \big\{S_{i,j}(F) : F \in \mathcal{F}\big\}$ where

$$S_{i,j}(F) = \begin{cases} F' \overset{\text{def}}{=} (F \setminus \{j\}) \cup \{i\} & \text{if } j \in F,\ i \notin F \text{ and } F' \notin \mathcal{F}, \\ F & \text{otherwise.} \end{cases}$$

From the definition $\left|S_{i,j}(\mathcal{F})\right| = |\mathcal{F}|$ and $\left|S_{i,j}(F)\right| = |F|$ should be obvious. More importantly, if $\mathcal{F}$ is t-intersecting then $S_{i,j}(\mathcal{F})$ is t-intersecting as well.

If $S_{i,j}(\mathcal{F}) = \mathcal{F}$ for all $1 \leq i < j \leq n$ then $\mathcal{F}$ is called *shifted*.

Let us use the notation $(a_1, a_2, \ldots, a_r)$ to denote the set $\{a_1, a_2, \ldots, a_r\}$ where $a_1 < a_2 < \cdots < a_r$. For two subsets $F = (a_1, \ldots, a_r)$ and $G = (b_1, \ldots, b_r)$ we say that F is smaller than G if $a_i \leq b_i$ for all $1 \leq i \leq r$. We denote this by $F \prec G$.

It is not hard to see that $\mathcal{F}$ is shifted iff for all pairs of F, G with $F \prec G$, $G \in \mathcal{F}$ implies $F \in \mathcal{F}$. For the proof of this and many other useful properties of shifting cf. [F87].

We shall need the following simple result.

Proposition 1.2 ([F78]). *$\mathcal{F} \subset 2^{[n]}$ be a shifted t-intersecting family. Then* (i) *and* (ii) *hold.*

(i) *For every $F \in \mathcal{F}$ there exists an integer $\ell \geq t$ such that*

$$|F \cap [2\ell - t]| \geq \ell.$$

(ii) *For all $F, G \in \mathcal{F}$ there exists an integer $h \geq t$ such that*

$$|F \cap [h]| + |G \cap [h]| \geq h + t. \tag{1.1}$$

Note that (1.1) *implies $|F \cap G \cap [h]| \geq t$.*

For $F \in \mathcal{F}$ define $\ell(F) = \left\{\max \ell, t \leq \ell \leq \frac{n}{2} : |F \cap [2\ell]| \geq \ell\right\}$. Note that if $2|F| \leq n$ then the maximality of $\ell(F)$ implies

$$|F \cap [2\ell(F)]| = \ell(F). \tag{1.2}$$

Let $k \geq s \geq 2$ be integers. Let $\binom{[n]}{k}$ denote the collection of all k-subsets of $[n]$.

Example 1.3. Define $\mathcal{E}(n, k, s) = \left\{E \in \binom{[n]}{k} : 1 \in E, E \cap [2, s+1] \neq \emptyset\right\} \cup \left\{F \subset \binom{[2n]}{k} : [2, s+1] \subset F\right\}$.

Note that $\mathcal{E}(n, k, s)$ is intersecting, $E \cap [2, s+1] \neq \emptyset$ for all $E \in \mathcal{E}(n, k, s)$ and

$$|\mathcal{E}(n, k, s)| = \binom{n-1}{k-1} - \binom{n-s-1}{k-1} + \binom{n-s-1}{k-s}.$$

Theorem 1.4. *Let $n \geq 2k \geq 2s \geq 4$. Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is a shifted intersecting family satisfying $F \cap [2, s+1] \neq \emptyset$ for all $F \in \mathcal{F}$. Then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-s-1}{k-1} + \binom{n-s-1}{k-s}. \tag{1.3}$$

This result is somewhat technical but its proof is rather special. We are going to prove it through an explicit injection from $\mathcal{F}$ into $\mathcal{E}(n, k, s)$.

For sets A, B let $A \triangle B$ denote their symmetric difference. Let us define the map $\alpha : \mathcal{F} \to \mathcal{E}(n, k, s)$ by

$$\alpha(F) = \begin{cases} F & \text{if } 1 \in F \text{ or if } [2, s+1] \subset F, \\ F \triangle [2\ell(F)] & \text{otherwise.} \end{cases}$$

To prove (1.3) it is sufficient to prove the following.

Proposition 1.5. *The map α is an injection into $\mathcal{E}(n, k, s)$.*

Let us recall two important results concerning intersecting families of k-sets.

Erdős–Ko–Rado Theorem ([EKR]). *Suppose that $n \geq 2k > 0$, $\mathcal{F} \subset \binom{[n]}{k}$ is an intersecting family. Then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}. \tag{1.4}$$

Taking all k-sets containing a fixed element shows that (1.4) is best possible.

An intersecting family is called *non-trivial* if there is no element common to all its members. For $k = 1$ there is no non-trivial k-intersecting family. For $k = 2$ the only such family is the triangle: $\binom{[3]}{2}$.

Hilton–Milner Theorem ([HM]). *Suppose that $n \geq 2k \geq 4$ and $\mathcal{F} \subset \binom{[n]}{k}$ is a non-trivial intersecting family. Then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1. \tag{1.5}$$

Recently Hurlbert and Kamat [HK] gave an injective proof for (1.4). We extend their work by providing an injective proof for (1.5). For this we need the following proposition.

Proposition 1.6 ([F87]). *Suppose that $n \ge 2k \ge 4$, $\mathcal{F} \subset \binom{[n]}{k}$ is a non-trivial intersecting family of maximal size. Then there exists a non-trivial intersecting family $\widetilde{\mathcal{F}} \subset \binom{[n]}{k}$ such that $|\widetilde{\mathcal{F}}| = |\mathcal{F}|$ and $\widetilde{\mathcal{F}}$ is shifted.*

Once one has Proposition 1.6, to establish (1.5) is easy. One only needs to apply the case $s = k$ of Theorem 1.4 to the family $\widetilde{\mathcal{F}}$. Indeed, since $\widetilde{\mathcal{F}}$ is non-trivial and shifted, $[2, k+1] \in \widetilde{\mathcal{F}}$ and being intersecting $F \cap [2, k+1] \ne \emptyset$ holds for all $F \in \widetilde{\mathcal{F}}$.

Since the proof of Proposition 1.6 is quite short and somewhat hidden in [F87], we reproduce it in Section 2.

Let us mention that there are several other, known proofs of the Hilton–Milner Theorem, cf. [FF], [FT], [M] or [KZ].

2 The proof of Propositions 1.5 and 1.6

We divide the proof into two lemmas. The first shows that for $F \in \mathcal{F} \setminus \mathcal{E}(n, k, s)$ the image $\alpha(F)$ is in $\mathcal{E}(n, k, s) \setminus \mathcal{F}$.

The second shows that α is an injection.

Lemma 2.1. *Suppose that $F \in \mathcal{F}(\bar{1})$ and $[2, s+1] \not\subset F$. Then* (i), (ii) *and* (iii) *hold.*

(i) $1 \in \alpha(F)$;

(ii) $\alpha(F) \notin \mathcal{F}$;

(iii) $\alpha(F) \cap [2, s+1] \ne \emptyset$.

Proof. Recall that $\alpha(F) = F \triangle [2\ell(F)]$. As $1 \notin F$ implies $1 \in \alpha(F)$, (i) is true.

(ii) Suppose for contradiction that $\alpha(F) \in \mathcal{F}$. Apply Proposition 1.2 to F and $\alpha(F)$. By (1.2), $F \cap [2\ell(F)]$ and $\alpha(F) \cap [2\ell(F)]$ are complementary ℓ-element subsets of $[2\ell(F)]$. Consequently $h > 2\ell(F)$.

However, for $h \ge 2\ell$, $|F \cap [h]| = |\alpha(F) \cap [h]|$. Thus $2|F \cap [h]| \ge h + 1$ implies

$$|F \cap [h]| \ge (h+1)/2. \tag{2.1}$$

Thus for $h + 1$ as well

$$|F \cap [h+1]| \ge (h+1)/2$$

and we get a contradiction with the maximality of $\ell(F)$.

(iii) Define $i(F) = \min\{i : 2 \leq i \leq n,\ i \notin F\}$. As $\ell(F) \geq 2$, (1.2) implies $i(F) \leq 2\ell(F)$. Also, $[2, s+1] \not\subset F$ implies $i(F) \leq s+1$. Consequently $i(F) \in [2\ell(F)]$ and $i(F) \in [2, s+1]$ hold. Therefore $i(F) \in \alpha(F) \cap [2, s+1]$. □

Lemma 2.2. *For distinct $F, F' \in \mathcal{F} \setminus \mathcal{E}(n, k, s)$ $\alpha(F) \neq \alpha(F')$ holds.*

Proof. Since $F, F' \notin \mathcal{E}(n, k, s)$, $\alpha(F) = F \triangle [2\ell(F)]$ and $\alpha(F') = F' \triangle [2\ell(F')]$. If $\ell(F) = \ell(F')$ then $\alpha(F) \neq \alpha(F')$ is evident from $F \neq F'$.

By symmetry suppose $\ell(F) < \ell(F')$. The maximality of $\ell(F)$ implies $|F \cap [2\ell'(F)]| < \ell'(F)$. Using $|F \cap [2\ell(F)]| = \ell(F) = |\alpha(F) \cap [2\ell(F)]|$, $|\alpha(F) \cap 2\ell'(F)| < \ell'(F) = |\alpha(F') \cap [2\ell(F')]|$ follows. This proves $\alpha(F) \neq \alpha(F')$. □

Since $\alpha(F) = F$ for $F \in \mathcal{F} \cap \mathcal{E}(n, k, s)$, Lemmas 2.1 and 2.2 prove that α is an injection into $\mathcal{E}(n, k, s)$. □

The proof of Proposition 1.6. Starting with a non-trivial intersecting family $\mathcal{F} \subset \binom{[n]}{k}$ of maximal size we can keep on applying the S_{ij} shift for various pairs until we run into trouble. The possible trouble is that $S_{ij}(\mathcal{F})$ ceases to be non-trivial, i.e., all its members contain the element i. Then $\{i, j\} \cap F \neq \emptyset$ must hold for all $F \in \mathcal{F}$. By symmetry let $i = 1$, $j = 2$.

The maximality of $|\mathcal{F}|$ implies that *all* k-sets G with $\{1, 2\} \subset G \subset [n]$ are in $\mathcal{F}$. Therefore continuing with the $S_{a,b}$ shift for $3 \leq a < b \leq n$ will never produce a trivial intersecting family. Eventually we obtain a non-trivial intersecting family $\mathcal{H}$, $|\mathcal{H}| = |\mathcal{F}|$ such that $S_{a,b}(\mathcal{H}) = \mathcal{H}$ for all $3 \leq a < b \leq n$.

Consequently, both $\{1, 3, 4, \ldots, k+1\}$ and $\{2, 3, 4, \ldots, k+1\}$ are in $\mathcal{H}$. Since all $G \in \binom{[n]}{k}$ with $\{1, 2\} \subset G \subset [n]$ are unchanged under the shift $S_{a,b}$ for $3 \leq a < b \leq n$, we infer that $\binom{[k+1]}{k} \subset \mathcal{H}$.

Noting that $\binom{[k+1]}{k}$ is not affected by $S_{i,j}$ for $1 \leq i < j \leq n$, we can continue shifting and eventually obtain a shifted, non-trivial intersecting family of the same size. □

3 Concluding remarks

For a family $\mathcal{F} \subset 2^{[n]}$ let $\triangle(\mathcal{F})$ be its *maximum degree*, that is, $\max_i |\mathcal{F}(i)|$. Then $\varrho(\mathcal{F}) = |\mathcal{F}| - \triangle(\mathcal{F})$ is called the *diversity* of $\mathcal{F}$. With this terminology, for intersecting families $\mathcal{F}$, $\mathcal{F} \subset \binom{[n]}{k}$, $n \geq 2k$, the Hilton–Milner Theorem shows that $\varrho(\mathcal{F}) \geq 1$ implies $|\mathcal{F}| \leq |\mathcal{E}(n,k,k)| = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$.

In [F87a] the author proved that $\varrho(\mathcal{F}) \geq \binom{n-s-1}{k-s}$ $(3 \leq s \leq k)$ implies $|\mathcal{F}| \leq |\mathcal{E}(n,k,s)|$. Kupavskii and Zakharov [KZ] gave a new proof of this result. It would be desirable to have a proof by injection. Let us note that for $\mathcal{F} \subset \mathcal{G}$ necessarily $\varrho(\mathcal{F}) \leq \varrho(\mathcal{G})$ holds.

In the case of Theorem 1.4, we may replace $\mathcal{F}$ by another family $\mathcal{G}$, $\mathcal{F} \subset \mathcal{G} \subset \binom{[n]}{k}$ where $\mathcal{G}$ is shifted, intersecting and all $G \in \binom{[n]}{k}$ with $[2, s+1] \subset G$ are members of $\mathcal{G}$. For such a special case Theorem 1.4 provides an injective proof. However the general case seems to be harder.

The proofs in [F87a] and [KZ] rely heavily on the Kruskal–Katona Theorem (cf. [Kr], [Ka]). Therefore we feel that it would be desirable to have a proof by injection for this important result as well.

References

[EKR] P. Erdős, C. Ko, and R. Rado, Intersection theorems for systems of finite sets, *Quart. J. Math. Oxford Second Series* **12** (1961), 313–320.

[F78] P. Frankl, The Erdős–Ko–Rado Theorem is true for $n = ckt$, *Combinatorics* (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. I, pp. 365–375, Colloq. Math. Soc. Jnos Bolyai **18**, North-Holland, Amsterdam–New York, 1978.

[F87] P. Frankl, The shifting technique in extremal set theory, *Surveys in combinatorics 1987*, pp. 81–110, London Math. Soc. Lecture Note Ser., 123, Cambridge Univ. Press, Cambridge, 1987.

[F87a] P. Frankl, Erdős–Ko–Rado theorem with conditions on the maximal degree, *J. Comb. Theory, Ser. A* **46** (2) (1987), 252–263.

[FF] P. Frankl, Z. Füredi, Non-trivial intersecting families, *J. Comb. Theory, Ser A* **41** (1986), 150–153.

[FT] P. Frankl, N. Tokushige, Some best possible inequalities concerning cross-intersecting families, *J. Comb. Theory, Ser. A* **61** (1) (1992), 87–97.

[HM] A. J. W. Hilton, E. C. Milner, Some intersection theorems for systems of finite sets, *Quart. J. Math. Oxford* **18** (1967), 369–384.

[HK] G. Hurlbert, V. Kamat, A new injective proof of the Erdős–Ko–Rado theorem, submitted.

[Ka] G. O. H. Katona, A theorem on finite sets, in: *Theory of Graphs, Proc. Colloq. Tihany*, pp. 187–207, Budapest, 1968.

[Kr] J. B. Kruskal, The number of simplices in a complex, in: *Math. Optimization Techniques*, pp. 251–278, California Press, Berkeley, 1963.

[KZ] A. Kupavskii, D. Zakharov, Regular bipartite graphs and intersecting families, to appear (in Russian).

[M] M. Mörs, A generalization of a theorem of Kruskal, *Graphs and Comb.* **1** (1985), 167–183.