Counting intersecting and pairs of cross-intersecting families

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Abstract

A family of subsets of $\{1,\ldots,n\}$ is called *intersecting* if any two of its sets intersect. A classical result in extremal combinatorics due to Erdős, Ko, and Rado determines the maximum size of an intersecting family of k-subsets of $\{1,\ldots,n\}$. In this paper we study the following problem: how many intersecting families of k-subsets of n are there? Improving the result of Balogh et al., we determine the asymptotic of this quantity for $n\ge 2k+2+2\sqrt{k\log k}$ and $k\to\infty$. Moreover, in the same assumptions we also determine the asymptotic of the number of *non-trivial* intersecting families, that the intersecting families in which the intersection of all sets is empty. We obtain analogous results for pairs of cross-intersecting families.

1 Introduction

Let $[n]=\{1,\ldots,n\}$ be the standard n-element set and for an integer $0\le k\le n$ let $\binom{[n]}{k}$ denote all its k-element subsets. Subsets $\mathcal{F}\subset\binom{[n]}{k}$ are called k-uniform families.

A family $\mathcal{F}$ is called *intersecting* if $F\cap F'\ne\emptyset$ holds for all $F,F'\in\mathcal{F}$. Similarly, if $\mathcal{F}\subset\binom{[n]}{k}$ and $\mathcal{G}\subset\binom{[n]}{l}$ then they are called *cross-intersecting* if for all $F\in\mathcal{F}$, $G\in\mathcal{G}$ one has $F\cap G\ne\emptyset$.

For $i\in[n]$ and $\mathcal{F}\subset\binom{[n]}{k}$ we use the standard notation

$$\mathcal{F}(i):=\{F-\{i\}: i\in F\in\mathcal{F}\}\subset\binom{[n]-\{i\}}{k-1}$$
$$\mathcal{F}(\bar{i}):=\{F\in\mathcal{F}: i\notin F\}\subset\binom{[n]-\{i\}}{k}.$$

Note that if $\mathcal{F}$ is intersecting then $\mathcal{F}(i)$ and $\mathcal{F}(\bar{i})$ are cross-intersecting.

The research concerning intersecting families was initiated by Erdős, Ko and Rado who determined the maximum size of intersecting families.

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Theorem A (Erdős, Ko, Rado [3]). *Suppose that $n \geq 2k > 0$ and $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting. Then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}. \tag{1}$$

the family of all k-sets containing a fixed element shows that (1) is best possible. Hilton and Milner proved in a stronger form that for $n > 2k$ these are the only families on which the equality is attained.

Theorem B (Hilton, Milner [7]). *Let $n > 2k > 0$ and suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting but $\bigcap_{F \in \mathcal{F}} F = \emptyset$. Then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1. \tag{2}$$

For $n = 2k+1$ the difference between the upper bounds is only $k-1$. However, as $n-2k$ increases, this difference gets much larger. Considering the number of subfamilies $2^{|\mathcal{F}|}$ of $\mathcal{F}$, the above difference turns into ratio and serves as an indication that most intersecting families are *trivial*, i.e., satisfy $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$.

In an important recent paper Balogh et al. [1] proved this in the following quantitative form. Let $I(n,k)$ denote the *total* number of intersecting families $\mathcal{F} \subset \binom{[n]}{k}$.

Theorem C (Balogh et al [1]). *If $n \geq 3k + 8\log k$ then*

$$I(n,k) = (n + o(1))2^{\binom{n-1}{k-1}}, \tag{3}$$

where $o(1) \to 0$ as $k \to \infty$.

One of the main tools of the proof of (3) is a nice bound on the number of *maximal* (i.e., non-extendable) intersecting families. We are going to explain it later. Let us simply point out that they obtain these bounds using the following fundamental result of Bollobás.

Theorem D (Bollobás [2]). *Suppose that $\mathcal{A} \subset \binom{[n]}{a}, \mathcal{B} \subset \binom{[n]}{b}$ with $\mathcal{A} = \{A_1, \ldots, A_m\}$, $\mathcal{B} = \{B_1, \ldots, B_m\}$ satisfy $A_i \cap B_j = \emptyset$ iff $i = j$. Then*

$$m \leq \binom{a+b}{a}. \tag{4}$$

Note that the bound (4) is independent of n. In [2] it is proved in a more general setting, not requiring uniformity, i.e., $\mathcal{A}, \mathcal{B} \subset 2^{[n]}$. The uniform version (4) was rediscovered several years later by Jaeger-Payan [8] and Katona [10].

We are going to use (4) to obtain an upper bound on the number of maximal pairs of cross-intersecting families. Let us denote by $CI(n,a,b,t)$ ($CI(n,a,b,[t_1,t_2])$) the *number of pairs of cross-intersecting families* $\mathcal{A} \subset \binom{[n]}{a}, \mathcal{B} \subset \binom{[n]}{b}$ and $|\mathcal{A}| = t$ ($t_1 \leq |\mathcal{A}| \leq t_2$). We also denote $CI(n,a,b) := \sum_t CI(n,a,b,t)$.

We prove the following bound for the number of pairs of cross-intersecting families:

Theorem 1. *Choose $a, b, n \in \mathbb{N}$ and put $c := \max\{a, b\}$, $T := \binom{n-a+b-1}{n-a}$. For $n \geq a + b + 2\sqrt{c \log c} + 2\max\{0, a - b\}$, $a, b \to \infty$, and $b \gg \log a$ we have*

$$CI(n, a, b) = (1 + \delta_{ab} + o(1))2^{\binom{n}{c}}, \tag{5}$$

$$CI(n, a, b, [1, T]) = (1 + o(1))\binom{n}{a}2^{\binom{n}{b}-\binom{n-a}{b}}, \tag{6}$$

where $\delta_{ab} = 1$ if $a = b$ and 0 otherwise.

Note that if $a = b$, then twice the right hand side of (6) gives the number of pairs of cross-intersecting families with both families being nonempty, and if $a < b$, then the RHS of (6) gives the number of cross-intersecting pairs with $\mathcal{A}$ being nonempty.

Let us show that (6) implies (5). Assuming for this paragraph that $c = b \geq a$, one notes that $CI(n, a, b) = \delta_{ab}(CI(n, a, b, 0) + CI(n, a, b, [1, T]))$ and that $CI(n, a, b, 0) = 2^{\binom{n}{b}}$. Thus, it is sufficient to show that the right hand side of (6) is $o(2^{\binom{n}{b}})$. It is done by a simple calculation:

$$\binom{n}{a}2^{-\binom{n-a}{b}} \leq 2^{n-\binom{b+\sqrt{n}}{b}} = o(1).$$

For a family $\mathcal{F} \subset \binom{[n]}{k}$ we define the *diversity* $\gamma(\mathcal{F})$ of $\mathcal{F}$ to be $|\mathcal{F}| - \Delta(\mathcal{F})$, where $\Delta(\mathcal{F}) := \max_{i\in[n]} \big|\{F : i \in F \in \mathcal{F}\}\big|$. For an integer t denote by $I(n, k, t)$ ($I(n, k, \geq t)$) the number of intersecting families with diversity t (at least t). In particular, $I(n, k, \geq 1)$ is the number of so-called *non-trivial* intersecting families, that is, families that cannot be pierced by a single point. With the help of (6) we obtain a refinement of Theorem C.

Theorem 2. *For $n \geq 2k + 2 + 2\sqrt{k \log k}$ and $k \to \infty$ we have*

$$I(n, k) = (n + o(1))2^{\binom{n-1}{k-1}}, \tag{7}$$

$$I(n, k, \geq 1) = (1 + o(1))n\binom{n-1}{k}2^{\binom{n-1}{k-1}-\binom{n-k-1}{k-1}}. \tag{8}$$

Again, it is easy to see that (8) implies (7). Indeed, in the assumptions of the theorem

$$n\binom{n-1}{k}2^{-\binom{n-k-1}{k-1}} \leq 2^{n+1-\left(\frac{n-k}{k}\right)^{k-1}} \leq 2^{n+1-2^{n-2k}} = o(1).$$

Therefore, $I(n, k, \geq 1) = o(2^{\binom{n-1}{k-1}})$. On the other hand, it is easy to see that $I(n, k, 0) = (n + o(1))2^{\binom{n-1}{k-1}}$ (for the easy proof cf. [1]).

In the next section we present the proof of (6), and in Section 3 we give the proof of (8).

2 Cross-intersecting families

Let us define the *lexicographic* order on the k-subsets of $[n]$. We have $F \prec G$ in the lexicographic order if $\min F \setminus G < \min G \setminus F$ holds. E.g., $\{1, 10\} \prec \{2, 3\}$. For $0 \le m \le \binom{n}{k}$ let $\mathcal{L}^{(k)}(m)$ denote the family of first m k-sets in the lexicographic order. E.g., $\mathcal{L}^{(k)}\left(\binom{n-1}{k-1}\right) = \{F \in \binom{[n]}{k} : 1 \in F\}$.

Next we state the Kruskal-Katona Theorem [11], [9], which is one of the most important results in extremal set theory.

Theorem E (Kruskal, [11], Katona, [9])**.** *If $\mathcal{A} \subset \binom{[n]}{a}$ and $\mathcal{B} \subset \binom{[n]}{b}$ are cross-intersecting then $\mathcal{L}^{(a)}(|\mathcal{A}|)$ and $\mathcal{L}^{(b)}(|\mathcal{B}|)$ are cross-intersecting as well.*

Computationwise, the bounds arising from the Kruskal-Katona Theorem are not easy to handle. Lovász [13] found a slightly weaker but very handy form, which may be stated as follows:

Theorem F (Lovász, [13])**.** *If $n \ge a + b$ and $\mathcal{A} \subset \binom{[n]}{a}$ and $\mathcal{B} \subset \binom{[n]}{b}$ are cross-intersecting, and $|\mathcal{A}| = \binom{x}{n-a}$ for a real number $x \ge n - a$, then*

$$|\mathcal{B}| \le \binom{n}{b} - \binom{x}{b} \qquad \textit{holds.} \tag{9}$$

Note that for $x \ge k - 1$ the polynomial $\binom{x}{k}$ is a monotone increasing function of x. Thus x is uniquely determined by $|\mathcal{A}|$ and a.

We would also need the following result, whose proof is based on Theorem E and which is a combination of a result of Frankl and Tokushige [5] (Theorem 2 in [5]) and two results of Kupavskii and Zaharov [12] (Part 1 of Theorem 1 and Corollary 1).

Theorem G (Frankl, Tokushige, [5], Kupavskii, Zakharov [12])**.** *Let $n > a + b$ and suppose that the families $\mathcal{A} \subset \binom{[n]}{a}, \mathcal{B} \subset \binom{[n]}{b}$ are cross-intersecting. If for some real number $\alpha \ge 1$ we have $\binom{n-\alpha}{n-a} \le |\mathcal{A}| \le \binom{n-a+b-1}{n-a}$, then*

$$|\mathcal{A}| + |\mathcal{B}| \le \binom{n}{b} + \binom{n-\alpha}{a-\alpha} - \binom{n-\alpha}{b}. \tag{10}$$

We go on to the proof of Theorem 1. First we discuss the proof of the lower bound in (5). To obtain that many pairs of intersecting families, take $\mathcal{A} := \{A\}, A \in \binom{[n]}{a}$ and $\mathcal{B}(A) := \{B \in \binom{[n]}{b} : B \cap A \ne \emptyset\}$. Next, choose an arbitrary subfamily $\mathcal{B} \subset \mathcal{B}(A)$.

The only thing one has to show is that few of these pairs of subfamilies correspond to two $\mathcal{B}(A)$ with different A. Actually, the number of pairs of such subfamilies is equal to $I(n, a, b, 2)$, and in the proof of the upper bound we show that $I(n, a, b, 2) = o(I(n, a, b, 1))$.

$\boldsymbol{2 \le |\mathcal{A}| \le n - a}$. Applying Theorem E, the size of the (unique) maximal family $\mathcal{B}'$ that forms a cross-intersecting pair with $\mathcal{A}$ is maximized if $\mathcal{A}$ consists of two sets A_1, A_2 that intersect in $a-1$ elements. Therefore, $|\mathcal{B}'| \le \binom{n}{b} - \binom{n-a+1}{b} + \binom{n-a-1}{b-2}$. Any other family $\mathcal{B}$ that forms a cross-intersecting pair with $\mathcal{A}$ must be a subfamily of $\mathcal{B}'$.

So we can bound the number of pairs of cross-intersecting families $\mathcal{A}, \mathcal{B}$ with $2 \le |\mathcal{A}| \le n-a$ as follows:

$$\frac{\sum_{t=2}^{n-a} CI(n,a,b,t)}{2^{\binom{n}{b}-\binom{n-a}{b}}} \le \sum_{t=2}^{n-k-1} \binom{\binom{n}{a}}{t} \frac{2^{\binom{n}{b}-\binom{n-a+1}{b}+\binom{n-a-1}{b-2}}}{2^{\binom{n}{b}-\binom{n-a}{b}}} \le 2^{n^2} 2^{-\binom{n-a-1}{b-1}} = o(1).$$

$\boldsymbol{n-a+1 \le |\mathcal{A}| \le \binom{n-u}{n-a}}$, where $u = \sqrt{c \log c} + \max\{0, a-b\}$. Note that $n-a+1 = \binom{n-a+1}{n-a}$. In this case the bound is similar, but we use Theorem F to bound the size of $|\mathcal{B}|$. For $\mathcal{A}$ with $|\mathcal{A}| = t := \binom{n-u'}{n-a}$, where $u \le u' \le a-1$, we get $|\mathcal{B}| \le 2^{\binom{n}{b}-\binom{n-u'}{b}}$, and since $\binom{\binom{n}{a}}{t} \le 2^n t$, we have the following bound:

$$\frac{CI(n,a,b,t)}{2^{\binom{n}{b}-\binom{n-a}{b}}} \le \binom{\binom{n}{a}}{t} \frac{2^{\binom{n}{b}-\binom{n-u'}{b}}}{2^{\binom{n}{b}-\binom{n-a}{b}}} \le 2^{n\binom{n-u'}{n-a}} 2^{-\binom{n-u'-1}{b-1}}.$$

At the same time we have $n \ge a+b+2u$ and

$$\begin{aligned}\frac{\binom{n-u'}{n-a}}{\binom{n-u'-1}{b-1}} = \frac{n-u'}{b} \prod_{i=0}^{n-a-b-1} \frac{n-b-u'-i}{n-a-i} \le \\ \frac{n}{b} \prod_{i=0}^{n-a-b-1} \frac{n-a-\sqrt{c\log c}-i}{n-a-i} \le \frac{n}{b} e^{-\sqrt{c \log c}\left(\sum_{i=b+1}^{n-a} \frac{1}{i}\right)} \le \frac{1}{2n} \qquad (11)\end{aligned}$$

for sufficiently large c. Indeed, $\sqrt{c\log c}\sum_{i=b+1}^{n-a} \frac{1}{i} \ge \sqrt{c \log c} \sum_{i=b+1}^{b+2\sqrt{c \log c}} \frac{1}{i} \ge (1+o(1)) \frac{2(\sqrt{c\log c})^2}{c} = (2+o(1)) \log c$, which justifies (11) for $n \le b^{3/2}$. For $n > b^{3/2}$ we have $\sqrt{c \log c} \sum_{i=b+1}^{n-a} \frac{1}{i} \ge (1+o(1)) \sqrt{c \log c} \log \frac{n}{b} \ge (1+o(1)) \sqrt{c \log c} \log n^{2/3} \gg \log n$, which justifies (11) for $n > b^{3/2}$.

We conclude that

$$\sum_{t=n-a+1}^{\binom{n-u}{n-a}} \frac{CI(n,a,b,t)}{2^{\binom{n}{b}-\binom{n-b}{b}}} \le 2^{-\frac{1}{2}\binom{n-u'-1}{b-1}} = o(1).$$

$\boldsymbol{\binom{n-u}{n-a} < |\mathcal{A}| \le T}$, where $u = \sqrt{b \log b} + \max\{0, a-b\}$. Using the Bollobas set-pair inequality, it is not difficult to obtain the following bound on the number of maximal pairs of cross-intersecting families.

Lemma 3. *The number of maximal cross-intersecting pairs $\mathcal{A}' \subset \binom{[n]}{a}, \mathcal{B}' \subset \binom{[n]}{b}$ is at most* $[\binom{n}{a}\binom{n}{b}]^{\binom{a+b}{a}}$.

We note that the proof is very similar to the proof of an analogous statement for intersecting families from [1].

Proof. Find a minimal $\mathcal{B}'$-*generating* family $\mathcal{M} \subset \mathcal{A}'$ such that $\mathcal{B}' = \{B \in \binom{[n]}{b} : B \cap M \neq \emptyset \text{ for all } M \in \mathcal{M}\}$. We claim that $|\mathcal{M}| \le \binom{a+b}{a}$. Indeed, due to minimality, for each set $M' \in \mathcal{M}$ the family $\mathcal{B}'' := \{B \in \binom{[n]}{b} : B \cap M \neq \emptyset \text{ for all } M \in \mathcal{M} - \{M'\}\}$ strictly contains $\mathcal{B}'$. Therefore, there is a set B in $\mathcal{B}'' \setminus \mathcal{B}'$ such that $B \cap M' = \emptyset, B \cap M \neq \emptyset$ for all $M \in \mathcal{M} - \{M'\}$. Applying the inequality (4) to $\mathcal{M}$ and the collection of such sets B, we get that $|\mathcal{M}| \le \binom{a+b}{b}$.

Interchanging the roles of $\mathcal{A}'$ and $\mathcal{B}'$, we get that a minimal $\mathcal{A}'$-generating family has size at most $\binom{a+b}{b}$ as well. Now the bound stated in the lemma is just a crude upper bound on the number of ways one can choose these two generating families out of $\binom{[n]}{a}$ and $\binom{[n]}{b}$, respectively. $\square$

Combined with the bound (10) on the size of any maximal pair of families with such cardinalities, we get that

$$\frac{CI(n,a,b,\ge \binom{n-u}{n-a})}{2^{\binom{n}{b}-\binom{n-a}{b}}} \le \left[\binom{n}{a}\binom{n}{b}\right]^{\binom{a+b}{a}} \frac{2^{\binom{n}{b}+\binom{n-u}{n-a}-\binom{n-u}{b}}}{2^{\binom{n}{b}-\binom{n-a}{b}}} \le 2^{2n\binom{a+b}{b}} 2^{\binom{n-u}{n-a}-\binom{n-u-1}{b-1}}. \tag{12}$$

We also have

$$\frac{\binom{a+b}{b}}{\binom{n-u-1}{b-1}} = \frac{n-u}{b}\prod_{i=0}^{b-1}\frac{a+b-i}{n-u-i} \le n\Big(\frac{a+b}{n-u}\Big)^b \le \frac{1}{4n}.$$

Indeed, the last inequality is clearly valid for $n \ge (a+b)^2, b \to \infty$. If $n < (a+b)^2$, then the before-last expression is at most

$$e^{\log n - \frac{b(n-a-b-u)}{n-u}} \le e^{2\log(a+b) - \frac{b(u+\max\{0,a-b\})}{O(a+b)}} \le e^{2\log(a+b) - \Omega(\min\{b,u\})}.$$

Since by the assumption we have $b \gg \log(a+b)$ and also, obviously, $u \gg \log(a+b)$, the last expression is at most $e^{-4\log(a+b)} < \frac{1}{4n}$.

Taking into account (11), which is valid for $u' = u$, we conclude that the right hand side of (12) is $o(1)$.

3 Intersecting families

We need a theorem due to Frankl [4], proved in the following, slightly stronger, form in [12].

Theorem H ([4, 12]). *Let $\mathcal{F} \subset \binom{[n]}{k}$ be an intersecting family, and $n > 2k$. Then, if $\gamma(\mathcal{F}) \ge \binom{n-u-1}{k-u}$ for some real $3 \le u \le k$, then*

$$|\mathcal{F}| \le \binom{n-1}{k-1} + \binom{n-u-1}{k-u} - \binom{n-u-1}{k-1}. \tag{13}$$

We go on to the proof Theorem 2. Let us first prove the lower bound. For $S \in \binom{[n]}{k}, i \in [n] \setminus S$ define the family $\mathcal{H}(i,S) := \{S\} \cup \{H \in \binom{[n]}{k} : i \in H, H \cap S \neq \emptyset\}$. Due to Theorem B, these families are the largest non-trivial intersecting families. We have $|\mathcal{H}(i,s)| = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$, and each such family contains no less than

$$2^{\binom{n-1}{k-1}-\binom{n-k-1}{k-1}} - k2^{\binom{n-2}{k-2}} = (1+o(1))2^{\binom{n-1}{k-1}-\binom{n-k-1}{k-1}} \tag{14}$$

non-trivial intersecting subfamilies, as $k \to \infty$. Indeed, a subfamily of $\mathcal{H}(i,S)$ containing S is non-trivial unless all sets containing i contain also a fixed $j \in S$. In other words, they must be a subset of a family $\mathcal{I}(i,j,S) := \{S\} \cup \{I \in \binom{[n]}{k} : i,j \in S\}$. The number of subfamilies of $\mathcal{I}(i,j,S)$ containing S is $2^{\binom{n-2}{k-2}}$. Next, we have $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} - \binom{n-2}{k-2} \geq \binom{n-3}{k-2}$, and thus the last inequality in the displayed formula above holds since $2^{\binom{n-3}{k-2}} \gg k$. Denote the set of all non-trivial subfamilies of $\mathcal{H}(i,S)$ by $\tilde{\mathcal{H}}(i,S)$.

Therefore, $\sum_{S\in\binom{n}{k}, i\notin S} |\tilde{\mathcal{H}}(i,S)| = (1+o(1))n\binom{n-1}{k}2^{\binom{n-1}{k-1}-\binom{n-k-1}{k-1}}$. On the other hand, the pairwise intersections of these families are small: the families from $\tilde{\mathcal{H}}(i,S) \cap \tilde{\mathcal{H}}(i,S')$ form the set $I(n,k,2)$, and we do (somewhat implicitly) show in the proof that $I(n,k,2) = o(I(n,k,1))$. It could also be verified by a simple direct, but somewhat tedious calculation. Therefore, the lower bound is justified.

Next we prove the upper bound. We count the number of families with different diversity separately. The number of families $\mathcal{F}$ with i being the most popular element and $\gamma(\mathcal{F}) \leq \binom{n-3}{k-2}$ is at most the number of cross-intersecting pairs $\mathcal{F}(\bar{i})$, $\mathcal{F}(i)$ (recall the definition from the introduction).

Therefore, we may apply (6) with $n' := n-1, a := k, b := k-1$, and get that the number of such families $\mathcal{F}$ is at most $(1+o(1))\binom{n-1}{k}2^{\binom{n-1}{k-1}-\binom{n-k-1}{k-1}}$. Note that $n' \geq a+b+2\sqrt{a\log a}+2$ and, in terms of Theorem 1, we have $T = \binom{n-3}{k-2}$ for our case. Multiplying the number of such families by the number of choices of i, we get the claimed asymptotic.

We are only left to prove that there are few families with diversity larger than $\binom{n-3}{k-2}$. Using the upper bound $\binom{n}{k}^{\binom{2k-1}{k-1}}$ for the number of maximal intersecting families in $\binom{[n]}{k}$ obtained in [1] (see Lemma 3 for the proof of a similar statement), combined with the bound (13) on the size of any maximal family with such diversity, we get that

$$\frac{I(n,k,\geq\binom{n-3}{k-2})}{2^{\binom{n-1}{k-1}-\binom{n-k-1}{k-1}}} \leq \binom{n}{k}^{\binom{2k-1}{k-1}} \frac{2^{\binom{n-1}{k-1}+\binom{n-4}{k-3}-\binom{n-4}{k-1}}}{2^{\binom{n-1}{k-1}-\binom{n-k-1}{k-1}}} \leq 2^{n\binom{2k-1}{k-1}}2^{\binom{n-4}{k-3}-\binom{n-5}{k-2}}. \tag{15}$$

Putting $n = 2k+x$, we have $\binom{n-4}{k-3}/\binom{n-5}{k-2} = \frac{(n-4)(k-2)}{(n-k-1)(n-k-2)} \leq \frac{(2k+x)k}{(k+x-2)^2} \leq 1 - \frac{x^2}{(k+x)^2} \leq 1 - \frac{1}{k}$. On the other hand,

$$\frac{\binom{2k-1}{k-1}}{\binom{n-5}{k-2}} = \frac{n-4}{k-1}\prod_{i=1}^{k}\frac{2k-i}{n-3-i} \leq n\Big(\frac{2k}{n-3}\Big)^k \leq \frac{1}{2kn},$$

where the last inequality is clearly valid for $n \geq 2k + 2 + 2\sqrt{k \log k}$ and sufficiently large k. We conclude that the right hand side of (15) is at most $2^{\frac{1}{2k}\binom{n-5}{k-2}} = o(1)$.

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