Families of sets with no matchings of sizes 3 and 4

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Abstract

In this paper we study a classical problem of extremal set theory, which asks for the maximum size of a family of subsets of $[n]$ such that s sets are pairwise disjoint. This problem was first posed by Erdős and resolved for $n \equiv 0, -1 \pmod s$ by Kleitman in the 60s. Very little progress was made on the problem until recently. The only result was a very lengthy resolution of the case $s = 3$, $n \equiv 1 \pmod 3$ by Quinn, which was written in his PhD thesis and never published in a refereed journal. In this paper we give another, much shorter proof of Quinn's result, as well as resolve the case $s = 4$, $n \equiv 2 \pmod 4$. This complements the results in our recent paper, where we resolve the case $n \equiv -2 \pmod s$ for $s \geq 5$.

1 Introduction

Let $[n] := \{1, 2, \ldots, n\}$ be the standard n-element set and $2^{[n]}$ its power set. A subset $\mathcal{F} \subset 2^{[n]}$ is called a *family*. For $0 \leq k \leq n$ let $\binom{[n]}{k}$ denote the family of all k-subsets of $[n]$.

For a family $\mathcal{F}$ let $\nu(\mathcal{F})$ denote the maximum number of pairwise disjoint members of $\mathcal{F}$. Note that $\nu(\mathcal{F}) \leq n$ holds unless $\emptyset \in \mathcal{F}$. The fundamental parameter $\nu(\mathcal{F})$ is called the *independence number* or *matching number*.

Denote the size of the largest family $\mathcal{F} \subset 2^{[n]}$ with $\nu(\mathcal{F}) < s$ by $e(n, s)$. The following classical result was obtained by Kleitman.

Kleitman Theorem ([5]) Let $s \geq 2, m \geq 1$ be integers. Then the following holds.

$$\text{For } n = sm - 1 \quad \text{we have} \quad e(n, s) = \sum_{m \leq t \leq n} \binom{n}{t}, \tag{1}$$

$$\text{for } n = sm \quad \text{we have} \quad e(n, s) = \frac{s-1}{s}\binom{n}{m} + \sum_{m+1 \leq t \leq n} \binom{n}{t}. \tag{2}$$

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The value $e(ms-1,s)$ is attained on the family of all sets of size greater then or equal to m. The corresponding example for (2) was given by Kleitman:

$$\mathcal{K} := \{K \subset [sm] : |K| \geq m+1\} \cup \left\{K \in \binom{[sm-1]}{m}\right\}.$$

(Note that $\binom{sm-1}{m} = \frac{s-1}{s}\binom{sm}{m}$.) Let us mention that in the case $s=2$ both bounds (1) and (2) reduce to $e(n,2) = 2^{n-1}$. This easy statement was proved already by Erdős, Ko and Rado [1].

Although (1) and (2) are beautiful results, for $s \geq 3$ they leave open the cases of $n \not\equiv 0$ or $-1 (\text{mod } s)$. For $s = 3$ the only remaining case was solved by Quinn [6]. However, his argument is very lengthy and was never published in a refereed journal. In this paper we reprove his result, as well as extend it to the case $n = 4m+2, s = 4$. This bridges the gap that was left between Quinn's result and the result of the paper [2], where we proved the same result for $n = sm+s-2, s \geq 5$.

Theorem 1. *Fix an integer $m \geq 1$. Then for $s = 3, 4$ and $n = sm+s-2$ we have*

$$e(n,s) = \binom{n-1}{m-1} + \sum_{m+1 \leq t \leq n} \binom{n}{t}. \tag{3}$$

The proof is based on a non-trivial averaging technique somewhat in the spirit of Katona's circle method [4]: we choose a relatively simple configuration of sets, show that the intersection of a family satisfying the conditions of Theorem 1 with *each* such configuration cannot be too large, and then average over all such configurations, which gives the result. However, the configuration is quite complicated, the sets in the configuration actually have weights, and to count the weighted intersection of the family with each configuration we use some kind of a discharging method.

We note that (2), along with more general statements, is proved using a simpler version of our technique in [3].

2 Preliminaries

Recall that $\mathcal{F}$ is called *closed upward* if for any $F \in \mathcal{F}$ all sets that contain F are also in $\mathcal{F}$. When dealing with families $\mathcal{F}$ with $\nu(\mathcal{F}) < s$ we may restrict our attention to the families that are closed upward, which we assume for the rest of the paper.

We are going to use the following inequality in the proofs:

$$\sum_{j=1}^{k-1} \binom{sm+s-2}{j} \leq \frac{1}{s-2}\binom{sm+s-2}{k} \quad \text{for any } k \leq m,\ s \geq 3. \tag{4}$$

Indeed, we have $\frac{\binom{sm+s-2}{k-j}}{\binom{sm+s-2}{k-j-1}} = \frac{sm+s-2-k+j+1}{k-j} \geq s-1$ for any $j \geq 0, k \leq m$, so by the formula for the summation of a geometric progression,

$$\sum_{j=1}^{k-1} \binom{sm+s-2}{j} \leq \frac{\frac{1}{s-1}}{1-\frac{1}{s-1}} \binom{sm+s-2}{k} = \frac{1}{s-2} \binom{sm+s-2}{k}.$$

3 Proof of Theorem 1 for $s = 3$

We first give a proof for the case when $m \geq 3$. Assume that $m \geq 3$ and put $n = 3m+1$ for this section. Consider a family $\mathcal{F} \subset 2^{[n]}$ with $\nu(\mathcal{F}) < 3$. Take an arbitrary permutation σ (assumed in what follows to be the identity permutation for simplicity) and fix some three disjoint m-element sets that form arcs in that permutation. This is what we call a *triple*. The x-triple is a triple of m-sets that do not contain the element x, $x = 1, \ldots, 3m+1$. It is clear that there is a one-to-one correspondence between the x's and the triples. For each triple we define three groups of sets of sizes $1, \ldots, m+3$ and assign them weights. By the abuse of notation we call this ensemble of sets a triple as well. Note that the arithmetic operations in the definitions of the sets are performed modulo n.

We define three groups of sets, indexed by $i = 0, 1, 2$. In what follows we define the i-th group. The i-th m-set $H_i^{(m)}(x)$ in the x-triple has the form $\{im+x+1, \ldots, im+x+m\}$. The set $H_i^{(m-j)}(x)$ of size $m-j$ for $j < m$ has the form $\{im+j+x+1, \ldots, im+x+m\}$. That is, it consists of the last $m-j$ elements of the m-set, if seen in the clockwise order. The sets $\emptyset \subset H_i^{(1)}(x) \subset H_i^{(2)}(x) \subset \cdots \subset H_i^{(m)}(x)$ form a full chain. The definition of the sets that have size $\geq m+1$ is less obvious, and we have more of them. Each of the $\geq m+1$ sets in the i-th group contains the corresponding m-set. The $m+1$-set

$$H_i^{(m+1)}(x;x) := H_i^{(m)}(x) \cup \{x\}$$

in a group i is called *central*. Note that the extra element it has is the element that was left out by the m-sets and so $H_i^{(m+1)}(x;x)$ is disjoint with the m-set from both of the j-th groups, $j \neq i$. The two others

$$H_i^{(m+1)}(j;x) := H_i^{(m)}(x) \cup \{mj+x+1\} \quad \text{for } j \in \{0,1,2\} - \{i\}$$

are called *side* and are disjoint with the corresponding $H_j^{(m-1)}(x)$ and with one of the m-sets. For each $j \in \{0,1,2\} - \{i\}$, we define two $m+2$-element sets: the *central* set

$$H_i^{(m+2)}(x,j;x) := H_i^{(m)}(x) \cup \{jm+x+1, x\}$$

and the *side* set

$$H_i^{(m+2)}(j,j;x) := H_i^{(m)}(x) \cup \{jm+x+2, jm+x+3\}.$$

The former ones are disjoint with the $m-1$-set from the group j and the m-set from the remaining group, while the latter ones are disjoint with the $m-3$-set from the corresponding group j. Finally, we have one $m+3$-set for each group:

$$H_i^{(m+3)}(x) := H_i^{(m)}(3m+1) \cup \{x+1, m+x+1, 2m+x+1, 3m+x+1\}.$$

It is disjoint with all $m-1$-sets from other groups.

Each set in each group gets some weight. We denote by w_k the weight of k-element sets, with possible superscripts s, c, corresponding to whether the set is side or central, respectively. Put

$$\alpha := \frac{(3m+2)m}{4(2m+3)(2m+1)}, \qquad \alpha' = 1 - 2\alpha. \tag{5}$$

The weights are as follows ($j \geq 0$):

$$\begin{aligned} w_{m-j} &:= \binom{n}{m-j}; & w_{m+1}^s &:= \alpha\binom{n}{m+1}; & w_{m+1}^c &:= \alpha'\binom{n}{m+1}; \\ w_{m+3} &:= \binom{n}{m+3}; & w_{m+2}^s &:= \frac{1}{8}\binom{n}{m+2}; & w_{m+2}^c &:= \frac{3}{8}\binom{n}{m+2}. \end{aligned} \tag{6}$$

Thus, the total weight of all sets in a triple is $3\sum_{k=1}^{m+3}\binom{n}{k}$. Now each set F that appears in several triples accumulates all the weight $w(F)$ that it was assigned. On Fig. 1 we listed all types of sets that are assigned non-zero weight with the corresponding weights. We recommend the reader to verify Fig. 1, since we will use its information later in the proof! The elements of the ground set are placed on the circle and the sets are represented modulo rotation. The family of all sets that got a nonzero weight for a given permutation σ we denote $\mathcal{G}(\sigma)$. Note that for each $j = 1, \ldots, m+3$ we have

$$\sum_{G \in \mathcal{G}(\sigma) \cap \binom{[n]}{j}} w(G) = 3n\binom{n}{j}.$$

To prove the theorem for $s = 3$, it is sufficient to show that for any σ we have

$$\sum_{F \in \mathcal{F} \cap \mathcal{G}(\sigma)} w(F) \leq 3n\left(\binom{n-1}{m-1} + \sum_{k=m+1}^{m+3}\binom{n}{k}\right) = 3m\binom{n}{m} + 3n\sum_{k=m+1}^{m+3}\binom{n}{k}. \tag{7}$$

For an event A, denote by $I[A]$ its indicator random variable. Indeed, if we take a permutation σ uniformly at random, then for each $j = 1, \ldots, m+3$ we have

$$\mathrm{E}_\sigma\left[\sum_{F \in \mathcal{F} \cap \mathcal{G}(\sigma) \cap \binom{[n]}{j}} w(F)\right] = \sum_{F \in \mathcal{F} \cap \binom{[n]}{j}} \mathrm{E}_\sigma\left[\bigcup_{G \in \mathcal{G}(\sigma) \cap \binom{[n]}{j}} I[F = G]w(G)\right] =$$

$$= \sum_{F \in \mathcal{F} \cap \binom{[n]}{j}} \sum_{G \in \mathcal{G}(\sigma) \cap \binom{[n]}{j}} \Pr[F = G]w(G) = \sum_{F \in \mathcal{F} \cap \binom{[n]}{j}} \sum_{G \in \mathcal{G}(\sigma) \cap \binom{[n]}{j}} \frac{w(G)}{\binom{n}{j}} =$$

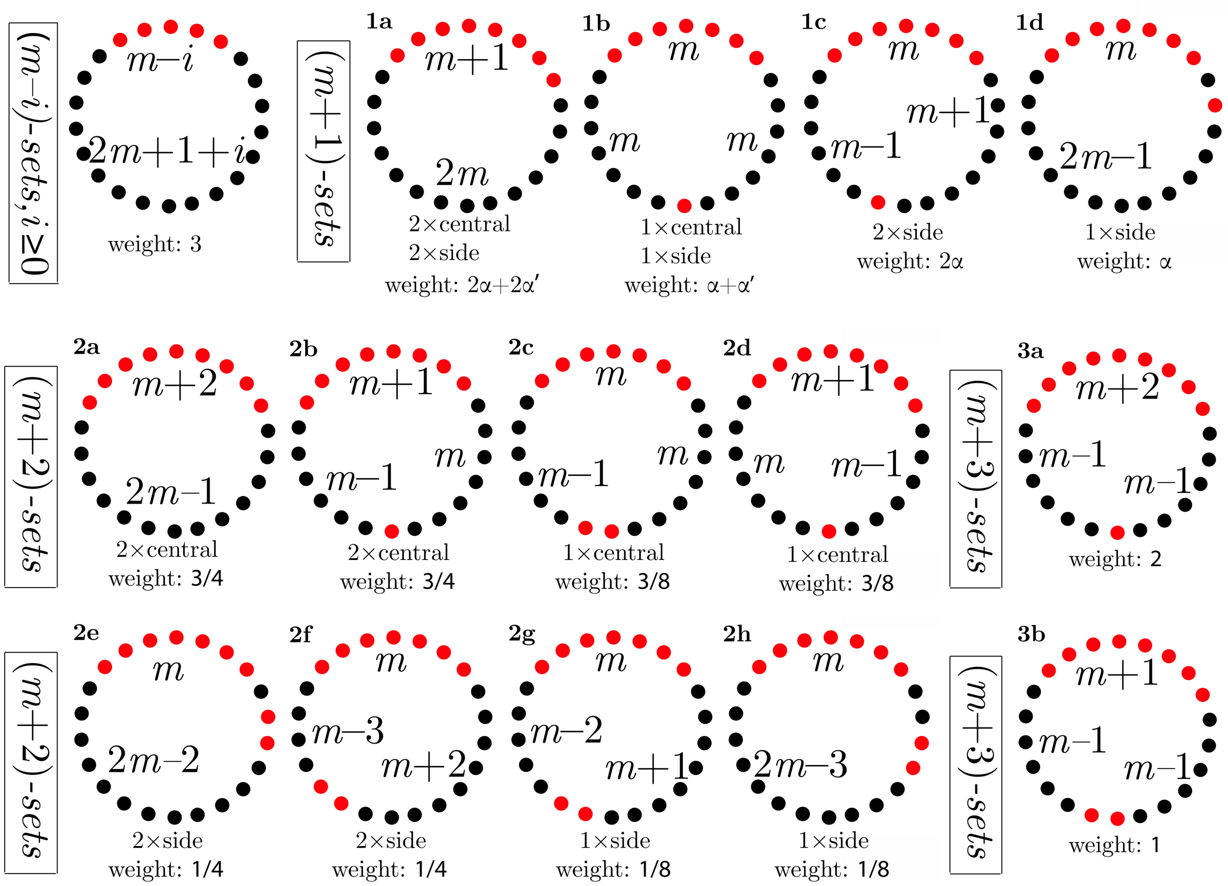

Fig. 1. All types of sets in $\mathcal{G}(\sigma)$ modulo rotation. The elements of the ground set are represented by circles, and the red circles represent the elements that are contained in the set. Inside the circles we specify the length of any interval of points that simultaneously either belong or do not belong to the set (excluding the intervals of length 1 and 2). The weights of k-sets are specified and divided by $\binom{n}{k}$ for shorthand. For $(m+1)$- and $(m+2)$-sets we also mention how many times a given set appears as a central and a side set.

$$= \sum_{F \in \mathcal{F} \cap \binom{[n]}{j}} 3n = 3n \left| \mathcal{F} \cap \binom{[n]}{j} \right|.$$

Therefore, (7) implies that

$$\sum_{j=1}^{m+3} 3n \left| \mathcal{F} \cap \binom{[n]}{j} \right| = \mathrm{E}_\sigma \left[\sum_{F \in \mathcal{F} \cap \mathcal{G}(\sigma)} w(F) \right] \leq 3m \binom{n}{m} + 3n \sum_{k=m+1}^{m+3} \binom{n}{k},$$

which implies the statement of the theorem for $s = 3$.

Our strategy to prove (7) is as follows. For a set $F \in \mathcal{G}(\sigma)$ we define the charge $c(F)$ to be equal to $w(F)$ if $F \in \mathcal{F}$, and to be 0 otherwise. Clearly, $\sum_{F \in \mathcal{G}(\sigma)} c(F) = \sum_{F \in \mathcal{F} \cap \mathcal{G}(\sigma)} w(F)$. If among the $(\leq m-1)$-sets in $\mathcal{G}(\sigma)$ there are no sets from $\mathcal{F}$, as well as there are at most

m m-sets, then we are done. Otherwise, having more $(\leq m)$-sets will result in certain larger sets not appearing in $\mathcal{F}$. Then we transfer (a part of) the charge of the $(\leq m)$-sets to the $(\geq m+1)$-sets that have zero charge. We show that the charge transferred to each $(\geq m+1)$-set is at most its weight. As a result of this procedure $(\leq m-1)$-sets will have zero total charge, the m-sets will have a total charge $3m\binom{n}{m}$, and each $(\geq m+1)$-set will have a charge not greater than its own weight. This will obviously conclude the proof of the theorem.

Next we design a charging scheme that satisfies the above requirements. For the sets of size $\leq m-1$ we transfer their charge triple by triple, assuring that the charge that we transferred to a bigger set in one triple is smaller than the weight that this bigger set got *from this triple*.

Stage 1. Transferring charge from the $(\leq \mathbf{m}-\mathbf{3})$-sets.
Assume that for some x and $i \in \{0,1,2\}$ the set $H_i^{(m-3)}(x)$ is in the x-triple. Choose j_1, j_2 such that $\{j_1, j_2, i\} = \{0,1,2\}$. Then one set from each of the two pairs $\big(H_{j_1}^{(m+2)}(i,i;x)$, $H_{j_2}^{(m+2)}(x,i;x)\big)$, $\big(H_{j_2}^{(m+2)}(i,i;x)$, $H_{j_1}^{(m+2)}(x,i;x)\big)$ is missing from $\mathcal{F}$. We transfer $\frac{1}{2}$ of the charge of the subsets $H_i^{(k)}(x)$, $k \leq m-3$, which is at most $\frac{1}{2}\sum_{k=1}^{m-3} w_k$, to each of these missing sets. Note again that we transfer only the weight got by the sets $H_i^{(k)}(x)$ in the x-triple. We have

$$\frac{1}{2}\sum_{k=1}^{t-3} w_k = \frac{1}{2}\sum_{k=1}^{m-3}\binom{n}{k} \overset{(4)}{\leq} \binom{n}{m-3} = \frac{\prod_{j=-2}^{2}(m+j)}{\prod_{j=0}^{4}(2m+j)}\binom{n}{m+2} < \frac{1}{16}\binom{n}{m+2}.$$

(We could have put $\frac{1}{32}$ instead of $\frac{1}{16}$, but it does not matter for the calculations.) Since each $m+2$-set in the x-triple may get this charge from each of the two groups to which it does not belong, the total charge transferred in that way is at most $\frac{1}{8}\binom{n}{m+2} \overset{(6)}{=} w_{m+2}^s$. The side $m+2$-sets are not going to get any more charge. As for the central $m+2$-sets, we have to make sure that they will get not more than $w_{m+2}^c - w_{m+2}^s \overset{(6)}{=} \frac{1}{4}\binom{n}{m+2}$ additional charge.

Stage 2. Transferring charge from pairs of $(\mathbf{m}-\mathbf{1})$-sets.
From now on we may assume that the charge of any $(\leq m-3)$-set is 0. Assume that for some x and some i, j_1, j_2, $\{j_1, j_2, i\} = \{0,1,2\}$, in the x-triple both $H_{j_1}^{(m-1)}(x)$ and $H_{j_2}^{(m-1)}(x)$ belong to $\mathcal{F}$. Then the set $H_i^{(m+3)}(x)$ is not in $\mathcal{F}$ and, consequently, has zero charge. Transfer the charge of the sets $H_{j_1}^{(k)}(x)$ and $H_{j_2}^{(k)}(x)$, $k = m-2, m-1$, to $H_i^{(m+3)}(x)$. The charge transferred is at most $2w_{m-2} + 2w_{m-1}$, which is

$$2\binom{n}{m-2} + 2\binom{n}{m-1} \leq \frac{3}{2}\binom{n}{m} = \frac{3(m+1)}{2(2m+1)}\binom{n}{m+1} \leq \binom{n}{m+1} \leq \binom{n}{m+3} \overset{(6)}{=} w_{m+3},$$

since $m \geq 3$. The $m+3$-sets are not going to get any more charge.

Stage 3. Transferring charge from pairs of $(\mathbf{m}-\mathbf{1})$- and $\mathbf{m}$-sets.
Assume that for some x and some i, j_1, j_2, $\{j_1, j_2, i\} = \{0,1,2\}$, in the x-triple both $H_{j_1}^{(m-1)}(x)$

and $H_{j_2}^{(m)}(x)$ belong to $\mathcal{F}$. Then the central $m+2$-set $H_i^{(m+2)}(x, j_1; x)$ is not in $\mathcal{F}$ and, consequently, has at most m_{m+2}^s charge in this triple (because of the charge possibly transferred on Stage 1). Transfer the charge of the sets $H_{j_1}^{(k)}(x)$, $k = m-2, m-1$, to $H_i^{(m+2)}(x, j_1; x)$. The charge transferred is at most $w_{m-2} + w_{m-1}$, which is

$$\binom{n}{m-2} + \binom{n}{m-1} = \frac{3m+2}{2m+3}\binom{n}{m-1} = \frac{3m+2}{2m+3}\prod_{j=0}^{2}\frac{m+j}{2m-j+2}\binom{n}{m+2} \leq \frac{1}{4}\binom{n}{m+2}.$$

The last inequality is valid for any $m \geq 1$ and is easy to verify by a direct calculation. The right hand side is exactly $w_{m+2}^c - w_{m+2}^s$, and so the total charge of the central $m+2$-sets is at most w_{m+2}^c in each triple. The $m+2$-sets are not going to get any more charge.

Stage 4. Transferring charge from single $(\mathbf{m-1})$-sets.
After the above redistribution of charges in each triple we have no $(\leq m-3)$-sets, at most one $m-2$ set and $m-1$-set with nonzero charges. Moreover, we do not have an $m-1$-set and two m-sets with nonzero charges in the same triple.

Assume that for some x and some i, j_1, j_2, $\{j_1, j_2, i\} = \{0, 1, 2\}$, in the x-triple the set $H_i^{(m-1)}(x)$ belongs to $\mathcal{F}$. Then one set from each of the two pairs $\big(H_{j_1}^{(m+1)}(x; x), H_{j_2}^{(m+1)}(i; x)\big)$, $\big(H_{j_2}^{(m+1)}(j_1; x), H_{j_1}^{(m+2)}(x; x)\big)$ is missing from $\mathcal{F}$. We transfer $\frac{1}{2}$ of the charge of the subsets $H_i^{(k)}(x)$, $k = m-2, m-1$, which is at most $\frac{1}{2}(w_{m-2} + w_{m-1})$, to each of these missing sets. We have

$$\frac{1}{2}(w_{m-2} + w_{m-1}) = \frac{1}{2}\binom{n}{m-2} + \frac{1}{2}\binom{n}{m-1} = \frac{(3m+2)m}{4(2m+3)(2m+1)}\binom{n}{m+1} \overset{(6)}{=} w_{m+1}^s. \quad (8)$$

Stage 5. Transferring charge from pairs of m-sets.
Denote the number of m-sets that have non-zero charge (that is, that are contained in $\mathcal{F} \cap \mathcal{G}(\sigma)$) by q. If $q \leq m$, then we are clearly done, since we have $\alpha' \geq \alpha$, and , therefore, $w_{m+1}^c > w_{m+1}^s$.

Assume that $q > m$. On the one hand, it makes an extra contribution $3(q-m)\binom{n}{m}$ to the left hand side of (7). On the other hand, the number of triples with two m-sets belonging to $\mathcal{F} \cap \mathcal{G}(\sigma)$ becomes non-zero. Indeed, if we denote by z_j the number of triples with j m-sets in the family, then $j \leq 2$ and we have $z_1 + 2z_2 = 3q$. Since $z_0 + z_1 + z_2 = n$, we have $z_2 \geq 3q - n$. We proceed as follows. Assume that for some x and some i, j_1, j_2, $\{j_1, j_2, i\} = \{0, 1, 2\}$, in the x-triple both the set $H_{j_1}^{(m)}(x)$ and $H_{j_2}^{(m)}(x)$ belong to $\mathcal{F}$. Then the central $m+1$-set $H_i^{(m+1)}(x; x)$ is not in the family. Moreover, there was no charge transferred to it in the x-triple, since we could not have two m-sets and an $m-1$ set with non-zero charges at the same time after Stage 3. We transfer $\frac{3(q-m)}{z_2}\binom{n}{m}$ charge to $H_i^{(m+1)}(x; x)$ from the two m-sets.

First note that we have transferred $z_2\frac{3(q-m)}{z_2}\binom{n}{m}$ charge from the m-sets to the central $m+1$-sets, which results in m-sets having total charge of $3m\binom{n}{m}$. This is exactly what we needed to have, and it is only left for us to verify that we do not have too much charge on the central $m+1$-sets. Unfortunately, we can run into problems in this situation, so we have to consider two cases. First, assume that $q \geq m+2$. Then the charge on each central set in each triple is at most

$$\frac{3(q-m)}{z_2}\binom{n}{m} \leq \frac{3(q-m)}{3q-n}\binom{n}{m} \leq \frac{6}{5}\binom{n}{m} = \frac{6(m+1)}{5(2m+1)}\binom{n}{m+1}. \tag{9}$$

The second inequality holds due to the fact that for $q \geq m+1$ the function $\frac{3(q-m)}{3q-n}$ decreases as q grows. Therefore, if $\alpha' = (1-2\alpha) \geq \frac{6(m+1)}{5(2m+1)}$, then we are done. Let us verify that (5) implies it. Adding to the right hand side of the inequality 2α, we get

$$\frac{6(m+1)}{5(2m+1)} + \frac{(3m+2)m}{2(2m+3)(2m+1)} = \frac{12(m+1)(2m+3)+5(3m+2)m}{10(2m+3)(2m+1)} =$$

$$= \frac{39m^2+70m+36}{40m^2+80m+30} < 1,$$

where the last inequality holds for any $m \geq 1$. Thus, we fulfilled all the requirements on the charging scheme and we are done in the case $q \geq m+2$.

In the case $q = m+1$, however, we are in trouble: the inequality $\alpha' \geq \frac{3(q-m)(m+1)}{z_2(2m+1)} = \frac{3(m+1)}{z_2(2m+1)}$ may not hold. We are fine if $z_2 \geq 3$, thus we only need to examine the case $z_2 = 2$, which we assume until the end of the section. The last equation says that there are exactly two triples with two m-sets from $\mathcal{F}$. This is the only case when we are not going to compare the amount of charge passed to the $m+1$-sets to the portion of its weight inside the triple. Instead, we compare it to the whole weight of the $m+1$-set.

We have two possible configurations in which $z_2 = 2$. One possibility is that we have two m-sets from $\mathcal{F}$ forming an interval of length $2m$ on the circle, and then the two triples that contribute to z_2 share the same two m-sets. In this case the central $m+1$-set that we forbid is the same in both triples, and it is of type 1a (see Fig. 1). Recall that this set has weight $2(\alpha+\alpha')\binom{n}{m+1}$. The other possibility is that we have two pairs of m-sets, with each pair separated on the two sides by a third m-set forming a triple with the pair, and by the element missing from the triple, respectively. In this case in each of the two triples we forbid a central $m+1$-set of type 2b. Each of these two sets (which are clearly different) has weight $(\alpha+\alpha')\binom{n}{m+1}$. In either case, we need to transfer $\frac{3(m+1)}{(2m+1)}\binom{n}{m+1}$ amount of charge from the m-sets to some of the $m+1$-sets.

Assume first that we do not have any $m-1$-element sets in $\mathcal{F} \cap \mathcal{G}(\sigma)$. Then in either of the possibilities described above the central $m+1$-sets have zero charge before Stage 5. Therefore, we are good if the weight of these (one or two) $m+1$-sets is greater than the amount of charge we transfer to them from the pairs of m-sets. Namely, we are good if

$$(\alpha+\alpha') = (2-2\alpha) \geq \frac{3(m+1)}{2m+1} \quad \Leftrightarrow \quad 2 \geq \frac{3(m+1)}{2m+1} + \frac{(3m+2)m}{2(2m+3)(2m+1)}. \tag{10}$$

We have

$$\frac{3(m+1)}{2m+1} + \frac{(3m+2)m}{2(2m+3)(2m+1)} = \frac{15m^2+32m+18}{8m^2+16m+6} < 2,$$

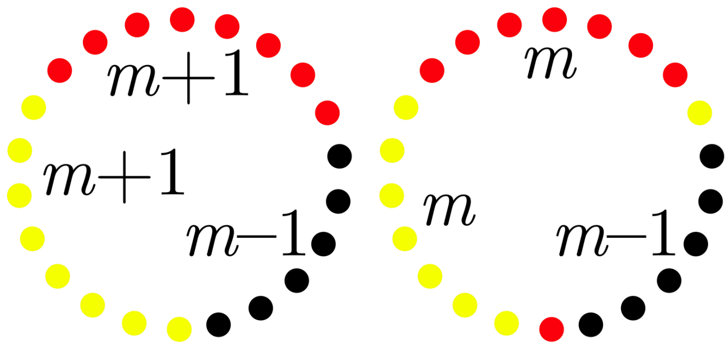

Fig. 2. In each of the two cases the $(m-1)$-set is represented by black points, and two $(m+1)$-sets, which form a matching with the $(m+1)$-set, are represented by red and yellow points.

where the last inequality holds for $m \geq 3$. Thus, this case is covered.

Finally, assume that there is at least one $m-1$-element set $F \in \mathcal{F} \cap \mathcal{G}(\sigma)$. Then, as one can see from Fig. 2, it forbids at least one $m+1$-set of each of the types 1a and 1b to appear. Denote them M_1, M_2. We have seen two paragraphs before that in either case of the arrangement of the m-sets we forbid sets of the same type (either one of type 1a, or two of type 1b). Therefore, there is at least one out of M_1, M_2 that did not get any charge at Stage 5. We assume that it is a set M_1 of type 1b (the other case is absolutely analogous, and even simpler). Since M_1 appears in two triples and got some charge only at Stage 4, the charge of M_1 after all five stages is at most $2\alpha\binom{n}{m+1}$. This, in turn, means that it has extra capacity of $(\alpha-\alpha')\binom{n}{m+1}$. We redistribute some part of the charge from the two $m+1$-sets that appeared in the triples with the pairs of m-sets to M_1. To be able to fulfil the requirements on the charging scheme we need the total capacity of these $m+1$-sets to be greater than the charge transferred to them. In other words, the following inequality must hold:

$$(3\alpha'-\alpha) \geq 3\frac{m+1}{2m+1} \quad \Leftrightarrow \quad 3-7\alpha \geq 3\frac{m+1}{2m+1}.$$

We have verified above (see (10)) that the same inequality holds if one replaces $3-7\alpha$ with $2-2\alpha$. One can easily see (cf. (5)) that $\alpha \leq \frac{3}{16}$, so we have $3-7\alpha > 2-2\alpha$. The case $q = m+1$ is examined in its entirety, and the proof of the theorem in the case $s = 3$ is complete.

3.1 The case $m \leq 2, s = 3$

In the argument above, we assumed that $m \geq 3$. However, we want to prove the theorem for $m \geq 1$, which leaves us two cases: $m = 1$ and $m = 2$. If $m = 1$, then we have $n = 4$, and we have to show that at least 4 sets, including the empty set, are missing from a family $\mathcal{F} \subset 2^{[4]}$ with $\nu(\mathcal{F}) \leq 2$. If there is at most one singleton in $\mathcal{F}$, then we are done. If there are at least two singletons, say, $\{1\}$ and $\{2\}$, then $\mathcal{F} \cap 2^{\{3,4\}}$ is empty, which gives 4 missing sets. The case $m=1$ is covered.

If $m=2$, then $n=7$ and we have to show that at least $1+\binom{7}{1}+\binom{6}{2}=23$ sets are missing from $\mathcal{F}$. If there is at least one singleton in $\mathcal{F}$, say $\{1\}$, then $\mathcal{F}\cap 2^{[2,7]}$ is intersecting, and so, by the Erdős-Ko-Rado theorem, a half of the sets are missing from it. This gives 32 missing sets.

Thus, we may assume that there are no sets of size smaller than 2 in $\mathcal{F}$. Now we may slightly modify the proof for the case $m\ge 3$ so that it works for $m\ge 2$. Namely, we give weights only to the central $m+1$-sets. Since we do not have sets of size smaller than m, we can go to part 5 of the analysis, where we want to show that with new weights (9) holds for any $q\ge m+1$ for $m=2, n=7$:

$$\frac{3(q-m)}{3q-n}\binom{n}{m}\le\frac{3}{2}\binom{n}{m}\le\binom{n}{m+1}\qquad\Leftrightarrow\qquad\frac{3}{2}\binom{7}{2}=\frac{63}{2}\le\binom{7}{3}=35.$$

The last inequality obviously holds. Thus, for $m=2$ we may terminate the proof right after (9). The proof is complete.

4 Proof of Theorem 1 for $s=4$

We first give a proof for the case when $m\ge 3$. We assume $n=4m+2$ for some $m\ge 3$ throughout this section. The proof we present is very similar in spirit to the proof in the case $n=3m+1$, and is in a sense even simpler. We present it somewhat more briefly, since the approach stays the same.

We fix an arbitrary permutation σ of the ground set. For simplicity, we assume that σ is the identity permutation. Quite predictably, we define four groups of sets, indexed by $i=0,1,2,3$, and forming a *quadruple*. In what follows we define the i-th group. The m-set $H_i^{(m)}(x)$ in an x-quadruple form an interval of length $4m$, leaving two contiguous elements $x-1,x$ out (thus, the quadruple is indexed by the last of the two missing elements in the clockwise order). The sets in the i-th group of size $m-j, j=0,\ldots,m$, form a full chain together with $H_i^{(m)}(x)$:

$$H_i^{(m-j)}(x):=\{x+1+j+im,\ldots,x+(i+1)m\}.$$

We again have both central and side $m+1$- and $m+2$-sets. The $m+1$-sets

$$H_i^{(m+1)}(x';x):=H_i^{(m)}(x)\cup\{x'\}\quad\text{ for } x'=x,x-1$$

in a group i are called *central*. Note that the extra element in both sets is left out by the m-sets and so $H_i^{(m+1)}(x';x)$ for both $x'=x-1,x$ is disjoint with the three m-sets from the j-th group, $j\ne i$. The three others

$$H_i^{(m+1)}(j;x):=H_i^{(m)}(x)\cup\{jm+x+1\}\quad\text{ for } j\in\{0,\ldots,3\}-\{i\}$$

are called *side* and are disjoint with the corresponding $H_j^{(m-1)}(x)$ and with two of the m-sets.

For each $j \in \{0, \dots, 3\} - \{i\}$, we define two side $m+2$-element sets:

$$H_i^{(m+2)}(x', j; x) := H_i^{(m+1)}(j; x) \cup \{x'\} \quad \text{for } x' = x, x-1$$

and for each i we define one central set:

$$H_i^{(m+2)}(x-1, x; x) := H_i^{(m)}(x) \cup \{x-1, x\}.$$

The former ones are disjoint with the $m-1$-set from the group j and the two m-set from the remaining groups, while the latter one is disjoint with the three m-sets from the groups $j, j \neq i$. Finally, we have one $m+5$-element set for each i :

$$H_i^{(m+5)}(x) := H_i^{(m)}(x) \cup \{x-1, x, x+1, m+x+1, 2m+x+1, 3m+x+1\}.$$

Each set in each group gets some weight. We denote by w_k the weight of k-element sets, with possible superscripts s, c, corresponding to whether the set is side or central, respectively. The weights are as follows ($j \geq 0$):

$$\begin{aligned} w_{m-j} &:= \binom{n}{m-j}; \quad & w_{m+1}^s &:= \frac{m}{5(3m+2)}\binom{n}{m+1}; \quad & w_{m+1}^c &:= \frac{1}{2}\binom{n}{m+1} - \frac{3}{2}w_{m+1}^s; \\ w_{m+5} &:= \binom{n}{m+5}; \quad & w_{m+2}^s &:= \frac{1}{22}\binom{n}{m+2}; \quad & w_{m+2}^c &:= \frac{8}{11}\binom{n}{m+2}. \end{aligned} \tag{11}$$

It is easy to check that for each i and $k = 1, \dots, m+2$ and $m+5$ in each group the weight of k-element sets sums up to $\binom{n}{k}$.

As before, each set F that appears in some quadruples accumulates all the weight $w(F)$ that it was assigned. The family of all the sets that got a nonzero weight for the permutation σ we denote $\mathcal{G}(\sigma)$. To prove the theorem in this case it is sufficient for us to show that for any σ we have

$$\sum_{F \in \mathcal{F} \cap \mathcal{G}(\sigma)} w(F) \leq 4n\Big(\binom{n-1}{m-1} + \sum_{k \in \{1,2,5\}} \binom{n}{m+k}\Big) = 4m\binom{n}{m} + 4n \sum_{k \in \{1,2,5\}} \binom{n}{m+k}. \tag{12}$$

For a set $F \in \mathcal{G}(\sigma)$ we define the charge $c(F)$ to be equal to $w(F)$ if $F \in \mathcal{F}$, and $c(F) := 0$ otherwise. Clearly, $\sum_{F \in \mathcal{G}(\sigma)} c(F) = \sum_{F \in \mathcal{F} \cap \mathcal{G}(\sigma)} w(F)$. We again design a scheme for the transfer of (a part of) the charge of the ($\leq m$)-sets to the ($\geq m+1$)-sets that have zero charge. We show that the charge transferred to each ($\geq m+1$)-set is at most its weight. As a result of this procedure ($\leq m-1$)-sets will have zero total charge, the m-sets will have a total charge $4m\binom{n}{m}$, and each ($\geq m+1$)-set will have a charge not greater than its own weight. This will obviously conclude the proof of the theorem.

Next we design the charging scheme that satisfies the above requirements. For $n = 4m+2$ it is sufficient in all cases to redistribute the charge within each quadruple, assuring that the charge that we transferred to a bigger set in one quadruple is smaller than the weight that this bigger set got from this quadruple.

Stage 1. Transferring charge from triples of $(\mathbf{m-1})$-sets.
Assume that for some x and some i_1,i_2,i_3,j, $\{i_1,i_2,i_3,j\}=\{0,1,2,3\}$, in the x-quadruple all $H_{i_u}^{(m-1)}(x)$, $u=1,2,3$, belong to $\mathcal{F}$. Then $H_j^{(m+5)}(x)$ is missing from $\mathcal{F}$, and, consequently, has zero charge. Transfer all the charge of the sets $H_{i_u}^{(k)}(x)$ to the missing $m+5$-set.

The charge transferred is at most $3\sum_{k=0}^{m-1} w_k$, which is

$$3\sum_{k=0}^{m-1}\binom{n}{k} \overset{(4)}{\leq} \frac{9}{2}\binom{n}{m-1} = \frac{9\prod_{p=0}^{5}(m+p)}{2\prod_{p=-2}^{3}(3m+p)}\binom{n}{m+5} \leq \binom{n}{m+5} \overset{(11)}{=} w_{m+5},$$

where the last inequality holds for any $m\geq 2$. The $m+5$-sets are not going to get any more charge.

Stage 2. Transferring charge from pairs of $(\mathbf{m-1})$-sets.
Assume that for some x and some i_1,i_2,j_1,j_2, $\{i_1,i_2,j_1,j_2\}=\{0,1,2,3\}$, in the x-quadruple both $H_{j_1}^{(m-1)}(x)$ and $H_{j_2}^{(m-1)}(x)$ belong to $\mathcal{F}$. Then in each of the four pairs $\big(H_{i_1}^{(m+2)}(j',x';x),$ $H_{i_2}^{(m+2)}(x'',j'';x)\big)$, where $\{j',j''\}=\{j_1,j_2\},\{x',x''\}=\{x-1,x\}$ one of the $m+2$-sets is missing from $\mathcal{F}$, and, consequently, has zero charge. Note that all these $m+2$-sets are side. Transfer one quarter of the charge of the sets $H_{j_1}^{(k)}(x)$ and $H_{j_2}^{(k)}(x)$, $k\leq m-1$, to each of these missing sets.

The charge transferred to each side $m+2$-set is at most $\frac{1}{2}\sum_{k=0}^{m-1} w_k$, which is

$$\frac{1}{2}\sum_{k=0}^{m-1}\binom{n}{k} \overset{(4)}{\leq} \frac{3}{4}\binom{n}{m-1} = \frac{3m(m+1)(m+2)}{4(3m+3)(3m+2)(3m+1)}\binom{n}{m+2} < \frac{1}{22}\binom{n}{m+2} \overset{(11)}{=} w^s_{m+2},$$

where the last inequality holds since $m\geq 3$ and since the fraction in front of $\binom{n}{m+2}$ decreases as $m\geq 3$ increases. The side $m+2$-sets are not going to get any more charge.

Stage 3. Transferring charge from single $(\mathbf{m-1})$-sets.
After the above redistribution of charges in each quadruple we have at most one $m-1$-set.

Assume that for some x and some i,j_1,j_2,j_3, $\{j_1,j_2,j_3,i\}=\{0,1,2,3\}$, in the x-quadruple the set $H_i^{(m-1)}(x)$ belongs to $\mathcal{F}$. Then one set from each of the six triples $\big(H_{j_1}^{(m+1)}(\pi(i);x),$ $H_{j_2}^{(m+1)}(\pi(x-1);x),H_{j_3}^{(m+1)}(\pi(x);x)\big)$, where π is a permutation of the set $\{i,x-1,x\}$, is missing from $\mathcal{F}$. It is not difficult to see that it means that at least three out of the listed sets are missing from $\mathcal{F}$. Note that among the possible missing sets there are both central and side $m+1$-sets.

We transfer $\frac{1}{3}$ of the charge of $H_i^{(k)}(x)$, $k\leq m-1$, to each of the three missing sets. This is at most $\frac{1}{3}\sum_{k=1}^{m-1} w_k$, which is

$$\frac{1}{3}\sum_{k=1}^{m-1}\binom{n}{k} \overset{(4)}{\leq} \frac{1}{2}\binom{n}{m-1} = \frac{m(m+1)}{2(3m+1)(3m+2)}\binom{n}{m+1} < \frac{m}{5(3m+2)}\binom{n}{m+1} \overset{(11)}{=} w^s_{m+1},$$

where the last equality holds since $\frac{m+1}{3m+1} \leq \frac{2}{5}$ for any $m \geq 3$. We are not going to transfer any more weight to the side $m+1$-sets.

Stage 4. Transferring charge from pairs and triples of m-sets.
At this stage only the sets of size greater than or equal to m have nonnegative charge. Denote the number of m-sets that have non-zero charge (that is, that are contained in $\mathcal{F} \cap \mathcal{G}(\sigma)$) by q. If $q \leq m$, then we are clearly done.

Assume that $q > m$. On the one hand, it makes an extra contribution $4(q-m)\binom{n}{m}$ to the left hand side of (12). On the other hand, the number of quadruples with two or three m-sets belonging to $\mathcal{F} \cap \mathcal{G}(\sigma)$ becomes non-zero. Indeed, if we denote by z_j the number of quadruples with j m-sets in the family, then $j \leq 3$ and we have $z_1 + 2z_2 + 3z_3 = 4q$. Since $z_0 + z_1 + z_2 + z_3 = n$, we have

$$z_2 + 2z_3 \geq 4q - n. \tag{13}$$

We proceed as follows.

(i) Pairs of m-sets. Assume that for some x and some i_1, i_2, j_1, j_2, $\{j_1, j_2, i_1, i_2\} = \{0, 1, 2, 3\}$, in the x-quadruple both the set $H^{(m)}_{j_1}(x)$ and $H^{(m)}_{j_2}(x)$ belong to $\mathcal{F}$. Then in each of the two pairs of central $m+1$-sets $\big(H^{(m+1)}_{i_1}(x'; x), H^{(m+1)}_{i_2}(x''; x)\big)$ for $\{x', x''\} = \{x-1, x\}$ one of the sets is not in the family. Moreover, each of them has received at most w^s_{m+1} charge (this was possible on Stage 3). We transfer $\frac{\mathbf{2(q-m)}}{\mathbf{4q-n}}\binom{\mathbf{n}}{\mathbf{m}}$ charge to each of the two missing central sets. We have to verify that the charge transferred is at most $w^c_{m+1} - w^s_{m+1}$. Note that $\frac{2(q-m)}{4q-n} \leq 1$, since this function for $q \geq m+1$ decreases as q grows. Therefore, we need to verify

$$\binom{n}{m} \leq w^c_{m+1} - w^s_{m+1} \quad \overset{(11)}{\Leftrightarrow} \quad 2\binom{n}{m} = \frac{2m+2}{3m+2}\binom{n}{m+1} \leq \binom{n}{m+1} - 5w^s_{m+1}. \tag{14}$$

The last inequality holds (with equality) since by (11) we have $5w^s_{m+1} = \frac{m}{3m+2}\binom{n}{m+1}$.

(ii) Triples of m-sets. Assume that for some x and some j_1, j_2, j_3, i, $\{j_1, j_2, j_3, i\} = \{0, 1, 2, 3\}$, in the x-quadruple the sets $H^{(m)}_{j_u}(x)$ belong to $\mathcal{F}$ for all $u = 1, 2, 3$. Then the central $m+2$-set $H^{(m+2)}_i(x-1, x; x)$ is not in the family $\mathcal{F}$. Moreover, it has zero charge. We transfer $\frac{\mathbf{8(q-m)}}{\mathbf{4q-n}}\binom{\mathbf{n}}{\mathbf{m}}$ charge to this set. We have $\frac{8(q-m)}{4q-n} \leq 4$ for $q \geq m+1$. Therefore, we have

$$4\binom{n}{m} = 4\frac{(m+1)(m+2)}{(3m+2)(3m+1)}\binom{n}{m+2} \leq \frac{8}{11}\binom{n}{m+2} \overset{(11)}{=} w^c_{m+2}. \tag{15}$$

The last inequality holds for $m \geq 3$.

Now we only have to make sure that we have transferred enough charge. Indeed, we have transferred the total amount of charge equal to

$$\frac{4(q-m)}{4q-n}\binom{n}{m} z_2 + \frac{8(q-m)}{4q-n}\binom{n}{m} z_3 = \frac{4(q-m)}{4q-n}\binom{n}{m}(z_2 + 2z_3) \overset{(13)}{\geq} 4(q-m)\binom{n}{m}.$$

Therefore, the total amount of charge that is left on the m-sets is at most $4m\binom{n}{m}$, all sets of size not greater than $m-1$ have zero charge, and none of the sets has the charge greater than their weight. The inequality (12) is verified, and the proof of Theorem 1 in the case $s=4$ is complete.

4.1 The case $m \le 2, s = 4$

In the argument above, we assumed that $m \ge 3$. However, we want to prove the theorem for $m \ge 1$, which leaves us two cases: $m = 1$ and $m = 2$. If $m = 1$, then we have $n = 6$, and we have to show that at least 6 sets, including the empty set, are missing from a family $\mathcal{F} \subset 2^{[6]}$ with $\nu(\mathcal{F}) \le 3$. If there is at most one singleton in $\mathcal{F}$, then we are done. If there are at least two singletons, say, $\{1\}$ and $\{2\}$, then $\mathcal{F} \cap [3,6]$ has no two pairwise disjoint sets, and, consequently, at least 8 sets are missing from $\mathcal{F}$ among the sets from $2^{[3,6]}$. The case $m = 1$ is covered.

If $m = 2$, then $n = 10$ and we have to show that at least $1 + \binom{10}{1} + \binom{9}{2} = 47$ sets are missing from $\mathcal{F}$. If there is at least one singleton in $\mathcal{F}$, say $\{1\}$, then, applying (2) to $\mathcal{F} \cap 2^{[2,10]}$, we get that at least $1 + \binom{9}{1} + \binom{9}{2} + \frac{1}{3}\binom{9}{3}$ sets are missing from $\mathcal{F}$, which is more than 47.

Thus, we may assume that there are no sets of size smaller than 2 in $\mathcal{F}$. Now we may slightly modify the proof for the case $m \ge 3$ so that it works for $m \ge 2$. Namely, we give weights only to the central $m+1$- and $m+2$-sets. As a result, we can go to part 4 of the analysis, where we have to verify the following analogues of (14) and (15) for $m = 2$, $n = 10$:

$$\frac{2m+2}{3m+2}\binom{n}{m+1} \le \binom{n}{m+1}, \qquad 4\frac{(m+1)(m+2)}{(3m+2)(3m+1)}\binom{n}{m+2} \le \binom{n}{m+2}.$$

Both hold for $m = 2$. The rest of the proof stays the same. The proof is complete.

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