Two problems of P. Erdős on matchings in set families

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Abstract

The families $\mathcal{F}_1, \ldots, \mathcal{F}_s \subset 2^{[n]}$ are called *q-dependent* if there are no pairwise disjoint $F_i \in \mathcal{F}_i,\ i = 1, \ldots, s$, satisfying $|F_1 \cup \ldots \cup F_s| \le q$. We determine $\max |\mathcal{F}_1| + \ldots + |\mathcal{F}_s|$ for *all* values $n \ge q, s \ge 2$. The result provides a far-reaching generalization of an important classical result of Kleitman. The uniform case $\mathcal{F}_1 = \ldots = \mathcal{F}_s \subset \binom{[n]}{k}$ of this problem is the so-called Erdős Matching Conjecture. After more than 50 years its full solution is still not in sight. In the present paper we provide a Hilton-Milner-type stability theorem for it in a relatively wide range, in particular, for $n \ge (2+o(1))sk$ with $o(1)$ depending on s. This is a considerable improvement of a result due to Bollobás, Daykin and Erdős.

We apply our results to advance in an anti-Ramsey-type problem, proposed by Özkahya and Young. They asked for the minimum number $ar(n, k, s)$ of colors in the coloring of the k-element subsets of $[n]$ that do not contain a *rainbow matching* of size s, that is, s sets of different colors that are pairwise disjoint. We prove a stability result for the problem, which allows to determine $ar(n, k, s)$ for all $k \ge 3$ and $n \ge sk + (s-1)(k-1)$. Some other consequences of our results are presented as well.

1 Introduction

Let $[n] := \{1, 2, \ldots, n\}$ be the standard n-element set and $2^{[n]}$ its power set. A subset $\mathcal{F} \subset 2^{[n]}$ is called a *family*. For $0 \le k \le n$ let $\binom{[n]}{k}$ denote the family of all k-subsets of $[n]$.

For a family $\mathcal{F}$ let $\nu(\mathcal{F})$ denote the maximum number of pairwise disjoint members of $\mathcal{F}$. Note that $\nu(\mathcal{F}) \le n$ holds, unless $\emptyset \in \mathcal{F}$. The fundamental parameter $\nu(\mathcal{F})$ is called the *independence number* or *matching number*.

Let us introduce an analogous notion for several families.

Definition 1. *Suppose that $\mathcal{F}_1, \ldots, \mathcal{F}_s \subset 2^{[n]}$, $2 \le s \le n$. We say that $\mathcal{F}_1, \ldots, \mathcal{F}_s$ are cross-dependent if there is no choice of $F_1 \in \mathcal{F}_1, \ldots, F_s \in \mathcal{F}_s$ such that $F_1, \ldots, F_s$ are pairwise disjoint.*

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Note that $\nu(\mathcal{F}) < s$ is equivalent to saying that $\mathcal{F}_1, \ldots, \mathcal{F}_s$, where $\mathcal{F}_i := \mathcal{F}$ for all i, are cross-dependent.

Example. Let $n = sm + s - l$ for some $l,\ 0 < l \leq s$. Let us define

$$\tilde{\mathcal{F}}_i := \begin{cases} \{F \subset [n] : |F| \geq m\}, & 1 \leq i < l, \\ \{F \subset [n] : |F| \geq m+1\}, & l \leq i \leq s. \end{cases}$$

Then $\tilde{\mathcal{F}}_1, \ldots, \tilde{\mathcal{F}}_s$ are easily seen to be cross-dependent.

The main result of the present paper is

Theorem 1. *Choose integers s, m, l satisfying $s \geq 2$, $m \geq 0$, and $1 \leq l \leq s$. Put $n = sm + s - l$ and suppose that $\mathcal{F}_1, \ldots, \mathcal{F}_s \subset 2^{[n]}$ are cross-dependent. Then*

$$\sum_{i=1}^{s} |\mathcal{F}_i| \leq (l-1)\binom{n}{m} + s \sum_{t \geq m+1} \binom{n}{t} \quad \left[= \sum_{i=1}^{s} |\tilde{\mathcal{F}}_i|\right]. \tag{1}$$

The inequality (1) extends the following important classical result of Kleitman.

Theorem (Kleitman, [14])**.** *Let $s \geq 2$ be an integer and $\mathcal{F} \subset 2^{[n]}$ a family satisfying $\nu(\mathcal{F}) < s$. Then the following holds.*

$$\text{If } n = s(m+1) - 1, \quad \text{then } |\mathcal{F}| \leq \sum_{t \geq m} \binom{n}{t}; \tag{2}$$

$$\text{if } n = sm, \quad \text{then } |\mathcal{F}| \leq \frac{s-1}{s}\binom{n}{m} + \sum_{t \geq m+1} \binom{n}{t}. \tag{3}$$

In the case $n = s(m+1) - 1$ the families $\tilde{\mathcal{F}}_i$ from the example above are all the same and thereby show that the bound (2) is best possible. The bound (3) is also best possible, as shown by Kleitman's example:

$$\mathcal{K} := \{K \subset [sm] : |K| \geq m+1\} \cup \binom{[sm-1]}{m}.$$

(Note that $\binom{sm-1}{m} = \frac{s-1}{s}\binom{sm}{m}$.) In the case $s = 2$ both bounds (2) and (3) reduce to $|\mathcal{F}| \leq 2^{n-1}$. This easy statement was proved already by Erdős, Ko and Rado [4].

Although (2) and (3) are beautiful results, for $s \geq 3$ they leave open the cases of $n \not\equiv 0$ or $-1 (\text{mod } s)$. For $s = 3$ the only remaining case was solved by Quinn [17]. Most recently, we made some further progress [8], [9].

Let us mention that, except for the case $s = 3$ and $l = 2$, the original methods of Kleitman could be used to prove Theorem 1. However, for that case they seem to fail. That forced us to find a very different proof. It is given in Section 3.

Next we discuss a generalization of the notion of cross-dependence.

Definition 2. *Let $2 \leq s \leq n$ and $1 \leq q \leq n$ be fixed integers. The families $\mathcal{F}_1, \ldots, \mathcal{F}_s \subset 2^{[n]}$ are called q-dependent if there are no pairwise disjoint $F_1, \ldots, F_s$, where $F_i \in \mathcal{F}_i$, satisfying $|F_1 \cup \ldots \cup F_s| \leq q$.*

For $q = n$ the notion of q-dependence reduces to that of cross-dependence. Quite surprisingly, one can determine the exact maximum of $|\mathcal{F}_1| + \ldots + |\mathcal{F}_s|$ for q-dependent families $\mathcal{F}_1, \ldots, \mathcal{F}_s \subset 2^{[n]}$ and *all* values of n, q, s.

Let $s \geq 2,\ m \geq 0,\ 1 \leq l \leq s$. If $n \geq q := sm + s - l$, then one can define

$$\tilde{\mathcal{F}}_i^{n,q} := \begin{cases} \{F \subset [n] : |F| \geq m\}, & 1 \leq i < l, \\ \{F \subset [n] : |F| \geq m+1\}, & l \leq i \leq s. \end{cases} \tag{4}$$

We prove the following generalization of Theorem 1.

Theorem 2. *Choose integers s, m, l satisfying $s \geq 2$, $m \geq 0$ and $1 \leq l \leq s$. Let $q = sm+s-l$ and $n \geq q$. If $\mathcal{F}_1, \ldots, F_s \subset 2^{[n]}$ are q-dependent, then*

$$\sum_{i=1}^{s} |\mathcal{F}_i| \leq (l-1)\binom{n}{m} + s \sum_{t=m+1}^{n} \binom{n}{t} \quad \left[= \sum_{i=1}^{s} |\tilde{\mathcal{F}}_i^{n,q}|\right]. \tag{5}$$

The proof of this theorem is given in Section 3.

Hilton-Milner-type result for Erdős Matching Conjecture

The Kleitman Theorem was motivated by a conjecture of Paul Erdős. Erdős himself worked on the uniform case, i.e., with the families $\mathcal{F} \subset \binom{[n]}{k}$. Let us make a formal definition.

Definition 3. *For positive integers n, k, s satisfying $s \geq 2$, $n \geq ks$, define*

$$e_k(n,s) := \max\left\{|\mathcal{F}| : \mathcal{F} \subset \binom{[n]}{k}, \nu(\mathcal{F}) < s\right\}.$$

Note that for $s = 2$ the quantity $e_k(n,s)$ was determined by Erdős, Ko and Rado.

Theorem (Erdős-Ko-Rado [4])**.**

$$e_k(n,2) = \binom{n-1}{k-1} \qquad \textit{for} \quad n \geq 2k. \tag{6}$$

The case $s \geq 3$ seems to be much harder.

There are some natural ways to construct a family $\mathcal{A} \subset \binom{[n]}{k}$, satisfying $\nu(\mathcal{A}) = s$ for $n \geq (s+1)k$. Following [5], let us define the families $\mathcal{A}_i^{(k)}(n,s)$:

$$\mathcal{A}_i^{(k)}(n,s) := \left\{A \in \binom{[n]}{k} : \big|A \cap [(s+1)i-1]\big| \geq i\right\}, \quad 1 \leq i \leq k. \tag{7}$$

Conjecture 1 (Erdős Matching Conjecture [2])**.** *For $n \geq (s+1)k$ we have*

$$e_k(n, s+1) = \max\left\{|\mathcal{A}_1^{(k)}(n,s)|, |\mathcal{A}_k^{(k)}(n,s)|\right\}. \tag{8}$$

The conjecture (8) is known to be true for $k \leq 3$ (cf. [3], [15] and [7]). Improving earlier results of [2], [1], [12] and [10], in [6]

$$e_k(n, s+1) = \binom{n}{k} - \binom{n-s}{k} \qquad \text{is proven for} \qquad n \geq (2s+1)k - s. \tag{9}$$

In the case of $s = 1$ (that is, the case of the Erdős-Ko-Rado Theorem) one has a very useful stability theorem due to Hilton and Milner [11]. Below we discuss one natural generalization of the Hilton-Milner theorem to the case $s > 1$.

Let us define the following families.

$$\mathcal{H}^{(k)}(n,s) := \left\{H \in \binom{[n]}{k} : H \cap [s] \neq \emptyset\right\} \cup \left\{[s+1, s+k]\right\} -$$
$$- \left\{H \in \binom{[n]}{k} : H \cap [s] = \{s\}, H \cap [s+1, s+k] = \emptyset\right\}.$$

Note that $\nu(\mathcal{H}^{(k)}(n,s)) = s$ for $n \geq sk$ and

$$|\mathcal{H}^{(k)}(n,s)| = \binom{n}{k} - \binom{n-s}{k} + 1 - \binom{n-s-k}{k-1}. \tag{10}$$

The covering number $\tau(\mathcal{H})$ of a hypergraph is the minimum of $|T|$ over all T satisfying $T \cap H \neq \emptyset$ for all $H \in \mathcal{H}$. Recall the definition (7). If $n \geq k + s$, then the equality $\tau(\mathcal{A}_1^{(k)}(n,s)) = s$ is obvious. At the same time, if $n \geq k+s$, then $\tau(\mathcal{H}^{(k)}(n,s)) = s+1$ and $\tau(\mathcal{A}_i^{(k)}(n,s)) > s$ for $i \geq 2$.

Let us make the following

Conjecture 2. *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ satisfies $\nu(\mathcal{F}) = s, \tau(\mathcal{F}) > s$. Then*

$$|\mathcal{F}| \leq \max\left\{\left\{|\mathcal{A}_i^{(k)}(n,s)| : i = 2, \ldots, k\right\},\ |\mathcal{H}^{(k)}(n,s)|\right\} \qquad \textit{holds.} \tag{11}$$

The Hilton-Milner Theorem shows that (11) is true for $s = 1$.

Theorem (Hilton-Milner [11])**.** *Suppose that $n \geq 2k$ and let $\mathcal{F} \subset \binom{[n]}{k}$ be a family satisfying $\nu(\mathcal{F}) = 1$ and $\tau(\mathcal{F}) \geq 2$. Then*

$$|\mathcal{F}| \leq |\mathcal{H}^{(k)}(n,1)| \qquad \textit{holds.}$$

We mention that for $n > 2sk$ the maximum on the RHS of (11) is attained on $|\mathcal{H}^{(k)}(n,s)|$. For $n > 2k^3 s$ (11) was shown by Bollobás, Daykin and Erdős [1].

One of the main results of this paper is the proof of (11) for a much wider range.

Theorem 3. *The conjecture is true provided $k \geq 2$, $n \geq (s + \max\{24, 2s+2\})k$ and for $k \geq 3, n \geq \big(2 + o(1)\big)sk$, where $o(1)$ depends on s only. More precisely, for any $\mathcal{G} \subset \binom{[n]}{k}$ with $\nu(\mathcal{G}) = s < \tau(\mathcal{G})$ we have $|\mathcal{G}| \leq |\mathcal{H}^{(k)}(n,s)|$.*

We prove Theorem 3 in Section 2.

We recall the definition of the *left shifting* (left compression), which we would simply refer to as *shifting*. For a given pair of indices $i < j \in [n]$ and a set $A \in 2^{[n]}$ we define the (i,j)-shift $S_{i,j}(A)$ of A in the following way.

$$S_{i,j}(A) := \begin{cases} A & \text{if } i \in A \text{ or } \quad j \notin A; \\ (A - \{j\}) \cup \{i\} & \text{if } i \notin A \text{ and } \quad j \in A. \end{cases}$$

Next, we define the (i,j)-shift $S_{i,j}(\mathcal{F})$ of a family $\mathcal{F} \subset 2^{[n]}$:

$$S_{i,j}(\mathcal{F}) := \{S_{i,j}(A) : A \in \mathcal{F}\} \cup \{A : A, S_{i,j}(A) \in \mathcal{F}\}.$$

We call a family $\mathcal{F}$ *shifted*, if $S_{i,j}(\mathcal{F}) = \mathcal{F}$ for all $1 \leq i < j \leq n$.

Recall that $\mathcal{F}$ is called *closed upward* if for any $F \in \mathcal{F}$ all sets that contain F are also in $\mathcal{F}$. When dealing with cross-dependent and q-dependent families, we may restrict our attention to the families that are closed upward and shifted (see, e.g. [8], Claim 17), which we assume for the rest of the paper.

2 Proof of Theorem 3

For $s = 1$ the theorem follows from Hilton-Milner theorem, therefore we may assume that $s \geq 2$. Choose a number u so that

$$n = (u + s - 1)(k - 1) + s + k. \tag{12}$$

Consider a family $\mathcal{G}$ satisfying the requirements of the theorem.

The case of shifted $\mathcal{G}$

First we prove Theorem 3 in the assumption that $\mathcal{G}$ is shifted. Following [6], we say that the families $\mathcal{F}_1, \ldots, \mathcal{F}_s$ are *nested*, if $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \ldots \supset \mathcal{F}_s$. The following lemma is the crucial tool for the proof and may be obtained by a straightforward modification of the proof of Theorem 3.1 from [6]:

Lemma 4 (Frankl [6]). *Let $N \geq (u + s - 1)(k - 1)$, and suppose that $\mathcal{F}_1, \ldots, \mathcal{F}_s \subset \binom{[N]}{k-1}$ are cross-dependent and nested. Then*

$$|\mathcal{F}_1| + |\mathcal{F}_2| + \ldots + |\mathcal{F}_{s-1}| + u|\mathcal{F}_s| \leq (s-1)\binom{N}{k-1}. \tag{13}$$

We use the following notation. For any $p \in [n]$ and a subset $Q \subset [1,p]$ define

$$\mathcal{G}(Q,p) := \{G \setminus Q : G \in \mathcal{G}, G \cap [1,p] = Q\}.$$

The first step of the proof of Theorem 3 is the following lemma.

Lemma 5. *Assume that* $|\mathcal{G}| - |\mathcal{G}(\emptyset, s)| \le \binom{n}{k} - \binom{n-s}{k} - C$ *for some* $C > 0$ *and that* $\nu(\mathcal{G}(\emptyset, s)) = x$ *for some* $1 \le x \le s$. *Then*

$$|\mathcal{G}| \le \binom{n}{k} - \binom{n-s}{k} - \frac{u-x-1}{u}C. \tag{14}$$

If $\nu(\mathcal{G}(\emptyset,s)) = \tau(\mathcal{G}(\emptyset,s)) = 1$, *then*

$$|\mathcal{G}| \le \binom{n}{k} - \binom{n-s}{k} - \frac{u-1}{u}C. \tag{15}$$

Proof. Recall the definition of the immediate shadow

$$\partial\mathcal{H} := \big\{H : \exists H' \in \mathcal{H}, H \subset H', |H' - H| = 1\big\}. \tag{16}$$

For every $H \in \partial\mathcal{G}(\emptyset, s+1)$ we have $H \in \mathcal{G}(\{s+1\}, s+1)$, since $\mathcal{G}$ is shifted. Combining this with the inequality $x|\partial\mathcal{H}| \ge |\mathcal{H}|$ from ([6], Theorem 1.2), valid for any $\mathcal{H}$ with $\nu(\mathcal{H}) \le x$, we get

$$|\mathcal{G}(\emptyset, s+1)| \le x'|\mathcal{G}(\{s+1\}, s+1)|, \tag{17}$$

where $x' = x$ if $\tau(\mathcal{G}(\emptyset,s)) > 1$ and $x' = 0$ if $\tau(\mathcal{G}(\emptyset,s)) = 1$.

For any $Q \subset [1, s+1], |Q| \ge 2$, we have $\mathcal{A}_1^{(k)}(n,s)(Q, s+1) = \binom{[s+2,n]}{k-|Q|}$, and so we have $|\mathcal{G}(Q, s+1)| \le |\mathcal{A}_1^{(k)}(n,s)(Q,s+1)|$. We also have $\mathcal{A}_1^{(k)}(n,s)(\emptyset, s+1) = \emptyset$ and $\sum_{i=1}^{s+1} |\mathcal{A}_1^{(k)}(n,s)(\{i\}, s+1)| = s\binom{n-s-1}{k-1}$. Using (17) and (13), we have

$$|\mathcal{G}(\emptyset, s+1)| + \sum_{i=1}^{s+1} |\mathcal{G}(\{i\}, s+1)| \le \sum_{i=1}^{s} |\mathcal{G}(\{i\}, s+1)| + (x'+1)|\mathcal{G}(\{s+1\}, s+1)| \le$$
$$\le s\binom{n-s-1}{k-1} - (u - x' - 1)|\mathcal{G}(\{s+1\}, s+1)|.$$

Thus, $|\mathcal{A}_1^{(k)}(n,s)| - |\mathcal{G}| \ge (u-x'-1)|\mathcal{G}(\{s+1\}, s+1)| \overset{(17)}{\ge} \frac{u-x'-1}{x'+1}|\mathcal{G}(\emptyset, s)|$. On the other hand, the inequality from the formulation of the lemma tells us that $|\mathcal{A}_1^{(k)}(n,s)| - |\mathcal{G}| \ge C - |\mathcal{G}(\emptyset, s)|$. Adding these two inequalities (the second one taken with coefficient $\frac{u-x'-1}{x'+1}$), we get that $|\mathcal{A}_1^{(k)}(n,s)| - |\mathcal{G}| \ge \frac{u-x'-1}{u}C$. $\square$

Therefore, to prove Theorem 3, it is sufficient to to obtain good bounds on C from the formulation of Lemma 5. We do that in the next two propositions. We use the following simple observation:

Observation 6. *If for some $C > 0$, $S \subset [s]$, and $\mathcal{B} \subset \binom{[s+1,n]}{k-1}$ we have $\sum_{i\in S}\big|\mathcal{G}(\{i\},s)\cap\mathcal{B}\big| \le |S||\mathcal{B}| - C$, then both $\sum_{i\in S}\big|\mathcal{G}(\{i\},s)\big| \le |S|\binom{n-s}{k-1} - C$ and $|\mathcal{G}| - |\mathcal{G}(\emptyset,s)| \le \binom{n}{k} - \binom{n-s}{k} - C$.*

We are going to use the next proposition and lemma for the case $\nu(\mathcal{G}(\emptyset,s)) \ge 2$. Assume that $\mathcal{G}(\emptyset,s)$ contains x pairwise disjoint sets $F_1,\dots,F_x$ for some $1 \le x \le s$. Put $\mathcal{B}_j := \{B \in \binom{[s+1,n]}{k-1} : B \cap F_j = \emptyset\}$.

Proposition 7. *In the assumptions above, choose a positive integer q and integers $0 =: p_0 < p_1 < p_2 < \ldots < p_q := x$. Put $f := \prod_{j=1}^{q}(p_j - p_{j-1})$. Then for $u \ge qf$ we have*

$$\sum_{i=1}^{s}\big|\mathcal{G}(\{i\},s)\big| \le s\binom{n-s}{k-1} - q\Big|\bigcap_{j=0}^{q-1}\Big(\bigcup_{z=p_j+1}^{p_{j+1}}\mathcal{B}_z\Big)\Big|. \tag{18}$$

Proof. Denote $\mathcal{I}(\{i\},s) := \mathcal{G}(\{i\},s) \cap \bigcap_{j=0}^{q-1}\Big(\cup_{z=p_j+1}^{p_{j+1}}\mathcal{B}_z\Big)$, $i = 1,\dots,s$.

Assume that $|\mathcal{I}(\{s-q+1\},s)| = y$. Then, since $\mathcal{I}(\{i\},s) \supset \mathcal{I}(\{i+1\},s)$ for any $0 < i < s$, we have

$$\sum_{i=s-q+1}^{s}|\mathcal{I}(\{i\},s)| \le qy. \tag{19}$$

Applying Observation 6 with $S = [s-q+1,s]$ and $\mathcal{B} = \bigcap_{j=0}^{q-1}\Big(\bigcup_{z=p_j+1}^{p_{j+1}}\mathcal{B}_z\Big)$, we get

$$\sum_{i=s-q+1}^{s}\big|\mathcal{G}(\{i\},s)\big| \le q\binom{n-s}{k-1} - q\Big|\bigcap_{j=0}^{q-1}\Big(\bigcup_{z=p_j+1}^{p_{j+1}}\mathcal{B}_z\Big)\Big| + qy. \tag{20}$$

On the other hand, since $|\mathcal{I}(\{s-q+1\},s)| = y$, by pigeon-hole principle we have

$$\Big|\mathcal{G}(\{s-q+1\},s) \cap \bigcap_{j=0}^{q-1}\mathcal{B}_{z_i}\Big| \ge \frac{y}{f} \qquad \text{for some } z_i \in [p_j+1, p_{j+1}]. \tag{21}$$

Next, the families $\mathcal{G}_z(\{1\},s),\dots,\mathcal{G}_z(\{s-q+1\},s)$, where $\mathcal{G}_z(\{i\},s) := \mathcal{G}(\{i\},s)\cap\bigcap_{j=0}^{q-1}\mathcal{B}_{z_i}$, are cross-dependent and nested. From (13) we get the inequality

$$|\mathcal{G}_z(\{1\},s)| + \ldots + |\mathcal{G}_z(\{s-q\},s)| + u|\mathcal{G}_z(\{s-q+1\},s)| \le (s-q)\big|\cap_{j=0}^{q-1}\mathcal{B}_{z_i}\big|, \tag{22}$$

which, in view of (21) gives us

$$\sum_{i=1}^{s-q}|\mathcal{G}_z(\{i\},s)| \le (s-q)\big|\cap_{j=0}^{q-1}\mathcal{B}_{z_i}\big| - \frac{uy}{f}. \tag{23}$$

Applying Observation 6 with $S = [1, s-q]$ and $\mathcal{B} = \cap_{j=0}^{q-1} \mathcal{B}_{z_i}$, we get

$$\sum_{i=1}^{s-q} \big|\mathcal{G}(\{i\}, s)\big| \le (s-q)\binom{n-s}{k-1} - \frac{uy}{f}. \tag{24}$$

We get the statement of the proposition by summing (20) and (24) and noting that by the assumption $\frac{uy}{f} \ge qy$. ☐

The following lemma is an important technical ingredient in establishing good bounds on u for which the statement of Theorem 3 holds.

Lemma 8. *Assume that $\nu\big(\mathcal{G}(\emptyset, s)\big) = x$ for $1 \le x \le s$. Then we have*

$$\sum_{i=1}^{s} \big|\mathcal{G}(\{i\}, s)\big| \le s\binom{n-s}{k-1} - \gamma\binom{n-k-s}{k-1}, \tag{25}$$

where (i) $\gamma = \frac{4}{3}$ *for* $x = 2$ *and* $u \ge 2$, (ii) $\gamma = \frac{3}{2}$ *for* $x = 3$ *and* $u \ge 4$, (iii) $\gamma = \frac{16}{9}$ *for* $x = 4, 5$ *and* $u \ge 8$, (iv) $\gamma = 2$ *for* $x \ge 6$ *and* $u \ge 24$, (v) $\gamma = \Omega(x/\log^2 x)$ *for* $u \ge xe^x$.

Proof. The logic of the proofs of all five statements is similar. We combine the bounds from Proposition 7 with different parameters to get the bound of the form $\beta\big|\cup_{i=1}^{s} \mathcal{G}(\{i\}, s)\big| \le \beta s\binom{n-s}{k-1} - \sum_{z=1}^{x} |\mathcal{B}_z|$ for the smallest possible β. Then the constant γ from the statement of Lemma 8 is defined as $\gamma := x/\beta$. Since $|\mathcal{B}_z| = \binom{n-k-s}{k-1}$ for any z, we get the statement, as long as we can guarantee the claimed bounds on β. Therefore, we aim to find a linear combination of equations (18) with coefficients β_j, which satisfy the following two conditions:

$$\text{a) the sum of the subtrahends in the RHS is at least } \sum_{z=1}^{x} |\mathcal{B}_z|, \tag{26}$$

$$\text{b) } \beta := \sum \beta_j \text{ is as small as possible.} \tag{27}$$

We make of the following inclusion-exclusion-type decomposition:

$$\sum_{z=1}^{x} |\mathcal{B}_z| = |\cup_{z=1}^{x} \mathcal{B}_z| + \sum_{1 \le z_1 < z_2 \le x} |\mathcal{B}_{z_1} \overline{\cap} \mathcal{B}_{z_2}| + 2 \sum_{1 \le z_1 < z_2 < z_3 \le x} |\overline{\cap}_{j=1}^{3} \mathcal{B}_{z_j}| + \ldots + (x-1)|\overline{\cap}_{j=1}^{x} \mathcal{B}_j|, \tag{28}$$

where the sign $\overline{\cap}$ stands for the intersection of *precisely* the corresponding $\mathcal{B}_z$ and exclude the sets that belong to any other $\mathcal{B}_j$. Formally, for any $S \subset [x]$ we have

$$\overline{\bigcap_{j \in S}} \mathcal{B}_j := \Big(\bigcap_{j \in S} \mathcal{B}_j\Big) \setminus \Big(\bigcup_{j \in [x] \setminus S} \mathcal{B}_j\Big).$$

All the cardinalities of the intersections in (28) are determined by the number of intersecting families. The number of summands of the form $|\overline{\cap}_{j=1}^{l} \mathcal{B}_{z_j}|$ is $(l-1)\binom{x}{l}$ for any $l = 1, \ldots, x$.

We call each of the families of the form $\overline{\bigcap}_{j=1}^{l}\mathcal{B}_{z_j}$ an *l-intersection*. For shorthand we call the cardinality of the corresponding family the l-intersection as well.

The subtrahend in (18) also admits a decomposition into l-intersections, analogous to (28). In the proof of each statement we guarantee (26) by finding a linear combination, in which the number of l-intersections in the subtrahend is greater than that in (18) for each l. We remark that the term $|\cup_{z=1}^{x}\mathcal{B}_z|$ is somewhat special, and it appears in each linear combination below. Note that this expression is the subtrahend in (18) for $q = 1$. Finally, we mention that for each member of the linear combination we use the following notation: $\big[$parameters substituted in (18); the value of the coefficient$\big]$.

Since the proofs of the first four statements are almost identical, we present the proofs of the first and the fourth out of them only. We start with the statement (i). We sum up $\big[q = 1;$ coefficient $1\big]$ with $\big[q = 2,\ p_1 = 1;$ coefficient $\frac{1}{2}\big]$. We get an inequality

$$\frac{3}{2}\big|\cup_{i=1}^{s}\mathcal{G}(\{i\},s)\big| \le \frac{3}{2}s\binom{n-s}{k-1} - \big|\mathcal{B}_1\cup\mathcal{B}_2\big| - \big|\mathcal{B}_1\cap\mathcal{B}_2\big| = \frac{3}{2}\big|\cup_{i=1}^{s}\mathcal{G}(\{i\},s)\big| - |\mathcal{B}_1| - |\mathcal{B}_2|.$$

The condition on u, imposed by the application of (18), is simply $u \ge 2$. It is clear that $\gamma = \frac{2}{3/2} = \frac{4}{3}$ for this linear combination. This concludes the proof of (i).

The proof of (iv) is more cumbersome. It is sufficient to verify (iv) for $x = 6$. Take the following linear combination: $\big[q = 1;$ coefficient $1\big]$, $\big[q = 2,\ p_1 = 3;$ coefficient $\frac{3}{2}\big]$, $\big[q = 3,\ p_1 = 2,\ p_2 = 4;$ coefficient $\frac{1}{3}\big]$, and $\big[q = 6,\ p_i = i$ for $i = 0,\dots,6;$ coefficient $\frac{1}{6}\big]$. First, it is clear that for this combination we have $\beta = 1 + \frac{3}{2} + \frac{1}{3} + \frac{1}{6} = 3$, and $\gamma = x/\beta = 2$. Moreover, it is easy to see that the condition on u, imposed by the application of (18), is $u \ge 24$ and comes from the third summand. Therefore, we are left to verify that the number of l-intersections in the subtrahend is at least that of (28).

Excluding the first term, there are $15, 40, 45, 24, 5$ i-intersections in (28) for $x = 6$ with $i = 2,3,4,5,6$, respectively. Next, we count the number of the i-intersections in the linear combination. The term $\big[q = 2,\ p_1 = 3;$ coefficient $\frac{3}{2}\big]$ gives, as it is easy to check, $27, 54, 45, 18, 3$ i-intersections for $i = 2,3,4,5,6$, respectively. For $i \le 4$ this term alone has at least as much i-intersections as (28). It is left for us to "find" six 5-intersections and two 6-intersections in the remaining terms. Six 5-intersections and one 6-intersection are given by $\big[q = 3,\ p_1 = 2,\ p_2 = 4;$ coefficient $\frac{1}{3}\big]$. The remaining 6-intersection (the intersection of all families) is given by by $\big[q = 6,\ p_i = i$ for $i = 0,\dots,6;$ coefficient $\frac{1}{6}\big]$. The proof of (iv) is complete.

The proof of (v) is the most technical. For simplicity we assume that $x = 2^r$ for some positive integer r. Consider the following linear combination:

$$\big[q = 2^j,\ p_i = ix/q \text{ for } i = 0,\dots,q;\ \text{coefficient } 4(j+1)\big],\ \text{where } j = 0,\dots,r.$$

For shorthand we denote the member of the linear combination with $q = 2^j$ by M_j.

First we verify that the combination above has enough l-sets for each l. The union of all $\mathcal{B}_z$ corresponds to M_0. For larger l we need to do an auxiliary calculation.

What is the number of l-intersections that are contained in M_j? This is almost the same as asking, how many different l-intersections are contained in the family

$$\bigcap_{i=0}^{2^j-1}\left(\bigcup_{z=i2^{r-j}+1}^{(i+1)2^{r-j}} \mathcal{B}_z\right). \tag{29}$$

Assuming that $l \geq q := 2^j$, the l-intersections that are not contained in the family above are exactly the ones that do not have a $\mathcal{B}_y$ with the index $y \in \left[i2^{r-j}+1, (i+1)2^{r-j}\right]$ for some $i = 0, \ldots, q-1$ among the intersecting families. We can bound the fraction of such l-intersections by the expression $q\left(\frac{q-1}{q}\right)^l \leq qe^{-l/q}$. For $l \geq q\log(2q) =: l_q$ this fraction is at most $\frac{1}{2}$. We conclude that the family (29) contains at least $\frac{1}{2}\binom{n}{l}$ l-intersections for $l \geq q\log_2(2q) = 2^j(j+1)$.

Since M_j has as the subtrahend the size of the family (29), multiplied by the factor $42^j(j+1)$, we conclude that M_j contributes at least $2^{j+1}(j+1)\binom{n}{l}$ l-intersections for $l \geq 2^j(j+1)$.

Next, for each $l \geq 2$ find the largest j, such that $2^j(j+1) \leq l$. It is clear that $2^{j+1}(j+1) > l$. As we have shown above, M_j contributes at least $2^{j+1}(j+1)\binom{n}{l} > l\binom{n}{l}$ l-intersections, which is more than the number of l-intersections in (28). Thus, we have enough l-intersections.

Second, we calculate the sum of coefficients of the members of the linear combination. We have

$$\beta = \sum_{i=0}^{r} 4(i+1) = 4\binom{r+2}{2} = O(\log^2 x).$$

Thus, $\gamma = x/\beta = O(x/\log^2 x)$.

Finally, we verify the condition on u imposed by the (18). For M_j the restriction is $u \geq 2^j\left(2^{r-j}\right)^{2^j} = 2^{j+(r-j)2^j}$. This expression is clearly maximized when $j = r-1$, and in that case we have $u \geq 2^{r-1+2^{r-1}}$. Thus, the condition $u \geq 2^{2^r} = 2^x$ is clearly sufficient. □

In the case when $\nu(\mathcal{G}(\emptyset, s)) = 1$ we need a proposition which is more fine-grained than Proposition 7. For each $j = 1, \ldots, k+1$ define the k-sets $D_j := [s+1, s+k+1] \setminus \{s+j\}$ and the families $\mathcal{C}_j := \left\{C \in \binom{[s+1,\ldots,n]}{k-1} : C \cap D_j = \emptyset\right\}$.

Proposition 9. *Assume that* $\nu(\mathcal{G}(\emptyset, s)) = 1$ *and put* $v := \max\{1, k+2-u\}$.
1. *If* $\tau(\mathcal{G}(\emptyset, s)) > 1$, *then*

$$\sum_{i=1}^{s}\left|\mathcal{G}(\{i\}, s)\right| \leq s\binom{n-s}{k-1} - |\cup_{j=v}^{k+1}\mathcal{C}_j| \tag{30}$$

2. If $\tau(\mathcal{G}(\emptyset,s))=1$ and for some integer t, $v\le t\le k$, we have $|\mathcal{G}(\emptyset,s)|>\binom{n-s-t}{k-t}$, then

$$\sum_{i=1}^{s}\big|\mathcal{G}(\{i\},s)\big|\le s\binom{n-s}{k-1}-|\cup_{j=t}^{k+1}\mathcal{C}_j|. \qquad (31)$$

Proof. **1.** Since $\mathcal{G}(\emptyset,s)$ is shifted and $\tau(\mathcal{G}(\emptyset,s))>1$, the set D_1 is contained in $\mathcal{G}(\emptyset,s)$. Then by shiftedness all D_j for $j=1,\ldots,k+1$ are contained in $\mathcal{G}(\emptyset,s)$. Arguing as in the proof of Proposition 7, let $\big|\big(\cup_{j=v}^{k+1}\mathcal{C}_j\big)\cap\mathcal{G}(\{s\},s)\big|=y$. Then there is an index $j, v\le j\le k+1$, such that $\big|\mathcal{C}_j\cap\mathcal{G}(\{s\},s)\big|\ge\frac{y}{v}\ge\frac{y}{u}$. The rest of the proof is the same as in Proposition 7.

2. Similarly, since $\mathcal{G}(\emptyset,s)$ is shifted and $|\mathcal{G}(\emptyset,s)|>\binom{n-s-t}{k-t}$, the set D_t must be contained in $\mathcal{G}(\emptyset,s)$. Therefore, all the sets D_j for $j=t,\ldots,s+1$ are contained in $\mathcal{G}(\emptyset,s)$ and we conclude as before. □

We go on to the proof of Theorem 3. In the case when $|\mathcal{G}(\emptyset,s)|=1$ we get exactly the bound stated in the theorem, since $|\mathcal{G}|=|\mathcal{G}|-|\mathcal{G}(\emptyset,s)|+1=\binom{n}{k}-\binom{n-s}{k}-|\mathcal{B}_1|+1$. Thus, for the rest of the proof we assume that $|\mathcal{G}(\emptyset,s)|>1$.

Consider first the case $\boldsymbol{\nu(\mathcal{G}(\emptyset,s))=\tau(\mathcal{G}(\emptyset,s))=1}$. If $1<|\mathcal{G}(\emptyset,s)|\le\binom{n-s-k+1}{1}=n-s-k+1$, then for C from Lemma 5 have $C\ge|\mathcal{C}_k\cup\mathcal{C}_{k+1}|=\binom{n-k-s}{k-1}+\binom{n-k-s-1}{k-2}$. For $k\ge 4, s\ge 2$ we have

$$\binom{n-k-s-1}{k-2}\ge n-s-k+1\ge|\mathcal{G}(\emptyset,s)|,$$

thus the theorem holds in this case. For $k=2,3$ one can also verify that $C-|\mathcal{G}(\emptyset,s)|+1\ge\binom{n-k-s}{k-1}$. It is straightforward but a bit technical, and we omit these considerations.

In case when $|\mathcal{G}(\emptyset,s)|>n-s-k+1$ we use the following bound:

$$C\ge|\cup_{j=k-1}^{k+1}\mathcal{B}_j|=\binom{n-k-s}{k-1}+2\binom{n-k-s-1}{k-2}=$$
$$=\Big(1+\frac{2(k-1)}{n-k-s}\Big)\binom{n-k-s}{k-1}\overset{(12)}{=}\Big(1+\frac{2}{u+s-1}\Big)\binom{n-k-s}{k-1}. \qquad (32)$$

The last expression is at least $\frac{u}{u-1}\binom{n-k-s}{k-1}$ if $\frac{u+s+1}{u+s-1}\ge\frac{u}{u-1}$, which holds in case $u\ge s+1$. Since $u\ge s+1$, we can apply (15) and conclude that Theorem 3 holds in this case.

Next we consider the case when $\boldsymbol{\nu(\mathcal{G}(\emptyset,s))=1<\tau(\mathcal{G}(\emptyset,s))}$. By analogy with (32) we get from Proposition 9 that

$$C\ge|\cup_{j=z}^{k+1}\mathcal{B}_j|=\Big(1+\frac{\min\{k+1,u\}}{u+s-1}\Big)\binom{n-k-s}{k-1}.$$

The inequality $\frac{u+s-1+\min\{k+1,u\}}{u+s-1}\ge\frac{u}{u-2}$ holds for $u\ge 2s+4$ and $k\ge 2$, and we can apply (14). It also holds for $u\ge s+3$ and $k\ge 3$.

In the case $\boldsymbol{\nu(\mathcal{G}(\emptyset, s)) = x \geq 2}$ we make use of Lemma 8. We are done in this case as long as, in terms of Lemma 8,

$$\gamma \cdot \frac{u-x-1}{u} \geq 1. \tag{33}$$

Using the first four statements from Lemma 8, one can see that it holds provided $u \geq \max\{24, 2s+2\}$. Indeed, let us verify this technical claim. It is clearly sufficient to verify it for $u = \max\{24, 2s+2\}$.

- If $x = 2$, then $\gamma = \frac{4}{3}$ and the left hand side of (33) is at least $\frac{4}{3}\frac{21}{24} \geq 1$.
- If $x = 3$, then the LHS is at least $\frac{3}{2} \cdot \frac{20}{24} = 1$.
- If $x = 4$, then the LHS is at least $\frac{16}{9} \cdot \frac{19}{24} > 1$.
- If $x = 5$, then the LHS is at least $\frac{16}{9} \cdot \frac{18}{24} > 1$.
- If $x \geq 6$ (and $s = x$), then the LHS is at least $2\frac{x+1}{2x+2} = 1$.

To conclude this case, we remark that the inequalities $u \geq \max\{24, 2s+2\}, n \geq (s+u)k, k \geq 2$ are sufficient for all the considerations above to work.

Using the fifth statement from Lemma 8, we get that (33) is satisfied for $u = s + o(s)$. Indeed, let $u = s + \delta\frac{s(\log\log s)^2}{\log s}$ with some δ that will be determined later. If $2x + 24 \leq s$, then the condition $u \geq s$ is sufficient to satisfy (33) by the previous paragraph. Thus, we may assume that $x \geq (\log s)$. Then, applying the fifth point of Lemma 8 with $x = \log s$, we get

$$\gamma \cdot \frac{u-x-1}{u} \geq \Omega\Big(\frac{\log s}{(\log\log s)^2}\Big)\frac{u-s-1}{u} = \Omega\Big(\frac{\log s}{(\log\log s)^2}\Big)\frac{\delta\frac{s(\log\log s)^2}{\log s}}{s} > 1,$$

if δ is sufficiently large. Remark that for the application of Lemma 8 we need that $u \geq 2^x$, which clearly holds in this case, since $u \geq s$. To conclude the proof of the theorem, we note that in the case $\nu(\mathcal{G}(\emptyset, s)) = 1$ the condition $u \geq s+3, k \geq 3$ was sufficient. Thus, $n \geq (2s + o(s))k, k \geq 3$ is a sufficient condition in this case. The proof of Theorem 3 for shifted families is complete.

The case of not shifted $\mathcal{G}$

Consider an arbitrary family $\mathcal{G}$ satisfying the requirements of the theorem. Since the property $\tau(\mathcal{G}) > s$ is not necessarily maintained by shifting, we cannot make the family $\mathcal{G}$ shifted right away. However, each (i,j)-shift, $1 \leq i < j \leq n$, decreases $\tau(\mathcal{G})$ by at most 1, and so we perform the (i,j)-shifts $(1 \leq i < j \leq n)$ one by one until either $\mathcal{G}$ becomes shifted or $\tau(\mathcal{G}) = s+1$. In the former case we fall into the situation of the previous subsection.

Assume w.l.o.g. that $\tau(\mathcal{G}) = s+1$ and that each set from $\mathcal{G}$ intersects $[s+1]$. Then all families $\mathcal{G}(\{i\}, s+1)$, $i = 1, \ldots, s+1$, are nonempty. Make the family $\mathcal{G}$ shifted in coordinates $s+2, \ldots, n$ by performing all the (i,j)-shifts for $s+2 \leq i < j \leq n$. Denote the new family by $\mathcal{G}$ again. Since the shifts do not increase the matching number, we have $\nu(\mathcal{G}) \le s$ and $\tau(\mathcal{G}) \le s+1$. Each family $\mathcal{G}(\{i\}, s+1)$ contains the set $[s+2, s+k]$.

Next, perform all possible shifts on coordinates $1, \ldots, s+1$, and denote the resulting family by $\mathcal{G}'$. We have $|\mathcal{G}'| = |\mathcal{G}|, \nu(\mathcal{G}') \le s$, and, most importantly, $\mathcal{G}'(\{i\}, s+1)$ are nested and non-empty for all $i = 1, \ldots, s+1$. The last claim is true due to the fact that all of the families contained the same set before the shifting.

We can actually apply the proof of the previous subsection to $\mathcal{G}'$. Indeed, the main consequence of the shiftedness we were using is that $\mathcal{G}'(\{i\}, s+1), i = 1, \ldots, s+1$, are all non-empty and nested. We do have it for $\mathcal{G}'$. The other consequence was the bound (17), which we do not need in this case as $\mathcal{G}'(\emptyset, s+1)$ is empty since each set from $\mathcal{G}'$ intersects $[s+1]$. The proof of Theorem 3 is complete.

3 Proof of Theorems 1 and 2

Proof of Theorem 1

Take s cross-dependent families $\mathcal{F}_1, \ldots, \mathcal{F}_s$. For $s = 2$ the bound (1) states that $|\mathcal{F}_1| + |\mathcal{F}_2| \le 2^{[n]}$, which follows from the following trivial observation: if $A \in \mathcal{F}_1$, then $[n] \setminus A \notin \mathcal{F}_2$. Thus, we may assume that $s \ge 3$. Also, the case of $m = 0$ is very easy to verify for any s, so we assume that $m \ge 1$.

Put $n = s(m+1) - l$ for the rest of this section. We assume that the families in question are closed upward.

We first reduce Theorem 1 to the following statement, which proof is given at the end of the section:

Proposition 10. *For $n' = 2s' - l', 0 < l \le s'$, and s' cross-dependent families $\mathcal{F}'_1, \ldots, \mathcal{F}'_{s'} \subset \binom{[n']}{1} \cup \binom{[n']}{2}$ we have*

$$\sum_{i=1}^{s'} |\mathcal{F}'_i| \le (l'-1)n' + s'\binom{n'}{2}. \tag{34}$$

Take s pairwise disjoint sets $H_1, \ldots, H_s$ of size $m-1$ at random. W.l.o.g. assume that the s elements of $[n] \setminus \cup_{i=1}^s H_i$ form the set $[2s-l]$. For $S \subset [2s-l]$ define $H_i(S) := H_i \cup S$.

Let $\emptyset =: H_i^{(0)} \subset \ldots \subset H_i^{(m-1)} := H_i$ be a randomly chosen full chain in $H_i, i = 1, \ldots, s$.

For each value of $i = 1, \ldots, s, j = 0, \ldots, m-2$, and $S \subset [2s-l]$ define the random variables $\beta_i^{(j)}$ and $\beta_i(S)$:

$$\beta_i^{(j)} = \begin{cases} 1 & \text{if } \ H_i^{(j)} \in \mathcal{F}_i, \\ 0 & \text{if } \ H_i^{(j)} \notin \mathcal{F}_i; \end{cases} \qquad \beta_i(S) = \begin{cases} 1 & \text{if } \ H_i(S) \in \mathcal{F}_i, \\ 0 & \text{if } \ H_i(S) \notin \mathcal{F}_i. \end{cases} \tag{35}$$

Note that S may be the empty set. The cross-dependence of $\mathcal{F}_i$ implies

$$\beta_1(S_1)\beta_2(S_2) \cdot \ldots \cdot \beta_s(S_s) = 0 \quad \text{whenever} \quad S_1, \ldots, S_s \subset [2s-l] \ \text{ are pairwise disjoint.} \tag{36}$$

The expectations $\mathrm{E}[\beta_i^{(j)}], \mathrm{E}[\beta_i(S)]$ satisfy

$$\mathrm{E}[\beta_i^{(j)}] = \frac{\left|\mathcal{F}_i \cap \binom{[n]}{j}\right|}{\binom{n}{j}}, \qquad \mathrm{E}[\beta_i(S)] = \frac{\left|\mathcal{F}_i \cap \binom{[n]}{m-1+|S|}\right|}{\binom{n}{m-1+|S|}}. \tag{37}$$

Our aim is to prove

Lemma 11. *Let $n = s(m+1) - l$. For every choice of $H_1, \ldots, H_s$ and the full chains one has*

$$\sum_{i=1}^{s} \left[\sum_{j=1}^{m-2} \binom{n}{j} \beta_i^{(j)} + \sum_{S \subsetneq [2s-l], |S| \le 3} \frac{\binom{n}{m-1+|S|}}{\binom{2s-l}{|S|}} \beta_i(S) \right] \le s \sum_{j=m+1}^{m+2} \binom{n}{j} + (l-1)\binom{n}{m}. \tag{38}$$

Passing to the expectations in (38), it is straightforward to see that from (37) we have

$$\sum_{i=1}^{s} \sum_{j=1}^{m+2} \left| \mathcal{F}_i \cap \binom{[n]}{j} \right| \le s \sum_{j=m+1}^{m+2} \binom{n}{j} + (l-1)\binom{n}{m},$$

from which (1) follows.

Proof of Lemma 11. We have $\boldsymbol{\sum_{i=1}^{s} \beta_i(\emptyset) = p}$ for some $0 \le p \le s-1$. W.l.o.g. assume that $\beta_1(\emptyset) = \ldots = \beta_p(\emptyset) = 1$.

Assume that $\boldsymbol{l/2 \le p \le s-2}$. In this case we have $s - p \le s - l/2 = \frac{1}{2}(2s-l)$. Then $\prod_{i=p+1}^{s} \beta_i(S_i) = 0$ for any $s-p$ pairwise disjoint S_i of cardinality two. By a simple averaging argument we immediately get that

$$\beta_i(S) = 0 \text{ for at least } \binom{2s-l}{2} \text{ pairs } (i,S), \text{ where } |S| = 2 \text{ and } i \in [p+1, s].$$

Since the families $\mathcal{F}_i$ are closed upward, we also get that

$$\beta_i(S) = 0 \text{ for at least } 2s-l \text{ pairs } (i,S), \text{ where } |S| = 1 \text{ and } i \in [p+1, s].$$

Therefore, the left hand side of (38) is at most $p\sum_{j=0}^{m-1}\binom{n}{j} + (s-1)\sum_{j=m}^{m+1}\binom{n}{j} + s\binom{n}{m+2}$. Consider the difference between the right and the left hand side of (38). The difference is at least

$$\binom{n}{m+1} - (s-l)\binom{n}{m} - p\sum_{j=0}^{m-1}\binom{n}{j}. \tag{39}$$

Next we show that this expression is always nonnegative.

For $m = 0$ the inequality (39) obviously holds, so we assume that $m \ge 1$. Remark that we have

$$\frac{\binom{n}{m-j-1}}{\binom{n}{m-j}} = \frac{m-j}{n-m+j+1} > \begin{cases} \frac{m-1}{(s-1)m} & \text{for } j \ge 1, \\ \frac{m}{(s-1)m+s-l+1} & \text{for } j \ge 0. \end{cases} \tag{40}$$

We have

$$\binom{n}{m+1} - (s-l)\binom{n}{m} = \Big(\frac{m(s-1)+s-l}{m+1} - (s-l)\Big)\binom{n}{m} = \frac{m(l-1)}{m+1}\binom{n}{m}. \qquad (41)$$

On the other hand, using (40) we may obtain that $\sum_{j=1}^{m} p\binom{n}{m-j}$ is at most

$$\sum_{j=1}^{m}(s-2)\binom{n}{m-j} \le \frac{m(s-1)(s-2)}{(s-2)m+1}\binom{n}{m-1} \le \frac{m(s-1)}{(s-1)m+s-l+1}\binom{n}{m}. \qquad (42)$$

It is easy to see that the right hand side of (41) is at least the right hand side of (42) for both $l=2$ and $l\ge 3$, which proves (38) in this case.

If $\boldsymbol{p = s-1}$, then $\beta_s(S)=0$ for all $S\subset[2s-l], |S|\le 3$. Therefore, the left hand side of (38) is at most $(s-1)\sum_{j=0}^{m+2}\binom{n}{j}$, which is smaller than $\binom{n}{m}+s\binom{n}{m+1}+s\binom{n}{m+2}$ for any $s\ge 3$. Indeed, the difference is

$$\binom{n}{m+2}+\binom{n}{m+1}-(s-l)\binom{n}{m}-(s-1)\sum_{j=0}^{m-1}\binom{n}{j},$$

which by the calculations in the previous case is at least $\binom{n}{m+2}-\sum_{j=0}^{m-1}\binom{n}{j}$. We have $\binom{n}{m+2}\ge\binom{n}{m}$ since $n=s(m+1)-2\ge 2(m+1)$ for any $s\ge 3, m\ge 1$. Finally, $\binom{n}{m}\ge\sum_{j=0}^{m-1}\binom{n}{j}$ by (40).

Assume that $\boldsymbol{p < l/2}$. Then again $\prod_{i=p+1}^{s}\beta_i(S_i)=0$ for any $s-p$ pairwise disjoint S_i. Consider the families $\mathcal{F}_i' := \{S\subset[2s-l] : \beta_i(S)=1, |S|\le 2\}$, $i=p+1,\dots,s$. These families are cross-dependent. Applying (34) to $\mathcal{F}_i'$ (with $n' := 2(s-p)-(l-2p)$, $s'=s-p$, $l'=l-2p$), we get that

$$\sum_{i=p+1}^{s}\Big|\mathcal{F}_i'\cap\binom{[2s-l]}{\le 2}\Big| \le (l-2p-1)(2s-l)+(s-p)\binom{2s-l}{2}.$$

We conclude that out of all coefficients $\beta_i(S)$, $i=1,\dots,s, 1\le|S|\le 2$, there are at least $(2s-l)(s-l+p+1)$ that are equal to zero. The following is verified by simple calculations.

Observation 12. *Let $s\ge 3, m\ge 1$. In the summation over S in (38) the coefficient in front of $\beta_i(S_1)$ for $|S_1|=1$ is not bigger than the coefficient in front of $\beta_i(S_2)$ for $|S_2|=2$.*

Using the observation, we get that the left hand side of (38) is at most

$$p\sum_{j=0}^{m-1}\binom{n}{j}+(l-p-1)\binom{n}{m}+s\binom{n}{m+1}+s\binom{n}{m+2}.$$

Since $\binom{n}{m}\ge\sum_{j=0}^{m-1}\binom{n}{j}$, the last expression is at most $(l-1)\binom{n}{m}+s\binom{n}{m+1}$. □

Proof of Proposition 10. The proof follows the same logic as the one given above. Choose s' distinct elements of $[n']$. W.l.o.g. assume that these elements form a set $[s']$. By analogy to (35), put $\beta_i(S) = 1$ if $S \in \mathcal{F}'_i$ and $\beta_i(S) = 0$ otherwise. Similarly to (38), it is sufficient to prove

$$\sum_{i=1}^{s'} \left[n'\beta_i(\{i\}) + \sum_{x=s'+1}^{n'} \frac{\binom{n'}{2}}{n'-s'} \beta_i(\{i,x\}) \right] \le (l'-1)n' + s'\binom{n'}{2}. \tag{43}$$

Assume that $\sum_{i=1}^{s'} \beta_i(\{i\}) = p$ and that w.l.o.g. $\beta_1(\{1\}) = \ldots = \beta_p(\{p\}) = 1$. If $p \le l' - 1$, then we are done. In the case $l' = s'$ the statement of the proposition is obvious since at least one of $\beta_i(\emptyset)$ must be equal to 0. Thus, we assume that $l \le p \le s' - 1$.

Recall that $n' = s' + (s' - l')$. For any set of distinct $x_{p+1}, \ldots, x_{s'} \in [s'+1, n']$ we have $\prod_{i=p+1}^{s'} \beta_i(\{i, x_i\}) = 0$. By simple averaging, there are at least $n' - s' = s' - l'$ pairs (i, x), $i = p+1, \ldots, s'$, $x \in [s'+1, n']$, for which we have $\beta_i(\{i, x\}) = 0$. Therefore, the left hand side of (43) in this case does not exceed $(s'-1)\left[\binom{n'}{2} + n'\right]$, which is at most the right hand side of (43), since $\binom{n'}{2} \ge (s'-l')n'$. □

Proof of Theorem 2

We prove the theorem by double induction. We apply induction on m, and for fixed m the induction on n. The case $m = 0$ of (5) is very easy to verify. The case $n = q$ is the bound (1).

We may assume that all the $\mathcal{F}_i$ are shifted. The following two families on $[n-1]$ are typically defined for a family $\mathcal{F} \subset 2^{[n]}$:

$$\mathcal{F}(n) := \{A - \{n\} : n \in A, A \in \mathcal{F}\},$$
$$\mathcal{F}(\bar{n}) := \{A : n \notin A, A \in \mathcal{F}\}.$$

It is clear that $\mathcal{F}_1(\bar{n}), \ldots, \mathcal{F}_s(\bar{n})$ are q-dependent. Next we show that $\mathcal{F}_1(n), \ldots, \mathcal{F}_s(n)$ are $(q-s)$-dependent. Assume for contradiction that $F_1, \ldots, F_s$, where $F_i \in\in \mathcal{F}_i(n)$, are pairwise disjoint and that $H := F_1 \cup \ldots \cup F_s$ has size at most $q - s$. Since $n \ge q$, $n - (q-s) \ge s$. That is, we can find distinct elements $x_1, \ldots, x_s \in [n] - H$. Since $\mathcal{F}_i$ are shifted, we have $F_i \cup \{x_i\} \in \mathcal{F}_i$ for $i = 1, \ldots, s$, and the sets $F_i \cup \{x_i\}$ are pairwise disjoint. Their union $H \cup \{x_1, \ldots, x_s\}$ has the size $|H| + s \le q$, a contradiction.

Recall the definition (4). The induction hypothesis for $\mathcal{F}_i(\bar{n})$ gives

$$\sum_i |\mathcal{F}_i(\bar{n})| \le \sum_i |\tilde{\mathcal{F}}_i^{n-1,q}|. \tag{44}$$

Applying the induction hypothesis to $\mathcal{F}_i(n)$ with $(q-s) = s(m-1) - l$ gives

$$\sum_i |\mathcal{F}_i(n)| \le \sum_i |\tilde{\mathcal{F}}_i^{n-1,q-s}|. \tag{45}$$

Adding (44) and (45), we get

$$\sum_i |\mathcal{F}_i| = \sum_i \big(|\mathcal{F}_i(n)| + |\mathcal{F}_i(\bar{n})|\big) \le \sum_i \big(|\tilde{\mathcal{F}}_i^{n-1,q}| + |\tilde{\mathcal{F}}_i^{n-1,q-s}|\big).$$

We have $\tilde{\mathcal{F}}_i^{n-1,q} = \tilde{\mathcal{F}}_i^{n,q}(\bar{n})$ and $\tilde{\mathcal{F}}_i^{n-1,q-s} = \tilde{\mathcal{F}}_i^{n,q}(n)$. Thus, for any i

$$|\tilde{\mathcal{F}}_i^{n-1,q}| + |\tilde{\mathcal{F}}_i^{n-1,q-s}| = |\tilde{\mathcal{F}}_i^{n,q}(\bar{n})| + |\tilde{\mathcal{F}}_i^{n,q}(n)| = |\tilde{\mathcal{F}}_i^{n,q}|.$$

4 Application of Theorem 3 to an anti-Ramsey problem

Let $\binom{[n]}{k} = \mathcal{F}_0 \cup \ldots \cup \mathcal{F}_{M-1}$ be a coloring. The following quantity was studied by Özkahya and Young [16]: the minimum value $ar(n,k,s)$ of M such that in any coloring as above with M colors there is a *rainbow* s-matching, that is, a set of s pairwise disjoint k-sets from pairwise distinct $\mathcal{F}_i$. They have conjectured that $ar(n,k,s) = e_k(n,s-1)+2$ and proved the conjecture for $s=3$ and for $n \ge 2k^3 s$. They also obtained the bound $ar(n,k,s) \le e_k(n,s-1)+s$ for $n \ge sk + (s-1)(k-1)$.

It is not difficult to see that $ar(n,k,s) \ge e_k(n,s-1)+2$ for any n,k,s. Indeed, consider the maximal family of k-sets with no $(s-1)$-matching and assign a different color to each of these sets. Next, assign the same color to all the remaining sets. This is a coloring of $\binom{[n]}{k}$ without a rainbow matching.

In this section we state and prove a result (unfortunately, only in a certain range), which is much stronger than the conjecture from [16]. We say that the coloring of $\binom{[n]}{k}$ is *star-like* if there exists a set $Y \subset \binom{[n]}{s-2}$ and a number $i, 0 \le i \le M-1$, such that each set $F \in \mathcal{F}_j, j \in \{0, M-1\} - \{i\}$, intersects Y. Clearly, each star-like coloring has at most $e_k(n,s-1)+1$ colors. For the convenience of the forthcoming formulation of the theorem we define the quantity

$$h(n,k,s) := \max\{|\mathcal{F}| : \mathcal{F} \subset \binom{[n]}{k}, \nu(\mathcal{F}) < s, \tau(\mathcal{F}) \ge s\},$$

which was determined for a certain range in Theorem 3.

Theorem 13. *Let $s \ge 3$, $k \ge 2$, $n \ge 2sk$ be some integers. Consider a coloring of $\binom{[n]}{k}$ into M colors. Then either this coloring is star-like, or $M \le h(n,k,s-1)+s$.*

The aforementioned conjecture of Özkahya and Young follows from Theorem 13, once we can apply Theorem 3 or an analogous statement. Indeed, it is shown in Theorem 3 that $h(n,k,s-1)$ is much smaller than $e_k(n,s-1)$. Moreover, we do not need a precise Hilton-Milner-type result here, so we can use a weaker form of Theorem 3 that was proven in [8]. Theorem 5 in [8] implies, in particular, that for $n \ge sk + (s-1)(k-1)$ we have $e_k(n,s-1) - h(n,k,s-1) \ge \frac{1}{s}\binom{n-s-k+2}{k-1}$. Since for such n and $k \ge 3$ we have $\frac{1}{s}\binom{n-s-k+2}{k-1} > s-2$, we conclude that the conjecture is verified for this range.

Corollary 14. *We have $ar(n,k,s) = e_k(n,s-1)+2$ for $n \ge sk+(s-1)(k-1)$, $k \ge 3$.*

Note that this is the same range in which Özkahya and Young got a weaker bound $ar(n,k,s) \le e_k(n,s-1)+s$. We remark that the case $k=2$ has already been settled for all values of parameters (see [16] for the history of the problem).

Fix a coloring in $M = h(n,k,s-1)+s$ colors. We may assume that there is a rainbow $(s-1)$-matching $F_1,\dots,F_{s-1}$ with, say, $F_i \in \mathcal{F}_{M-i}, i=1,\dots,s-1$.

Let $G_i \in \mathcal{F}_i, i = 0\dots,M-s$, be an arbitrary system of distinct representatives of these color classes. Note that $M-s+1 > h(n,k,s)$. Thus, either $\mathcal{G} := \{G_0,\dots,G_{M-s}\} \subset \binom{[n]}{k} - \binom{U}{k}$ for a suitable $U \subset \binom{[n]}{n-s+2}$, or there is an $s-1$-matching, say $G_1,\dots,G_{s-1}$, in $\mathcal{G}$.

In the latter case we can apply the argument of Özkahya and Young: since the colors of $F_1,\dots,F_{s-1},G_1,\dots,G_{s-1}$ are all distinct, $H_i \cap (F_1 \cup \ldots \cup F_{s-1}) \ne \emptyset$ holds for $1 \le i < s$. On the other hand, any k-set from $[n] \setminus \cup_{i=1}^{s-1} H_i \cup F_i$ will form a rainbow s-matching with one of these two $s-1$-matchings. Therefore, we are done in this case if $n \ge k+(2k-1)(s-1) = sk+(s-1)(k-1)$.

In the former case the family $\mathcal{G}$ must satisfy $\tau(\mathcal{G}) \le s-2$ for all choices of the representatives $G_i \in \mathcal{F}_i$. Let $T := T(\mathcal{G})$ be a *cover* of size $s-2$, that is, $T \cap G \ne \emptyset$ for all $G \in \mathcal{G}$.

Claim 15. *Fix $N \ge h(n,k,s-1)$ and pairwise disjoint families $\mathcal{H}_i, i = 0,\dots N$, of k-subsets of $[n]$. If for any set of representatives $\mathcal{H} := \{H_0,\dots,H_N\}$ with $H_i \in \mathcal{H}_i$ there is a cover $T = T(\mathcal{H})$ of size $s-2$, then T is the cover for the family $\cup_{i=0}^{N} \mathcal{H}_i$.*

Proof. Assume the contrary: that there is a set $H' \in H_0$ such that $H' \cap T = \emptyset$. The family $\mathcal{H}' := \{H', H_1,\dots,H_N\}$ also satisfies $\tau(\mathcal{H}') \le s-2$, and so there is a set $T' \ne T$, $|T'| = s-2$, such that $H_1,\dots,H_N$ all intersect T'. Define $m(T,T') := \left|\{F \subset \binom{[n]}{k} : F \cap T \ne \emptyset \ne F \cap T'\}\right|$. We want to show that for any distinct T, T' of size $s-2$ the quantity $m(T,T')$ is smaller than $h(n,k,s-1)$. This will clearly lead to the contradiction.

Let us show that $m(T,T')$ is maximal when $|T \cap T'| = |T|-1 = s-3$. Indeed if $|T \cap T'| < |T|-1$, then we may choose $x \in T \setminus T', y \in T' \setminus T$ and define $T'' := (T - \{y\}) \cup \{x\}$. Let F be an arbitrary set satisfying $F \cap T \ne \emptyset$ and $F \cap T' \ne \emptyset$ and $F \cap T'' = \emptyset$. This means that $F \cap (T' \cup T'') = \{y\}$, $F \cap ((T - \{x\}) \setminus T') \ne \emptyset$.

Setting $|T \cap T'| = t$, the number of such sets $F \in \binom{[n]}{k}$ is $\binom{n-(s-2)-1}{k-1} - \binom{n-2(s-2)+t}{k-1}$.

On the other hand, the sets F satisfying $F \cap T \ne \emptyset, F \cap T'' \ne \emptyset$, and $F \cap T; = \emptyset$ are those with $F \cap (T' \cup T'') = \{x\}$. Their number is $\binom{n-(s-2)-1}{k-1}$, which is clearly bigger.

Assuming that $|T \cap T'| = s-3$, we get that $m(T,T') = \binom{n}{k} - \binom{n-(s-3)}{k} + \binom{n-(s-1)}{k-2}$. Since $h(n,k,s-1) \ge \binom{n}{k} - \binom{n-(s-2)}{k} - \binom{n-(s-2)-k}{k-1} + 1$, we have $h(n,k,s-1) - m(T,T') > \binom{n-(s-2)}{k-1} - \binom{n-(s-2)-k}{k-1} > 0$. This completes the proof of the claim. □

Applying Claim 15 to the first $M-s$ color classes, we get that they all intersect the set T of size $s-2$. To complete the proof we need to show that the same holds for some $M-1$ colors.

Note that since $\sum_{i=0}^{M-s}|\mathcal{F}_i| \le \binom{n}{k} - \binom{n-(s-2)}{k}$, one of the last $s-1$ color classes, say $\mathcal{F}_{M-1}$, has size at least $\frac{1}{s-1}\binom{n-s+2}{k}$.

Claim 16. *In every rainbow $(s-1)$-matching one of the k-sets belongs to $\mathcal{F}_{M-1}$.*

Proof. Assume the contrary. Let the color classes forming the $(s-1)$-matching in question be $\mathcal{F}_{M-s}, \dots, \mathcal{F}_{M-2}$. Applying Claim 15 to $\mathcal{F}_0, \dots, \mathcal{F}_{M-s-1}, \mathcal{F}_{M-1}$, we find a cover T of size $s-2$ for $\mathcal{F}_0 \cup \ldots \cup \mathcal{F}_{M-s-1} \cup \mathcal{F}_{M-1}$. We infer

$$M-s+\frac{1}{s-1}\binom{n-s+2}{k} \le |\mathcal{F}_0 \cup \ldots \cup \mathcal{F}_{M-s-1} \cup \mathcal{F}_{M-1}| \le \binom{n}{k} - \binom{n-s+2}{k}. \quad (46)$$

We have $M-s \ge h(n,k,s-1) > \binom{n}{k} - \binom{n-(s-3)}{k}$. Also, we have

$$\binom{n-(s-3)}{k} = \frac{n-s+2-k}{n-s-2}\binom{n-(s-2)}{k} \ge \frac{s-1}{s}\binom{n-(s-2)}{k},$$

provided $n-s+2 \ge sk$. The inequalities above contradict (46), and so the claim follows. □

We conclude that there is no rainbow $(s-1)$-matching in $\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_{M-2}$, which implies that there is a cover of size $s-2$ for any set of distinct representatives of the color classes. In turn, Claim 15 implies that $\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_{M-2}$ can be covered by a set T of size $s-2$, i.e., the coloring is star-like. Theorem 13 is proved.

5 Almost matchings

Let us say that the sets $F_1, \dots, F_s$ form an *almost matching* if $\{F_1, \dots, F_s\}$ has at most one vertex of degree greater than one and even that vertex has degree at most two.

Define

$$a(m,s) := \max\big\{|\mathcal{F}| : \mathcal{F} \subset 2^{[n]}, \mathcal{F} \text{ contains no almost matching of size } s\big\}.$$

Since almost matching includes matching, $a(m,s) \le e(m,s)$ is obvious.

Theorem 17. *The inequality $a(sm-2,s) = \sum_{t\ge m}\binom{sm-2}{t}$ holds for all $s \ge 2, m \ge 1$. Moreover, the equality holds iff $\mathcal{F} = \{F \subset [n], |F| \ge m\}$.*

Proof. We may suppose that $\mathcal{F}$ is closed upwards. Consider the families $\mathcal{F}_1, \dots, \mathcal{F}_s$, where $\mathcal{F}_i := \mathcal{F}$ for $i = 1, \dots, s-1$, and $\mathcal{F}_s := \partial\mathcal{F} \cup \{[n]\}$. (see (16)).

Claim 18. *The families $\mathcal{F}_1, \dots, \mathcal{F}_s$ are cross-dependent.*

Proof. Indeed, if $F_1, \ldots, F_s$, where $F_i \in \mathcal{F}_i$, are pairwise disjoint, then, replacing F_s by some $F \in \mathcal{F}$, $F_s \subset F, |F \setminus F_s| = 1$, we obtain s members $F_1, \ldots, F_{s-1}, F$ of $\mathcal{F}$ that are almost disjoint. $\square$

Applying Theorem 1 yields

$$\sum_{i=1}^{s} |\mathcal{F}_i| \le \binom{n}{m-1} + s \sum_{t \ge m} \binom{n}{t}. \tag{47}$$

Now recall Harper's Theorem. In the special case $|\mathcal{F}| \ge \sum_{t=m}^{n} \binom{n}{t}$ it gives $|\partial \mathcal{F}| \ge \sum_{t=m-1}^{n-1} \binom{n}{t}$. Therefore, $|\partial \mathcal{F} \cup \{[n]\}| \ge \sum_{t=m-1}^{n} \binom{n}{t}$. Moreover, by Harper's theorem, both equalities are strict for $\mathcal{F}$ upward closed unless $\mathcal{F} = \{F \subset [n] : |F| \ge m\}$. Together with (47) this yields the statement of the theorem. $\square$

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