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Intersection theorems for $\{0, \pm 1\}$-vectors and s-cross-intersecting families

Peter Frankl, Andrey Kupavskii*

Abstract

In this paper we pursue two possible directions for the extension of the classical Erdős-Ko-Rado theorem which states that any family of k-element, pairwise intersecting subsets of the set $[n] := \{1, \ldots, n\}$ has cardinality at most $\binom{n-1}{k-1}$.

In the first part of the paper we study families of $\{0, \pm 1\}$-vectors. Denote by $\mathcal{L}_k$ the family of all vectors $\mathbf{v}$ from $\{0, \pm 1\}^n$ such that $\langle \mathbf{v}, \mathbf{v} \rangle = k$. For any k and l and sufficiently large n, as well as for small k and any l, n, we determine the maximal size of a family $\mathcal{V} \subset \mathcal{L}_k$ satisfying $\langle \mathbf{v}, \mathbf{w} \rangle \geq l$ for any $\mathbf{v}, \mathbf{w} \in \mathcal{V}$.

In the second part of the paper we study cross-intersecting pairs of families. We say that two families $\mathcal{A}, \mathcal{B}$ are *s-cross-intersecting*, if for any $A \in \mathcal{A}, B \in \mathcal{B}$ we have $|A \cap B| \geq s$. We also say that a family $\mathcal{A}$ of sets is *t-intersecting*, if for any $A_1, A_2 \in \mathcal{A}$ we have $|A_1 \cap A_2| \geq t$. For a pair of nonempty s-cross-intersecting t-intersecting families $\mathcal{A}, \mathcal{B}$ of k-sets, we determine the maximal value of $|\mathcal{A}| + |\mathcal{B}|$ for sufficiently large n .

1 Introduction

Let $[n] = \{1, \ldots, n\}$ be an n-element set and for $0 \leq k \leq n$ let $\binom{[n]}{k}$ denote the collection of all its k-element subsets. Further, let $2^{[n]}$ denote the power set of $[n]$. Any subset $\mathcal{F} \subset 2^{[n]}$ is called a *set family* or *family* for short. We say a "k-set" instead of a "k-element set" for shorthand. If $F \cap F' \neq \emptyset$ for all $F, F' \in \mathcal{F}$, then $\mathcal{F}$ is called *intersecting*. The following classical result is one of the cornerstones of extremal set theory.

Theorem (Erdős-Ko-Rado theorem [2])**.** *If $\mathcal{A} \subset \binom{[n]}{k}$ is intersecting and $n \geq 2k$, then $|\mathcal{A}| \leq \binom{n-1}{k-1}$ holds.*

This is one of the first results in extremal set theory and probably the first result about intersecting families.

*Moscow Institute of Physics and Technology, École Polytechnique Fédérale de Lausanne; Email: `kupavskii@yandex.ru` Research supported in part by the Swiss National Science Foundation Grants 200021-137574 and 200020-14453 and by the grant N 15-01-03530 of the Russian Foundation for Basic Research.

Numerous results extended the Erdős-Ko-Rado theorem in different ways. One of the directions is concerned with *nontrivial* intersecting families. We say that an intersecting family is *nontrivial* if the intersection of *all* sets from the family is empty. Note that the size of a family of all k-sets containing a single element matches the bound from the Erdős-Ko-Rado theorem. Probably, the best known result in this direction is the Hilton-Milner theorem [12], which determines the maximum size of a non-trivial intersecting family.

In the same paper Hilton and Milner dealt with pairs of *cross-intersecting* families. We say that the families $\mathcal{A}, \mathcal{B} \subset 2^{[n]}$ are *cross-intersecting* if for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have $A \cap B \neq \emptyset$. Hilton and Milner proved the following inequality:

Theorem (Hilton and Milner [12])**.** *Let $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$ be non-empty cross-intersecting families with $n \geq 2k$. Then $|\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{k} - \binom{n-k}{k} + 1$.*

This inequality was generalized by Frankl and Tokushige [10] to the case $\mathcal{A} \subset \binom{[n]}{a}$, $\mathcal{B} \subset \binom{[n]}{b}$ with $a \neq b$, as well as to more general constraints on the sizes of $\mathcal{A}, \mathcal{B}$. A simple proof of the theorem above may be found in [8].

Another natural direction for the generalizations deals with more general constraints on the sizes of pairwise intersections. A family $\mathcal{F}$ is called *t-intersecting* if for any $F_1, F_2 \in \mathcal{F}$ we have $|F_1 \cap F_2| \geq t$. The Complete Intersection Theorem due to Alswede and Khachatrian [1], continuing previous work done by Frankl [3] and Frankl and Füredi [6], gave an exhaustive answer to the question of how big a t-intersecting family of k-sets could be. We discuss it in more detail in Section 3.

One more example of a result of such kind is a theorem due to Frankl and Wilson [11]. It states that, if n is a prime power and $\mathcal{F} \subset \binom{[4n-1]}{2n-1}$ satisfies $|F_1 \cap F_2| \neq n-1$ for any $F_1, F_2 \in \mathcal{F}$, then the size of $\mathcal{F}$ is at most $\binom{4n-1}{n-1}$, which is exponentially smaller than $\binom{4n-1}{2n-1}$. This theorem has several applications in geometry.

The third direction is to generalize the EKR theorem to more general classes of objects. Out of the numerous generalizations of such kind we are going to talk only about the ones concerned with $\{0, \pm 1\}$-vectors, that is, vectors that have coordinates from the set $\{-1, 0, 1\}$. In this case a natural replacement for the size of the intersection is the scalar product. A generalization of [11] to the case of $\{0, \pm 1\}$-vectors due to Raigorodskii [15] led to improved bounds in several geometric questions, such as Borsuk's problem and the chromatic number of the space. However, it seems plausible to improve Raigorodskii's result, once one achieves a better understanding of "intersecting" families of $\{0, \pm 1\}$-vectors. Partly motivated by that, in our recent work [7] we managed to extend EKR theorem to the case of $\{0, \pm 1\}$-vectors with fixed number of 1's and -1's.

The contribution of this paper is two-fold. First, in several scenarios we determine the maximal size of a family of $\{0, \pm 1\}$-vectors of fixed length and with restrictions on the scalar product. The main results are stated in Section 2.2. Second, we prove results analogous to the Hilton-Milner theorem stated above, but for pairs of families that are s-cross-intersecting and t-intersecting. For details and precise definitions see Section 3.

2 Families of $\{0, \pm 1\}$-vectors

Denote by $\mathcal{L}_k$ the family of all vectors $\mathbf{v}$ from $\{0, \pm 1\}^n$ such that $\langle \mathbf{v}, \mathbf{v} \rangle = k$. Note that $|\mathcal{L}_k| = 2^k \binom{n}{k}$. This section is mostly devoted to the study of the quantity below.

$$F(n, k, l) := \max\{|\mathcal{V}| : \mathcal{V} \subset \mathcal{L}_k, \forall \mathbf{v}, \mathbf{w} \in \mathcal{V} \;\; \langle \mathbf{v}, \mathbf{w} \rangle \geq l\}. \tag{1}$$

Recall the following theorem of Katona:

Theorem (Katona, [13]). *Let $n > s > 0$ be fixed integers. If $\mathcal{U} \subset 2^{[n]}$ is a family of sets such that for any $U, V \in \mathcal{U}$ we have $|U \cup V| \leq s$ then*

$$|\mathcal{U}| \leq f(n, s) := \begin{cases} \displaystyle\sum_{i=0}^{s/2} \binom{n}{i} & \text{if } s \text{ is even,} \\ \displaystyle 2\sum_{i=0}^{(s-1)/2} \binom{n-1}{i} & \text{if } s \text{ is odd.} \end{cases} \tag{2}$$

Moreover, for $n \geq s + 2$ the equality is attained only for the following families. If s is even, than it is the family $\mathcal{U}^s$ of all sets of size at most $s/2$. If s is odd, then it is one of the families U_j^s of all sets that intersect $[n] - \{j\}$ in at most $(s-1)/2$ elements, where $1 \leq j \leq n$.

Given two sets U, V, we denote the symmetric difference of these two sets by $U \triangle V$, that is, $U \triangle V := U \backslash V \cup V \backslash U$. A theorem due to Kleitman states that the bound (2) holds for a more general class of families.

Theorem (Kleitman, [14]). *If for any two sets U, V from a family $\mathcal{U} \subset 2^{[n]}$ we have $|U \triangle V| \leq s$, then the bound (2) holds for $\mathcal{U}$.*

Note that there is no uniqueness counterpart in Kleitman's theorem.

2.1 Simple properties of $F(n, k, l)$

First we state and prove some simple observations concerning $F(n, k, l)$.

Proposition 1. *Fix any $n \geq k \geq 1$. Then $F(n, k, -k+1) = 2^{k-1}\binom{n}{k} = |\mathcal{L}_k|/2$.*

Proof. We split vectors from $\mathcal{L}_k$ in pairs $\mathbf{v}, \mathbf{w}$ so that $\langle \mathbf{v}, \mathbf{w} \rangle = -k$. We can take exactly one vector out of each pair in the family. □

For any $\mathbf{v} = (v_1, \ldots, v_n)$ and any vector family $\mathcal{V}$ define $S(\mathbf{v}) = \{i : v_i \neq 0\}$ and $\mathcal{V}(S) = \{\mathbf{v} \in \mathcal{V} : S(\mathbf{v}) = S\}$. We also define $N(\mathbf{v}) = \{i : v_i = -1\}$.

Proposition 2. *We have $F(2t, 2t, 2s) = F(2t, 2t, 2s-1) = f(2t, t-s)$ and $F(2t+1, 2t+1, 2s) = F(2t+1, 2t+1, 2s+1) = f(2t+1, t-s)$.*

Proof. For any $\mathbf{v}, \mathbf{w} \in \mathcal{L}_{2t}$ we have $\langle \mathbf{v}, \mathbf{w} \rangle = 2t - 2|N(\mathbf{v}) \triangle N(\mathbf{w})|$. Therefore, the statement of the proposition follows from Kleitman's Theorem. □

We say that a family $\mathcal{V} \subset \mathcal{L}_k$ of vectors is *homogeneous*, if for every $i \in [n]$ the i'th coordinates of vectors from $\mathcal{V}$ are *all* non-negative or *all* non-positive.

Proposition 3. *For $n > k$ we have $F(n, k, k-1) = \max\{k+1, n-k+1\}$.*

Proof. First of all note that $u_i v_i = -1$ forces $\langle \mathbf{u}, \mathbf{v} \rangle \leq k - 2$. Thus $\mathcal{V}$ is homogeneous and the condition translates into $|S(\mathbf{u}) \cap S(\mathbf{v})| \geq k - 1$.

Since the only $(k-1)$-intersecting families of k-sets are stars around a $(k-1)$-element set and subsets of $\binom{[k+1]}{k}$, we are done. □

Given a vector family $\mathcal{V}$, for any $i \in [n]$ we denote by d_i the *degree* of i, that is, the number of vectors from $\mathcal{V}$ that have a nonzero coordinate on position i.

Proposition 4. *For $n > k \geq 2$ the following holds:*
(i) $F(3, 2, 0) = 4$.
(ii) $F(n, k, k-2) = \binom{n}{k}$ if $n = k+2$ or $k \geq 3$ and $n = k+1$.
(iii) $F(n, k, k-2) = \max\left\{\binom{n-k+2}{2}, k(n-k)+1\right\}$ for $n \geq k+3$.

Proof. The first part is very easy to verify. As for the other parts, assume that $u_i = -v_i$ for some $\mathbf{u}, \mathbf{v} \in \mathcal{V}$. Then $S(\mathbf{u}) = S(\mathbf{v})$ and $\mathbf{u}$ and $\mathbf{v}$ must agree in the remaining coordinate positions. Thus, $d_i = 2$ in this case. Since $F(k, k, k-2) = 2$, for $n = k+1$ a vertex of degree 2 would force $|\mathcal{V}| \leq 2 + 2 = 4$. On the other hand, if we assume that $\mathcal{V}$ is homogeneous, then $|\mathcal{V}| \leq \binom{k+1}{k}$ for $k \geq 3$, as desired. Since $\binom{k+2}{k} - \binom{k+1}{k} \geq 2$ for $k \geq 2$, the same argument works for $n = k+2, k \geq 2$ as well.

If $n > k+2$, the same argument as above implies that the family $\{S(\mathbf{u}) : \mathbf{u} \in \mathcal{V}\}$ is $(k-2)$-intersecting. If $|\mathcal{V}|$ is maximal, then it is homogeneous. The bound then follows from the Complete Intersection Theorem (see Section 3). □

2.2 Results

Our main result concerning $\{0, \pm 1\}$-vectors is the following theorem, which determines $F(n, k, l)$ for all k, l and sufficiently large n and which shows the connection of $F(n, k, l)$ with the above stated theorems of Katona and Kleitman.

Theorem 5. *For any k and $n \geq n_0(k)$ we have*

1. $$F(n, k, l) = \binom{n-l}{k-l} \qquad \text{for } 0 \leq l \leq k.$$

2. $$F(n, k, -l) = f(k, l)\binom{n}{k} \quad \text{for } 0 \leq l \leq k.$$

Note that $f(k,0) = 1$ and so the values for $l = 0$ in part 1 and $l = 0$ in part 2 coincide. We remark that the statement of part 2 of the theorem for $l = k$ is obvious and for $l = -k+1$ it was already derived in Proposition 1.

Using the same technique, we may extend the result of part 2 of Theorem 5 in the following way:

Theorem 6. *Let $\mathcal{V} \subset \mathcal{L}_k$ be the set of vectors such that for any $\mathbf{v}, \mathbf{w} \in \mathcal{V}$ we have $\langle \mathbf{v}, \mathbf{w} \rangle \neq -l-1$ for some $0 \leq l < k$. Then we have*

$$\max_{\mathcal{V}} |\mathcal{V}| = f(k,l)\binom{n}{k} + O(n^{k-1}).$$

For $k = 3$ the values not covered by Propositions 1, 3, 4 and Theorem 5 are $F(n,3,0)$ and $F(n,3,-1)$ for n not too large. We determine $F(n,3,0)$ in the following theorem, but first we need some preparation.

Let us use the notation (a,b,c) for a set $\{a,b,c\}$ if we know that $a < b < c$. Also for $(a,b,c) \subset [n]$ let $\mathbf{u}(a,b,c) = (u_1, \dots, u_n)$ with $u_i = 1$ for $i \in \{a,b,c\}$ and $u_i = 0$ otherwise. Further let $\mathbf{v}(a,b,c) = (v_1, \dots, v_n)$ be the vector with $v_a = v_b = 1$, $v_c = -1$ and $v_i = 0$ otherwise.

Let us show two lower bounds for $F(n,3,0)$. Taking all non-negative vectors gives

$$F(n,3,0) \geq \binom{n}{3}.$$

The other one is based on the following family:

$$\mathcal{V}(n) = \{\mathbf{u}(a,b,c), \mathbf{v}(a,b,c) : |(a,b,c) \cap [3]| \geq 2\}.$$

Note that $\langle \mathbf{v}, \mathbf{v}' \rangle \geq 0$ for all $\mathbf{v}, \mathbf{v}' \in \mathcal{V}(n)$ and the first few values of $|\mathcal{V}(n)|$ are $|\mathcal{V}(3)| = 2, |\mathcal{V}(4)| = 8, |\mathcal{V}(5)| = 14, |\mathcal{V}(6)| = 20$.

Theorem 7. *We have*
(1) $F(n,3,0) = |\mathcal{V}(n)|$ for $n = 3,4,5$.
(2) $F(6,3,0) = 21$.
(3) $F(n,3,0) = \binom{n}{3}$ for $n \geq 7$.

As we will see, the proof of (2) is the most difficult.

3 Cross-intersecting families

We say that two families $\mathcal{A}, \mathcal{B} \subset 2^{[n]}$ are *s-cross-intersecting*, if for any $A \in \mathcal{A}, B \in \mathcal{B}$ we have $|A \cap B| \geq s$. In the proof of Theorem 5 we need to estimate the sum of sizes of two t-intersecting families that are s-cross-intersecting. For the proof some relatively crude bounds are sufficient but we believe that this problem is interesting in its own right. As a matter of fact the case of non-uniform families was solved by Sali [16] (cf. [4] for an extension with a simpler proof).

To state our results for the k-uniform case let us make some definitions.

Definition 1. *For $k \geq s > t \geq 1$ and $k \geq 2s - t$ define the Frankl-family $\mathcal{A}_i(k,s,t)$ by*

$$\mathcal{A}_i(k,s,t) = \{A \subset \binom{[k]}{s} : |A \cap [t+2i]| \geq t+i\}, \quad 0 \leq i \leq s-t.$$

Note that for $A, A' \in \mathcal{A}_i(k,s,t)$ one has $|A \cap A' \cap [t+2i]| \geq t$, in particular, $\mathcal{A}_i(k,s,t)$ is t-intersecting. Also, for $i < s-t$ every vertex of $[k]$ is covered by some $A \in \mathcal{A}_i(k,s,t)$.

The following result was conjectured by Frankl [3] and nearly 20 years later proved by Ahlswede and Khachatrian.

Theorem (Complete Intersection Theorem ([1]))**.** *Suppose that $\mathcal{A} \subset \binom{[k]}{s}$ is t-intersecting, $k \geq 2s-t$. Then*

$$|\mathcal{A}| \leq \max_{0 \leq i \leq s-t} |\mathcal{A}_i(k,s,t)| =: m(k,s,t) \qquad \textit{holds.} \tag{3}$$

Moreover, unless $k = 2s, t = 1$ or $\mathcal{A}$ is isomorphic to $\mathcal{A}_i(k,s,t)$, the inequality is strict.

Definition 2. *For $k \geq 2s-t$, $0 \leq i < s-t$ define*

$$\mathcal{M}_i(k,s,t) = \{A \subset [k] : |A| \geq s, |A \cap [t+2i]| \geq t+i\} \cup \{A \subset [k] : |A| \geq k-s+t\}.$$

Note that $\{A \in \binom{[k]}{s} : A \in \mathcal{M}_i(k,s,t)\} = \mathcal{A}_i(k,s,t)$ for $k > 2s-t$ and that $\mathcal{M}_i(k,s,t)$ is t-intersecting.

For fixed k, s, t and $0 \leq i < s-t$ let us define the pair $\mathcal{A}_i = \{A \in \binom{[n]}{k} : A \cap [k] \in \mathcal{M}_i(k,s,t)\}$, $\mathcal{B}_i = \{[k]\}$. Then these non-empty t-intersecting families are s-cross-intersecting.

Note that $\mathcal{A}_{s-t}(k,s,t) = \binom{[2s-t]}{s}$. For $i = s-t$ we define

$$\mathcal{A}_{s-t} = \{A \in \binom{[n]}{k} : |A \cap [2s-t]| \geq s\} \quad \text{and}$$
$$\mathcal{B}_{s-t} = \{B \in \binom{[n]}{k} : [2s-t] \subset B\}.$$

Again the non-empty t-intersecting families $\mathcal{A}_{s-t}$ and $\mathcal{B}_{s-t}$ are s-cross-intersecting.

With this terminology we prove

Theorem 8. *Let $k > s > t \geq 1$ be integers, $k \geq 2s-t$. Suppose that $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$ are non-empty t-intersecting families which are cross s-intersecting. Then for $n \geq n_0(s,t,k)$ we have*

$$\max_{\mathcal{A},\mathcal{B}} \{|\mathcal{A}| + |\mathcal{B}|\} = \max_{0 \leq i \leq s-t} \{|\mathcal{A}_i| + |\mathcal{B}_i|\}. \tag{4}$$

Moreover, unless $k = 2s, t = 1$ or $\mathcal{A}, \mathcal{B}$ are isomorphic to $\mathcal{A}_i, \mathcal{B}_i$, the equality above transforms into a strict inequality.

We note that the conclusive result for $t = 0$ was obtained by the authors in the paper [9], and therefore we do not consider this case here.

Remark. The assumption $k \geq 2s - t$ is in fact not restrictive, since if $A \in \mathcal{A}$ and $B \in \mathcal{B}$ then any two sets from $\mathcal{A}$ intersect in at least $2s - k$ elements inside B (and analogously for $\mathcal{B}$ and A). If $2s - k \geq t$, then the t-intersecting condition is implied by the s-cross-intersecting condition, and it reduces to the case studied in [9].

We also note that the problem makes sense only in case $s > t$. Otherwise, we may just take both $\mathcal{A}$ and $\mathcal{B}$ to be the same t-intersecting family of maximal cardinality. Thus, in case $t \geq s$ the problem reduces to a trivial application of the Complete Intersection Theorem.

4 Preliminaries

In the proof of Theorem 5 we use two types of shifting. The first one pushes bigger coordinates to the left. For a given pair of indices $i < j \in [n]$ and a vector $\mathbf{v} = (v_1, \ldots, v_n) \in \{0, \pm 1\}^n$ we define an (i, j)-shift $S_{i,j}(\mathbf{v})$ of $\mathbf{v}$ in the following way. If $v_i \geq v_j$, then $S_{i,j}(\mathbf{v}) = \mathbf{v}$. If $v_i < v_j$, then $S_{i,j}(\mathbf{v}) := (v_1, \ldots, v_{i-1}, v_j, v_{i+1}, \ldots, v_{j-1}, v_i, v_{j+1}, \ldots, v_n)$, that is, it is obtained from $\mathbf{v}$ by interchanging its i-th and j-th coordinate.

Next, we define an (i, j)-shift $S_{i,j}(\mathcal{W})$ for a family of vectors $\mathcal{W} \subset \{0, \pm 1\}^n$:

$$S_{i,j}(\mathcal{W}) := \{S_{i,j}(\mathbf{v}) : \mathbf{v} \in \mathcal{W}\} \cup \{\mathbf{v} : \mathbf{v}, S_{i,j}(\mathbf{v}) \in \mathcal{W}\}.$$

The second is the *up-shift*: $S_i(v_1, \ldots, v_n) = (v_1, \ldots, v_n)$ if $v_i = 0$ or 1 and $S_i(v_1, \ldots, v_n) = (v_1, \ldots, v_{i-1}, 1, v_{i+1}, \ldots, v_n)$ if $v_i = -1$. The shift $S_i(\mathcal{W})$ is defined similarly to $S_{i,j}(\mathcal{W})$:

$$S_i(\mathcal{W}) := \{S_i(\mathbf{v}) : \mathbf{v} \in \mathcal{W}\} \cup \{\mathbf{v} : \mathbf{v}, S_i(\mathbf{v}) \in \mathcal{W}\}.$$

We call a system $\mathcal{W}$ *shifted*, if $\mathcal{W} = S_{i,j}(\mathcal{W})$ for all $i < j \in [n]$ and $\mathcal{W} = S_i(\mathcal{W})$ for all $i \in [n]$. Any system of vectors may be made shifted by means of a finite number of (i, j)-shifts and i-up-shifts. Moreover, it was shown in [7] and [14] (and is easy to verify directly) that (i, j)-shifts and up-shifts do not decrease the minimal scalar product in $\mathcal{W}$.

5 Proof of Theorems 5 and 6

Part 1 of Theorem 5

First we show that the left hand side is at least the right hand side. Take the family of vectors $\mathcal{V} = \{\mathbf{v} = (v_1, \ldots, v_n) : v_1 = \ldots = v_l = 1, v_i \in \{0, 1\}\}$. It clearly satisfies the condition $\langle \mathbf{v}, \mathbf{w} \rangle \geq l$ for any $\mathbf{v}, \mathbf{w} \in \mathcal{V}$ and has cardinality $\binom{n-l}{k-l}$.

We proceed to the upper bound. Take any family $\mathcal{V}$ of $\{0, \pm 1\}$-vectors having minimal scalar product at least l and define its subfamilies $\mathcal{V}_i(\epsilon) = \{\mathbf{w} \in \mathcal{V} : w_i = \epsilon\}$ for $\epsilon = \pm 1$.

Claim 9. *If for some i both $\mathcal{V}_i(-1)$ and $\mathcal{V}_i(1)$ are nonempty, then*

$$|\mathcal{V}_i(1)| + |\mathcal{V}_i(-1)| \le 2^k \binom{k-1}{l+1}\binom{n-l-2}{k-l-2} = O(n^{k-l-2}).$$

We call any such $i \in [n]$ *bad.*

Proof. Consider the set families $\mathcal{A}^+ = \{S \in \binom{[n]-\{i\}}{k-1} : \exists \mathbf{w} \in \mathcal{V}_i(1)$ such that $S(\mathbf{w}) = S \cup \{i\}\}$ and $\mathcal{A}^- = \{S \in \binom{[n]-\{i\}}{k-1} : \exists \mathbf{w} \in \mathcal{V}_i(-1)$ such that $S(\mathbf{w}) = S \cup \{i\}\}$. Both $\mathcal{A}^+, \mathcal{A}^-$ are nonempty. Each of them is $(l-1)$-intersecting, moreover, they must be $(l+1)$-cross-intersecting, since otherwise we would have two vectors in $\mathcal{V}$ that have scalar product at most $l-1$.

Let $A \in \mathcal{A}^+$, $B \in \mathcal{A}^-$. Then $|A \cap B| \ge l+1$ for all $B \in \mathcal{A}^-$ implies $|\mathcal{A}^-| \le \binom{|A|}{l+1}\binom{n-l-2}{k-l-2}$ and similarly $|\mathcal{A}^+| \le \binom{|B|}{l+1}\binom{n-l-2}{k-l-2}$.

The statement of the claim follows from the trivial inequalities $|\mathcal{V}_i(1)| \le 2^{k-1}|\mathcal{A}^+|$, $|\mathcal{V}_i(-1)| \le 2^{k-1}|\mathcal{A}^-|$. ☐

Consider the set $I \subset [n]$ of all bad coordinate positions and define a subfamily $\mathcal{V}_b$ of all vectors that have non-zero values on the bad positions: $\mathcal{V}_b = \{\mathbf{v} \in \mathcal{V} : \exists i \in I : v_i \ne 0\}$. Put $\mathcal{V}_g = \mathcal{V} - \mathcal{V}_b$. Since $\mathcal{V}_g$ is homogeneous on $[n]-I$ and is l-intersecting, we have $|\mathcal{V}_g| \le \binom{n-|I|-l}{k-l}$ for $n \ge k^2$ due to the result of [3]. Therefore,

$$|\mathcal{V}| \le \binom{n-|I|-l}{k-l} + |I|O(n^{k-l-2}) \le$$

$$\le \binom{n-l}{k-l} - |I|\left(\binom{n-|I|-l}{k-l-1} - O(n^{k-l-2})\right) \le \binom{n-l}{k-l},$$

for n large enough, with the last inequality strict in case $|I| > 0$.

Part 2 of Theorem 5 and Theorem 6

First we show that there is such a family of vectors for which the inequality is achieved. For each $S \subset \binom{n}{k}$ we take in $\mathcal{V}$ all the vectors from $\mathcal{L}_k(S)$ that have at most $l/2$ -1's in case l is even and the sets that have at most $(l-1)/2$ -1's among the first $(k-1)$ nonzero coordinates in case l is odd (in other words, take the families of vectors $\mathbf{w}$ whose corresponding sets $N(\mathbf{w})$ duplicate the examples $\mathcal{U}^l$, $\mathcal{U}^l_k$ from the formulation of Katona's Theorem). It is not difficult to see that $\mathcal{V}$ satisfies the requirements of both Theorems 5 and 6 on the size and on the scalar product.

Now we prove the upper bound. The proof is somewhat similar to the previous case. Take any vector family satisfying the conditions of Theorem 5 or Theorem 6. For any set $X \subset [n]$ and any vector $\mathbf{w}$ of length n denote by $\mathbf{w}_{|X}$ the restriction of $\mathbf{w}$ to the coordinates $i, i \in X$. Obviously $\mathbf{w}_{|X}$ is a vector of length $|X|$. We look at all vectors $\mathbf{v} \in \{\pm 1\}^{l+1}$ and split them into groups of antipodal vectors $\mathbf{v}$, $\bar{\mathbf{v}}$, that is, vectors that satisfy $\langle \mathbf{v}, \bar{\mathbf{v}} \rangle = -l-1$. Fix a set of coordinates $I = \{i_1, \dots, i_{l+1}\}$ and define $\mathcal{V}(I, \mathbf{v}) = \{\mathbf{w} \in \mathcal{V} : \mathbf{w}_{|I} = \mathbf{v}\}$.

Claim 10. *If both $\mathcal{V}(I,\mathbf{v})$ and $\mathcal{V}(I,\bar{\mathbf{v}})$ are nonempty, then*

$$|\mathcal{V}(I,\mathbf{v})| + |\mathcal{V}(I,\bar{\mathbf{v}})| \le 2^{k-l-1}(k-l)\binom{n-l-2}{k-l-2} = O(n^{k-l-2}).$$

We call any such I *bad*. In particular, the claim gives that the total cardinality of all such "bad" subfamilies is $O(n^{k-1})$.

Proof. Consider the set families $\mathcal{A} = \{S \in \binom{[n]-I}{k-l-1} : \exists \mathbf{w} \in \mathcal{V}(I,\mathbf{v}) \text{ such that } S(\mathbf{w}) = S \cup I\}$ and $\bar{\mathcal{A}} = \{S \in \binom{[n]-I}{k-l-1} : \exists \mathbf{w} \in \mathcal{V}(I,\bar{\mathbf{v}}) \text{ such that } S(\mathbf{w}) = S \cup I\}$. Both $\mathcal{A}, \bar{\mathcal{A}}$ are nonempty. Moreover, they must be cross-intersecting, since otherwise we would have two vectors in $\mathcal{V}$ that have scalar product exactly $-l-1$. Therefore, by Hilton-Milner theorem,

$$|\mathcal{A}| + |\bar{\mathcal{A}}| \le \binom{n-l-1}{k-l-1} - \binom{n-k}{k-l-1} + 1 \le (k-l)\binom{n-l-2}{k-l-2}.$$

The statement of the claim follows from the trivial inequalities $|\mathcal{V}(I,\mathbf{v})| \le 2^{k-l-1}|\mathcal{A}|$, and $|\mathcal{V}(I,\bar{\mathbf{v}})| \le 2^{k-l-1}|\bar{\mathcal{A}}|$. □

Define $\mathcal{V}_b = \{\mathbf{w} \in \mathcal{V} : \exists I \subset \binom{[n]}{l+1} : \mathcal{V}(I, \mathbf{v}_{|I}) \ne \emptyset, \mathcal{V}(I, \bar{\mathbf{v}}_{|I}) \ne \emptyset\}$. Consider the remaining family of "good vectors" $\mathcal{V}_g = \mathcal{V} - \mathcal{V}_b$.

First we finish the proof of Theorem 6. The family $\mathcal{V}_g$ is good in the following sense. If we fix $S \in \binom{[n]}{k}$ and consider the family of sets $\mathcal{A} = \{N(\mathbf{w}) : w \in \mathcal{V}_g(S)\}$, then $\mathcal{A}$ satisfies $A_1 \triangle A_2 \le l$ for any $A_1, A_2 \in \mathcal{A}$. Therefore, $|\mathcal{V}_g(S)| \le f(k,l)$ by Kleitman's theorem and

$$|\mathcal{V}| \le \sum_{S \in \binom{[n]}{k}} f(k,l) + |\mathcal{V}_b| = f(k,l)\binom{n}{k} + O(n^{k-1}).$$

The proof of Theorem 6 is complete.

From now on we suppose that the family $\mathcal{V}$ is shifted. Note that, since shifting may increase scalar products, it is impossible to use it in the case of Theorem 6. It is not difficult to see that the family $\mathcal{V}_b$ is up-shifted, since if we deleted some $\mathbf{w} \in \mathcal{V}$, then we would have deleted all $\mathbf{u} \in \mathcal{V}$ with $S(\mathbf{u}) = S(\mathbf{w}), N(\mathbf{u}) \supset N(\mathbf{w})$. One can see in a similar way that $\mathcal{V}_b$ is shifted.

Claim 11. *For every $S \in \binom{[n]}{k}$ the family $\mathcal{V}_g(S)$ satisfies $N(\mathbf{w}) \cup N(\mathbf{v}) \le l$ for all $\mathbf{v}, \mathbf{w} \in \mathcal{V}_g(S)$.*

Proof. Suppose for contradiction $N(\mathbf{w}) \cup N(\mathbf{u}) \ge l+1$. Consider any set $I \subset N(\mathbf{w}) \cup N(\mathbf{u})$, $|I| = l+1$. Since $\mathcal{V}_g$ is up-shifted, there is a vector $\mathbf{u}'$ in $\mathcal{V}_g(S)$ such that $\langle \mathbf{w}_{|I}, \mathbf{u}'_{|I}\rangle = -l-1$, which means that for $\mathbf{v} = \mathbf{w}_{|I}$ both $\mathcal{V}_g(I,\mathbf{v})$ and $\mathcal{V}_g(I,\bar{\mathbf{v}})$ are nonempty, a contradiction. □

Claim 12. *If $I \subset \binom{[n]}{l+1}$ is bad, then for any $S \supset I$, $S \in \binom{[n]}{k}$, we have $|\mathcal{V}_g(S)| \le f(k,l) - 1$.*

Proof. Consider the family $\mathcal{A} \subset 2^S$ of subsets of S, that is defined in the following way: $\mathcal{A} = \{N(\mathbf{w}) \cap S : \mathbf{w} \in \mathcal{V}_g(S)\}$. For simplicity, we identify S with $[k]$, preserving the order of elements. In view of the uniqueness part of Katona's theorem, it is sufficient to show that $\mathcal{A}$ does not contain one of the sets from the extremal families: from $\mathcal{U}^l$ in case l is even and from any of $\mathcal{U}^l_i$, $i = 1, \dots, k$ in case l is odd.

First consider the case when l is even. If I is bad, it means that there exists a vector $\mathbf{v}$ of length $l+1$, such that both $\mathcal{V}(I, \mathbf{v}), \mathcal{V}(I, \bar{\mathbf{v}})$ are nonempty. Therefore, all vectors from these families are not present in $\mathcal{V}_g$. Assume w.l.o.g. that $|N(\mathbf{v})| \le l/2$. Then the set $N(\mathbf{v})$ is not present in $\mathcal{A}$, but at the same time it belongs to $\mathcal{U}^l$.

From now on, assume that l is odd. We remark that, due to the fact that $\mathcal{V}_g$ is shifted, the family $\mathcal{A}$ is shifted to the right, and, therefore, out of the extremal families $\mathcal{A}$ may coincide with $\mathcal{U}^l_k$ only. We argue similarly to the previous case. If $|N(\mathbf{v})| \le (l-1)/2$ or $|N(\bar{\mathbf{v}})| \le (l-1)/2$, then we are done as in the previous case. The only remaining case is when $|N(\mathbf{v})| = (l+1)/2$. If $k \in I$, then we are done again: one of $N(\mathbf{v}), N(\bar{\mathbf{v}})$ then contains k and therefore belongs to $\mathcal{U}^l_k$.

Suppose from now on that $k \notin I$. Fix some $i \in N(\mathbf{v})$ and take any $\mathbf{w} \in \mathcal{V}$ such that $\mathbf{w}_{|I} = \bar{\mathbf{v}}$. Since $\mathcal{V}$ is shifted, there is a vector $\mathbf{u} \in \mathcal{V}$ such that $\mathbf{u}_{|I-\{i\}} = \mathbf{w}_{|I-\{i\}}, u_k = 1$. Assume that $\mathcal{A}$ contains the set $N' = N(\mathbf{v}) - \{i\} \cup \{k\}$, and take $\mathbf{v}' \in \mathcal{V}_g(S)$ such that $N(\mathbf{v}') = N'$. Then, putting $I' = I - \{i\} \cup \{k\}$, we get that both $\mathcal{V}(I', \mathbf{v}')$ and $\mathcal{V}(I', \bar{\mathbf{v}}')$ are nonempty: the first family due to the assumption that $N' \in \mathcal{A}$, and the second one due to the fact that $\mathbf{u}_{|I'} = \bar{\mathbf{v}}'$. Therefore, $N' \notin \mathcal{A}$. But $N' \in \mathcal{U}^l_k$, and thus $\mathcal{A}$ does not coincide with $\mathcal{U}^l_k$. ☐

Assume now that there are t bad sets $I \subset \binom{[n]}{l+1}$. Then the number of sets $S \subset \binom{[n]}{k}$ that contain one of the bad sets I is at least $t\binom{n-l-1}{k-l-1}/\binom{k}{l+1}$. Therefore, we have

$$|\mathcal{V}| - f(k,l)\binom{n}{k} \le -t\frac{\binom{n-l-1}{k-l-1}}{\binom{k}{l+1}} + \frac{1}{2}\sum_{v \in \{\pm1\}^{l+1}} \sum_{\text{bad } I} \Big(|\mathcal{V}(I, \mathbf{v})| + |\mathcal{V}(I, \bar{\mathbf{v}})|\Big) \le$$

$$\le -t\Big(\frac{\binom{n-l-1}{k-l-1}}{\binom{k}{l+1}} - 2^k(k-l)\binom{n-l-2}{k-l-2}\Big) < 0,$$

provided $n > 2^k k^2 \binom{k}{l+1}$. We note that taking $n > 4^k k^2$ makes the choice of n from which the proof works independent of l.

6 Proof of Theorem 7

Let $\mathcal{W} \subset \{0, \pm1\}^n$ satisfy $\langle \mathbf{w}, \mathbf{w} \rangle = 3$, $\langle \mathbf{w}, \mathbf{v} \rangle \ge 0$ for all $\mathbf{w}, \mathbf{v} \in \mathcal{W}$ and suppose that $\mathcal{W}$ is shifted.

Claim 13. *If $S(\mathbf{w}) = (a,b,c)$ for $\mathbf{w} \in \mathcal{W}$ then either $\mathbf{w} = \mathbf{u}(a,b,c)$ or $\mathbf{w} = \mathbf{v}(a,b,c)$ holds.*

Proof. Assume the contrary and let (x, y, z) be the non-zero coordinate values of the vector $\mathbf{w}$ (at positions a, b, c). Using the up-shift w.l.o.g. $(x, y, z) = (-1, 1, 1)$ or $(1, -1, 1)$. In both cases by shifting $(1, 1, -1)$ is also in $\mathcal{W}$. However it has scalar product -1 with both, a contradiction. $\square$

From the claim $F(3, 3, 0) \leq 2$, and $F(4, 3, 0) \leq \binom{4}{3} \times 2 = 8$ follow.

From now on $n \geq 5$ and we use induction on n to prove the statement of the theorem. Set $\mathcal{G} = \{(a, b) \in \binom{[n-1]}{2} : \mathbf{u}(a, b, n) \in \mathcal{W}\}$, $\mathcal{H} = \{(a, b) \in \binom{[n-1]}{2} : \mathbf{v}(a, b, n) \in \mathcal{W}\}$. By shiftedness $\mathcal{H} \subset \mathcal{G}$ and by $\langle \mathbf{v}, \mathbf{w} \rangle \geq 0$, $\mathcal{G}$ and $\mathcal{H}$ are cross-intersecting. We may assume that $\mathcal{H}$ is non-empty, since otherwise $|\mathcal{W}| \leq \binom{n}{3}$ holds due to shiftedness.

Claim 14. *If $\mathbf{v}(2, 3, n) \in \mathcal{W}$ then $|\mathcal{H}| = |\mathcal{G}| = 3$.*

Proof. By shiftedness $\{(1, 2), (1, 3), (2, 3)\} \subset \mathcal{H} \subset \mathcal{G}$. The statement follows from the fact that no other 2-element set can intersect those three sets. $\square$

Since $\mathcal{W}(\bar{n}) \overset{def}{=} \{\mathbf{w} \in \mathcal{W} : w_n = 0\}$ satisfies $|\mathcal{W}(\bar{n})| \leq F(n-1, 3, 0)$, $\mathbf{v}(2, 3, n) \in \mathcal{W}$ and Claim 14 imply

$$
\begin{aligned}
|\mathcal{W}| &\leq F(n-1, 3, 0) + 6, && \text{which gives} \\
|\mathcal{W}| &\leq 8 + 6 = 14 && \text{for } n = 5, \\
|\mathcal{W}| &\leq 14 + 6 < 21 && \text{for } n = 6, \text{ and} \\
|\mathcal{W}| &\leq \binom{n-1}{3} + 6 < \binom{n}{3} && \text{for } n \geq 7.
\end{aligned}
$$

Consequently, we may suppose that the degree of n in $\mathcal{W}$ is at least 7, in particular, $\mathcal{H}$ is the star, $\mathcal{H} = \{(1, 2), (1, 3), \ldots, (1, p)\}$ for some $p \leq n - 1$.

The following lemma is obvious.

Lemma 15. *One of the following holds.*
(i) $p = 2$, $|\mathcal{H}| = 1$, $|\mathcal{G}| \leq (n-2) + (n-3) = 2n - 5$
(ii) $p = 3$, $|\mathcal{H}| = 2$, $|\mathcal{G}| \leq (n-2) + 1 = n - 1$
(iii) $p \geq 4$, $|\mathcal{H}| \leq |\mathcal{G}| \leq n - 2$.

Since for $n = 5$ in all cases $|\mathcal{H}| + |\mathcal{G}| \leq 6$ holds, the proof of part (1) of Theorem 7 is complete. Also, for $n \geq 7$ it follows that $|\mathcal{H}| + |\mathcal{G}| \leq 2n - 4$. Since $\binom{n}{3} - \binom{n-1}{3} = \binom{n-1}{2} > 2n - 4$ for $n \geq 7$, the induction step works fine in this case too.

The only case that remains is $n = 6$. Using shiftedness and $a = 1$ for all (a, b, n) with $\mathbf{v}(a, b, n) \in \mathcal{W}$ it follows that $a = 1$ holds for all (a, b, c) with $\mathbf{v}(a, b, c) \in \mathcal{W}$.

Define $\mathcal{B} = \{(b, c) \in \binom{[6]-\{1\}}{2} : \mathbf{v}(1, b, c) \in \mathcal{W}\}$ and $\mathcal{D} = \{(d, e, f) \in \binom{[2,6]}{3} : \mathbf{u}(a, b, c) \in \mathcal{W}\}$. Note that $(b, c) \in \mathcal{B}$ and $D \in \mathcal{D}$ cannot satisfy $D \cap \{b, c\} = \{c\}$, since otherwise $\langle \mathbf{u}(d, e, f), \mathbf{v}(1, b, c) \rangle = -1$. Let us note that one can put into $\mathcal{W}$ all the vectors $\mathbf{u}(1, b, c)$ with $2 \leq b < c \leq 6$ because all the vectors with a -1 position have 1 in the first coordinate.

Therefore $|\mathcal{W}| = 10 + |\mathcal{B}| + |\mathcal{D}|$ holds. We have to prove

$$|\mathcal{B}| + |\mathcal{D}| \le 11. \tag{5}$$

Let us first exhibit the system with 21 vectors showing $F(6,3,0) \ge 21$.

$$\mathcal{U}_6 = \{\mathbf{u}(1,b,c), \mathbf{v}(1,b,c) : (b,c) \subset [2,6]\} \cup \{\mathbf{u}(2,3,4)\}.$$

We have $|\mathcal{U}_6| = 2 \times \binom{5}{2} + 1 = 21$ and the only non-trivial scalar product to check is that $\langle \mathbf{v}(1,b,c), \mathbf{u}(2,3,4)\rangle \ge 0$. It is true automatically if $c \ge 5$. For $c = 3,4$ it follows that $b \in \{2,3\}$ making the scalar product equal to 0 as desired.

To avoid a tedious case by case analysis it is simplest to give a matching from the 9-element set $\{D \in \binom{[2,6]}{3} : D \ne (2,3,4)\}$ into the 10-element set $\{(b,c) : 2 \le b < c \le 6\}$ such that whenever D and (b,c) are matched, $D \cap (b,c) = \{c\}$ holds. This will prove (5). An example of the desired matching is exhibited below.

$$\begin{array}{lll} (2,3,5) - (4,5) & (2,4,5) - (3,5) & (2,3,6) - (4,6) \\ (2,4,6) - (5,6) & (2,5,6) - (3,6) & (3,4,5) - (2,3) \\ (3,4,6) - (2,4) & (3,5,6) - (2,5) & (4,5,6) - (2,6) \end{array}$$

A careful analysis shows that $\mathcal{U}_6$ is the only extremal configuration. The proof is complete.

7 Proof of Theorem 8

We may assume that the families $\mathcal{A}, \mathcal{B}$ are shifted and, thus, both contain $[k]$. We define $\mathcal{A}(T) = \{A \backslash T : A \in \mathcal{A}, A \supset T\}$. Note that we had a similar notation for families of vectors, but now we are working with sets only, so it should not cause any confusion. We call a set $T \in \binom{[n]}{s}$ a *kernel* for a set family $\mathcal{S}$, if $\mathcal{S}(T)$ contains $k+1$ pairwise disjoint edges. An inequality proved in [5] states that

$$|\mathcal{A}(T)| \le k\binom{n-s-1}{k-s-1}, \tag{6}$$

if T is not a kernel for $\mathcal{A}$, and similarly for $\mathcal{B}(T)$.

The following claim is an easy application of the pigeon-hole principle.

Claim 16. *Suppose that $C_0, \ldots, C_k$ form a sunflower with center T, i.e. $C_i \cap C_j = T$ for all $0 \le i < j \le k$. Suppose D is a k-element set. Then there exists an i such that $D \cap C_i = D \cap T$ holds.*

We immediately get the following corollary:

Corollary 17. *If $|T| = s$ and $|D \cap C_i| \ge s$ for all $0 \le i \le k$, then $T \subset D$.*

Lemma 18. *If T_1 and T_2 are kernels for $\mathcal{A}$ and $\mathcal{B}$ respectively, then $T_1 = T_2$ holds.*

Proof. From Corollary 17 it follows that for any $B \in \mathcal{B}$ we have $T_1 \subset B$. Applying it for $B_1, B_2 \in \mathcal{B}$, $B_1 \cap B_2 = T_2$, we get that $T_1 \subset B_1 \cap B_2 = T_2$, but since $|T_1| = |T_2|$, we have $T_1 = T_2$. ☐

From Corollary 17 and Lemma 18 it follows that if both $\mathcal{A}$ and $\mathcal{B}$ have kernels, then $|\mathcal{A}| + |\mathcal{B}| \le 2\binom{n-s}{k-s}$, which is smaller then the bound in (4). From now on we may w.l.o.g. assume that $\mathcal{B}$ does not have a kernel and that $\mathcal{B}$ contains the set $[k]$ as an element. Then any kernel of $\mathcal{A}$ must be a subset of $[k]$. Let $\mathcal{J} \subset \binom{[k]}{s}$ be the family of kernels of $\mathcal{A}$. Define $\tilde{\mathcal{A}} = \{A \in \mathcal{A} : \nexists T \in \mathcal{J}, A \supset T\}$.

Claim 19. *We have $|\tilde{\mathcal{A}}| = O(n^{k-s-1})$. Analogously, we have $|\mathcal{B}| = O(n^{k-s-1})$.*

Proof. Any set from $\tilde{\mathcal{A}}$ must intersect $[k]$ in at least s elements. Thus, $|\tilde{\mathcal{A}}| \le \sum_{T \in \binom{[k]}{s}} |\tilde{\mathcal{A}}(T)|$, which by inequality (6) is at most $\binom{[k]}{s} k \binom{n-s-1}{k-s-1} = O(n^{k-s-1})$. The same proof works for $\mathcal{B}$. ☐

Due to Claim 16 and the fact that $\mathcal{A}$ is t-intersecting, for any set $A \in \mathcal{A}$ and any $T \in \mathcal{J}$ we have $|A \cap T| \ge t$. Moreover, repeating the proof of Lemma 18, it is easy to see that for any $T_1, T_2 \in \mathcal{J}$ we have $|T_1 \cap T_2| \ge t$. Therefore, $|\mathcal{J}| \le m(k, s, t)$ (see the formulation of the Complete Intersection Theorem). Due to Claim 19, we have

$$|\mathcal{A}| + |\mathcal{B}| \le |\mathcal{J}| \binom{n-k}{k-s} + O(n^{k-s-1}). \tag{7}$$

Indeed, the only additional thing one has to note is that the number of sets from $\mathcal{A}$ that intersect $[k]$ in at least $s+1$ elements is $O(n^{k-s-1})$.

On the other hand, it is easy to exhibit an example of a family $\mathcal{A}$ that is t-intersecting and is s-cross-intersecting with $\{[k]\}$, and which has cardinality $m(k, s, t)\binom{n-k}{k-s}$. For that one just has to take a maximum t-intersecting family $\mathcal{J}'$ of s-element sets in $[k]$ and put $\mathcal{A} = \{A \in \binom{[n]}{k} : \exists T \in \mathcal{J}' : A \cap [k] = T\}$.

Therefore, if $|\mathcal{J}| < m(k, s, t)$, then by (7) and the previous construction $|\mathcal{A}| + |\mathcal{B}|$ cannot be maximal for large n. Therefore, from now on we may assume that $\mathcal{J} = \mathcal{A}_i(k, s, t)$ for some $0 \le i \le s - t$, where $|\mathcal{A}_i(k, s, t)| = m(k, s, t)$.

We remark that out of $\mathcal{A}_j(k, s, t)$ the only family that satisfies $|\mathcal{A}_j(k, s, t)| = |\mathcal{A}_j(k+1, s, t)|$ is the family $\mathcal{A}_{s-t}(k, s, t)$. Indeed, it consists of all the sets that have all their s elements among the first $2s - t$ elements and does not use elements j with $j > 2s - t$. For all the other families, as we have already mentioned, the degree of each $j \in [k]$ is positive.

Suppose first that $0 \le i < s - t$ and that $|\mathcal{A}_{s-t}(k, s, t)| < m(k, s, t)$. Most importantly for us, it means that $m(k-1, s, t) < m(k, s, t)$ due to the discussion in the previous paragraph. Then it is easy to see that $|\mathcal{B}| = 1$. Indeed, if $B_1, B_2 \in \mathcal{B}$, then due to Corollary 17 we know that $T \subset B_1 \cap B_2$ for each $T \in \mathcal{J}$. Since $|B_1 \cap B_2| \le k - 1$, we have $|\mathcal{J}| \le m(k-1, s, t) < m(k, s, t)$.

We showed that the extremal pair of families satisfies $\mathcal{B} = \{[k]\}$. To complete the proof in this case and to show that the pair of extremal families is one of $\mathcal{A}_i, \mathcal{B}_i$ for $0 \le i < s-t$, we have to verify that $\{A \cap [k] : A \in \mathcal{A}\} \subset \mathcal{M}_i(k,s,t)$ provided that $\mathcal{J} = \mathcal{A}_i(k,s,t)$.

Take any set A such that $A \notin \mathcal{M}_i(k,s,t)$, that is, $|A \cap [k]| < k-s+t$ and $|A \cap [t+2i]| < t+i$. The following claim completes the proof in the case $m(k,s,t)$ is attained on $\mathcal{M}_i(k,s,t)$ with $i < s-t$.

Claim 20. *In the notations above, there is $T \in \mathcal{J} = \mathcal{A}_i(k,s,t)$ such that $|T \cap A| \le t-1$.*

Proof. W.l.o.g., let $A = [1,l] \cup [m+1,k]$, where $l \le t+i-1$ and $l+m < k-s+t$. Set $T = [l', l'+s-1]$ with $l' \le i+1$ and with l' chosen in such a way that $|T \cap A|$ is minimized. It is easy to see that, first, $T \in \mathcal{J}$ and, second, $|\mathcal{T} \cap A| = \max\{l-i, l+m+s-k\} < t$. □

If the only maximal family is $\mathcal{A}_{s-t}(k,s,t)$, then $\mathcal{J} = \mathcal{A}_{s-t}(k,s,t)$ and $\mathcal{A}$ cannot contain a set A that satisfy $|A \cap [2s-t]| < s$. Indeed, then A intersects one of the $T \in \mathcal{J}$ in less than t elements, which by Claim 16 means that there exists $A' \in \mathcal{A}$ such that $|A' \cap A| < t$, a contradiction. Therefore, the family $\mathcal{A}$ is contained in $\mathcal{A}_{s-t}$, and we are only left to prove that $\mathcal{B} \subset \mathcal{B}_{s-t}$. Assume that $\mathcal{B}$ contains B, such that $B \not\supseteq [2s-t]$. But then there is $T \in \mathcal{J}$ such that $|B \cap T| \le s-1$, a contradiction.

The last case that remains to verify is when both $\mathcal{A}_{s-t}(k,s,t)$ and $\mathcal{A}_{s-t-1}(k,s,t)$ have size $m(k,s,t)$. But either $\mathcal{J} = \mathcal{A}_{s-t-1}(k,s,t)$ or $\mathcal{J} = \mathcal{A}_{s-t}(k,s,t)$ and we end up in one of the two scenarios described above.

It is clear from the proof that the families that we constructed are the only possible extremal families (excluding the case $n = 2k, s = 1$), which is mostly due to the uniqueness in the formulation of the Complete Intersection Theorem.

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