Partition-free families of sets

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Abstract

For an n-element set X and a real number $0 < p < 1$ there is a very natural probability measure w_p on the power set 2^X. The measure of a family $\mathcal{F} \subset 2^X$ is defined as the sum of the measure of its members, and for $F \subset X$ one sets $w(F) = p^{|F|} \cdot (1-p)^{n-|F|}$. A family $\mathcal{F} \subset 2^X$ is called partition-free if it has no pairwise disjoint members whose union is X. Denoting the maximum of $w_p(\mathcal{F})$ over all partition-free families $\mathcal{F} \subset 2^X$ by $m(n,p)$ we prove the rather surprising fact that while $m\left(n, \frac{1}{k}\right) = 1 - \frac{1}{k}$ for all integers $k \geq 2$, $m(n,p) \to 1$ as $n \to \infty$ for all other values of p. This problem was raised recently by Fan Wei.

1 Introduction

Let $[n] = \{1, 2, \ldots, n\}$ and let $\mathcal{F} \subset 2^{[n]}$ be a family of subsets of $[n]$. The family $\mathcal{F}$ is called *partition-free* if there is no collection of pairwise disjoint members $F_1, \ldots, F_r \in \mathcal{F}$ satisfying $F_1 \cup \ldots \cup F_r = [n]$. We would like to stress that there is no restriction on r. In fact, letting $r = 2$ implies that for every $F \in \mathcal{F}$, the complement $[n] \setminus F$ is missing from $\mathcal{F}$. This implies $|\mathcal{F}| \leq 2^{n-1}$, which was observed already by Erdős, Ko and Rado [EKR].

Since the family 2^Y is partition-free for every $Y \subset [n]$, $|Y| = n-1$, the bound 2^{n-1} is best possible. Let $0 < p < 1$ be a real number and consider the usual probability measure on $2^{[n]}$: The measure of a subset $F \subset [n]$ is $w_p(F) = p^{|F|}(1-p)^{n-|F|}$. The measure of a family $\mathcal{F} \subset 2^{[n]}$ is then

$$w_p(\mathcal{F}) = \sum_{F \in \mathcal{F}} w_p(F) \leq 1.$$

Fan Wei noted that for $Y \subset [n]$, $|Y| = n-1$ one has $p(2^Y) = 1 - p$ and she conjectured that for $p > \frac{1}{2}$ this is best possible. We prove that the situation is more complex.

Definition 1.1. Let $m(n,p)$ be the maximum of $w(\mathcal{F})$ over all partition-free families $\mathcal{F} \subset 2^{[n]}$.

For the case $p \geq 1/2$ it is easy to deduce the value of $m(n,p)$ and determine the extremal families (cf. Proposition 2.2). The case $p < 1/2$ is more interesting.

Theorem 1.2. (i) *$m(n,p) = 1-p$ if p is of the form $1/k$ for some integer $k \geq 2$.*

(ii) *$m(n,p) \to 1$ as $n \to \infty$ for all $p < 1/2$ that are not of the form $1/k$.*

2 The proof of the main bounds

Let us give some examples of partition-free families.

Example 2.1. Let k be a positive integer. Define

$$\mathcal{F}_k(n) = \left\{ F \subset [n] : \frac{n}{k+1} < |F| < \frac{n}{k} \right\} \cup \{\emptyset\}.$$

It is easy to see that $\mathcal{F}_k(n)$ is partition-free. Indeed, the union of any collection of at most k members have total size less than n. On the other hand, the total size of $k+1$ or more non-empty members surpasses n. Adding the empty set does not alter the partition-free property.

For the case n even let $\mathcal{B}(n) \subset \binom{[n]}{n/2}$ be *any* collection of half of all $\frac{n}{2}$-element sets, where we take exactly one of each pair of complementary $\frac{n}{2}$-sets.

Proposition 2.2. *Let $p > 1/2$ and let $\mathcal{F} \subset 2^{[n]}$ be partition-free. Then* (i) *or* (ii) *holds.*

(i) *n is odd and*

$$w_p(\mathcal{F}) \leq w_p(\mathcal{F}_1(n)). \tag{2.1}$$

Moreover, the inequality is strict unless $\mathcal{F} = \mathcal{F}_1(n)$.

(ii) *n is even and*

$$w_p(\mathcal{F}) \leq w_p\big(\mathcal{F}_1(n) \cup \mathcal{B}(n)\big). \tag{2.2}$$

Moreover, equality holds only if $\mathcal{F} = \mathcal{F}_1(n) \cup \mathcal{B}(n)$ for some choice of $\mathcal{B}(n)$.

Proof. Since $\mathcal{F}$ is partition-free, $[n] \notin \mathcal{F}$. Let A, B form a partition of $[n]$, $0 < |A| < |B| < n$. Then $|B| > n/2$ implies $B \in \mathcal{F}_1(n)$. The partition-free property implies $|\mathcal{F} \cap \{A, B\}| \leq 1$.

Since $p > \frac{1}{2}$, $w(A) = p^{|A|}(1-p)^{|B|} < p^{|B|} \cdot (1-p)^{|A|} = w(B)$ holds. Thus $\mathcal{F}_1(n)$ never loses to $\mathcal{F}$ on a complementary pair of sets.

$$\sum_{F \in \mathcal{F} \cap \{A,B\}} w_p(F) \leq \sum_{F \in \mathcal{F}_1(n) \cap \{A,B\}} w_p(F). \tag{2.3}$$

Adding (2.3) over all complementary pairs yields (2.1). This concludes the proof for case (i).

In case (ii), i.e., if n is even, $w_p(A) = w_p(B)$ holds for all complementary pairs of $n/2$-element sets. Thus (2.3) holds for the enlarged family $\mathcal{F}_1(n) \cup \mathcal{B}$ (instead of $\mathcal{F}_1(n)$). This proves (2.2)

In case of equality, equality must hold in (2.3) for all choices of A, B. Thus $\mathcal{F}_1(n) \subset \mathcal{F}$ and both statements concerning the extremal families follow. □

Let us mention that in [EFK] we proved essentially the same statement for the case of intersecting families (i.e., families without two disjoint sets).

Let us next consider the case $p = \frac{1}{k}$, $k \geq 2$. In this case there is a natural interpretation of the probabilistic measure on $2^{[n]}$ using $\mathbb{Z}_k^n$, the family of all integer sequences $\overrightarrow{a} = (a_1, \ldots, a_n)$ with $0 \leq a_i < k$. Obviously, $|\mathbb{Z}_k^n| = k^n$. Let $V(\overrightarrow{a}) = \{i : a_i = 0\} \subset [n]$ be the *null-set* of $\overrightarrow{a}$.

To a fixed $F \subset [n]$ there are *exactly* $(k-1)^{n-|F|}$ sequences $\overrightarrow{a} \in \mathbb{Z}_k^n$ satisfying $V(\overrightarrow{a}) = F$.

Starting with a partition-free family $\mathcal{F} \subset 2^{[n]}$ let us define $\mathcal{A} \subset \mathbb{Z}_k^n$ by

$$\mathcal{A} = \{\overrightarrow{a} : V(\overrightarrow{a}) \in \mathcal{F}\}.$$

Then

$$|\mathcal{A}| = \sum (k-1)^{n-|F|} = k^n \cdot w_{1/k}(\mathcal{F}).$$

Thus to prove $w_{1/k}(\mathcal{F}) \leq \frac{k-1}{k}$ it is sufficient to show

$$|\mathcal{A}| \leq (k-1)k^{n-1}. \tag{2.4}$$

We do it following [FT].

Let us divide $\mathbb{Z}_k^n$ into k^{n-1} groups of k sequences each. To this end define $\overrightarrow{a}(j) = (a_1 + j, \ldots, a_n + j)$ for $j = 0, \ldots, k-1$ where addition is modulo k. Set $C(\overrightarrow{a}) = \{\overrightarrow{a}(0), \ldots, \overrightarrow{a}(k-1)\}$. Then for the k^{n-1} sequences starting with 0, i.e., $(0, a_2, \ldots, a_n)$ the $C(\overrightarrow{a})$ form a partition of $\mathbb{Z}_k^n$. Noting that $V\big(\overrightarrow{a}(0)\big), \ldots, V\big(\overrightarrow{a}(k-1)\big)$ always form a partition of $[n]$ we infer that at most $k-1$ of these k sequences can be in $\mathcal{A}$. This yields (2.4) and concludes the proof of (i).

To prove (ii) is easy. Let $\frac{1}{k} > p > \frac{1}{k+1}$, $k \geq 2$. Consider the family $\mathcal{F}_k(n)$. We claim that $w(\mathcal{F}_k(n)) \to 1$ as $n \to \infty$.

Indeed, choosing a positive real ε satisfying $\frac{1}{k} > p + \varepsilon$ and $p - \varepsilon > \frac{1}{k+1}$ we see that $F \in \mathcal{F}_k(n)$ for all F with

$$\big||F| - pn\big| < \varepsilon n. \tag{2.5}$$

By Bernstein's inequality (cf., e.g., [R]) the proportion of $F \in 2^{[n]}$ *not* satisfying (2.5) is tending to 0 exponentially fast. □

3 Concluding remarks

Let $\mathcal{F} \subset 2^{[n]}$ be a partition-free family. If $F, G \in \mathcal{F}$ are disjoint then adding $F \cup G$ to $\mathcal{F}$ will not alter the partition-free property. This shows that for $k \geq 2$ the partition-free family $\mathcal{F}_k(n)$ is not maximal in general. Let $s, s+1, \ldots, s+t$ be the integers in the open interval $\left(\frac{n}{k+1}, \frac{n}{k}\right)$. Then one can add to $\mathcal{F}_k(n)$ all subsets of size q if $q = b_0 \cdot s + b_1 \cdot (s+1) + \cdots + b_t(s+t)$ for appropriately chosen nonnegative integers $b_0, \ldots, b_t$. For example, let $k = 2$, $n = 6\ell + 5$, $\ell \geq 1$. Then the integers in the interval $\left(2\ell + \frac{5}{3}, 3\ell + \frac{5}{2}\right)$ are $2\ell+2, \ldots, 3\ell+2$. Thus we can add to $\mathcal{F}_2(6\ell+5)$ also all sets of size $4\ell+4, 4\ell+5, \ldots, 6\ell+4$. Let $\widetilde{\mathcal{F}}_2(6\ell+5)$ denote this enlarged family. It is easy to see that $\big|\widetilde{\mathcal{F}}_2(6\ell+5)\big| = 2^{6\ell+4}$.

More generally, if $n \equiv 2k+1$ (mod $k(k+1)$, that is) $n = \ell k(k+1) + 2k + 1$ for some $\ell \geq 1$ then one can add to $\mathcal{F}_k(n)$ all sets of size s for $s \in \big[\ell ki + 2i, \ell(k+1)i + 2i\big]$ for $i = 2, \ldots, k$. Let again $\widetilde{\mathcal{F}}_k\big(\ell \cdot k(k+1) + 2k + 1\big)$ denote this enlarged family which is partition-free and of size 2^{n-1}.

Conjecture 3.1. *For p fixed, $\frac{1}{k+1} < p < \frac{1}{k}$ and $n = \ell k(k+1) + 2k + 1$ one has*

$$m(n,p) = \widetilde{\mathcal{F}}_k(n) \quad \textit{as long as} \quad \ell > \ell(p).$$

A related problem is the following. Let q, r be distinct positive integers and suppose that the integer n is of the form $aq + br$ for some positive integers a, b but n is not divisible neither by q nor by r.

Suppose that $\mathcal{F} \subset \binom{[n]}{q} \cup \binom{[n]}{r}$ and it is partition-free.

Problem 3.2. Determine or estimate $\max |\mathcal{F}|$ subject to the above conditions.

The easiest case is $n = q + r$ and the answer is $\binom{n}{q} = \binom{n}{r}$.

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