A near exponential improvement on a bound of Erdős and Lovász

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Abstract

Let $m(k)$ denote the maximum number of edges in a non-extendable, intersecting k-graph. Erdős and Lovász proved that $m(k) \le k^k$. An improved bound is provided.

1 Introduction

Let $k \ge 2$ be an integer. A collection $\mathcal{F} = \{F_1, \ldots, F_m\}$ of distinct k-element sets is usually called a *k-graph*, $|\mathcal{F}| = m$ is its *size*. The k-graph $\mathcal{F}$ is called *intersecting* if $F \cap F' \ne \emptyset$ for all $F, F' \in \mathcal{F}$. The intersecting k-graph $\mathcal{F}$ is called *maximal* or *saturated* if $\mathcal{F} \cup \{F_0\}$ ceases to be intersecting for all possible choices of a k-set $F_0 \notin \mathcal{F}$.

In their seminal paper [EL] Erdős and Lovász proved the following important finiteness result.

Erdős–Lovász Bound [EL]. If $\mathcal{F}$ is a maximal intersecting k-graph then

$$|\mathcal{F}| \le k^k. \tag{1.1}$$

Let $m(k)$ denote the maximum of $|\mathcal{F}|$. It is easy to see that the only maximal intersecting 2-graph is the triangle. This construction can be extended to $k \ge 3$.

Example 1.1 ([EL]). Let $E_1, E_2, \ldots, E_k$ be pairwise disjoint sets, $|E_i| = i$. Define $\mathcal{E}_i = \{E : |E| = k, E_i \subset E, |E_j \cap E| = 1, i < j \le k\}$. Then $\mathcal{E} = \mathcal{E}_1 \cup \ldots \cup \mathcal{E}_k$ is maximal intersecting,

$$|\mathcal{E}| = \sum_{1 \le i \le k} k!/i! = \lfloor (e-1)k! \rfloor. \tag{1.2}$$

For $k = 3$ one has $|\mathcal{E}| = 10$. Although there are other non-isomorphic examples, no maximal intersecting 3-graph has more than 10 edges, i.e., $m(3) = 10$. For more than twenty years this construction was believed to be the largest possible (cf. [L]). However, in [FOT] a construction of size about $(k/2)^k$ was given. Since we are mostly interested in upper bounds, we reproduce it only for the even case.

Example 1.2 ([FOT])**.** Let $k = 2a+2$, $a \geq 1$. Let x be a vertex and choose $2a+1$ pairwise disjoint $(a+2)$-element sets A_i, $0 \leq i \leq 2a$, $x \notin A_i$. Define $\mathcal{A}_i = \{A : |A| = k,\ A_i \subset A,\ |A \cap A_j| = 1,\ i+1 \leq j \leq i+a\}$ (computation is modulo $2a+1$). Define also $\mathcal{B} = \{B : |B| = k,\ x \in B,\ |B \cap A_i| = 1,\ 0 \leq i \leq 2a\}$. Set $\mathcal{A} = \mathcal{B} \cup \mathcal{A}_0 \cup \ldots \cup \mathcal{A}_{2a+1}$. Then $\mathcal{A}$ is maximal intersecting, $|\mathcal{A}| = (a+2)^{k-1} + (k-1) \cdot (a+2)^a \sim \left(\frac{k}{2}\right)^{k-1} \cdot e$.

To improve the bound (1) considerably seems to be difficult. In 1994 Tuza [T] proved $m(k) \leq \left(1 - \frac{1}{e} + o(1)\right) k^k$ but no progress was made for another 20 years.

In 2016 Arman and Retter [AR] proved

$$m(k) \leq (1 + o(1))k^{k-1}. \tag{1.3}$$

The aim of the present paper is to provide a near-exponential improvement of the previous upper bounds.

Theorem 1.3. *For $k \geq 81$ one has*

$$m(k) < k^k \cdot e^{-k^{1/4}/6}. \tag{1.4}$$

For a family of sets $\mathcal{F}$ and a set D we use the following standard notations.

$$\mathcal{F}(D) = \{F \setminus D :\ D \subset F \in \mathcal{F}\}, \quad \mathcal{F}(\overline{D}) = \{F \in \mathcal{F} : F \cap D = \emptyset\}.$$

In the case $D = \{x\}$ we simply write $\mathcal{F}(x)$ and $\mathcal{F}(\overline{x})$. Note the identity $|\mathcal{F}| = |\mathcal{F}(x)| + |\mathcal{F}(\overline{x})|$.

A set C is said to be a *cover* (for $\mathcal{F}$) if $F \cap C \neq \emptyset$ for all $F \in \mathcal{F}$. The covering number $\tau(\mathcal{F})$ is defined as

$$\tau(\mathcal{F}) = \{\min |C| : C \text{ is a cover for } \mathcal{F}\}.$$

If $\mathcal{F}$ is an intersecting k-graph then $\tau(\mathcal{F}) \leq k$. Indeed, every $F \in \mathcal{F}$ is a cover.

Two families $\mathcal{A}$ and $\mathcal{B}$ are said to be *cross-intersecting* if $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$.

Observation 1.4. *Suppose that $\mathcal{F}$ is an intersecting k-graph with $\tau(\mathcal{F}) = k$. Let x be an arbitrary vertex. Then* (i) *and* (ii) *hold.*

(i) *$\mathcal{F}(\overline{x})$ and $\mathcal{F}(x)$ are cross-intersecting.*

(ii) *$\tau(\mathcal{F}(\overline{x})) = k - 1$.*

Proof. (i) Take $H \in \mathcal{F}(x)$, $F \in \mathcal{F}(\overline{x})$. Since $H \cup \{x\} \in \mathcal{F}$ and $x \notin F \in \mathcal{F}$,

$$\emptyset \neq (H \cup \{x\}) \cap F = H \cap F.$$

(ii) By (i) any $H \in \mathcal{F}(x)$ is a cover for $\mathcal{F}(\overline{x})$ showing $\tau(\mathcal{F}(\overline{x})) \leq k - 1$. On the other hand if T covers $\mathcal{F}(\overline{x})$ then $T \cup \{x\}$ is a cover for $\mathcal{F}$. Thus

$$\tau(\mathcal{F}(\overline{x})) \geq \tau(\mathcal{F}) - 1 = k - 1.$$ □

We deduce Theorem 1.3 from the following result.

Theorem 1.5. *Let $\mathcal{F}$ be a maximal intersecting k-graph, $k \geq 81$ and x an arbitrary vertex. Then*

$$|\mathcal{F}(x)| \leq k^{k-1} e^{-k^{1/4}/6}. \tag{1.5}$$

To deduce (1.4) form (1.5) is immediate. Choose an arbitrary edge F of a maximal intersecting family $\mathcal{F}$ with $|\mathcal{F}| = m(k)$. Since $\mathcal{F}$ is intersecting,

$$|\mathcal{F}| \leq \sum_{x \in F} |\mathcal{F}(x)| - (k - 1).$$

Applying (1.5) to each term $|\mathcal{F}(x)|$ yields (1.4). □

The paper is organised as follows. The next section introduces the notion of a t-broom. This is a simple k-graph that can be found as a subgraph in every k-graph with large covering number.

In Section 3 we consider a pair of cross-intersecting families. The main result is Proposition 3.3 that shows that the existence of brooms in the first implies the existence of relatively slim *s-cuts* (cf. Definition 3.1) for the second.

In Section 4 we use this result to prove Theorem 1.5 and thereby Theorem 1.3 as well.

2 Brooms

Definition 2.1. Let $t \geq 2$, $s \geq 2$ be integers. A k-graph $\mathcal{B} = \{B_1, \dots, B_s\}$ is called a *t-broom* of size s if $1 \leq |B_i \cap B_j| < t$ for $1 \leq i < j \leq s$ and $\mathcal{B}$ has no vertex of degree more than two (i.e., $B_u \cap B_v \cap B_w = \emptyset$ for all $1 \leq u < v < w \leq s$).

Proposition 2.2. *Suppose t and s are positive integers, $s \geq 3$, $\mathcal{G}$ is an intersecting k-graph, $\tau(\mathcal{G}) \geq \binom{s}{2}t$. Then either $\mathcal{G}$ contains a t-broom of size $s+1$ or there exist $G, G' \in \mathcal{G}$ such that*

$$t \leq |G \cap G'| \leq k - t.$$

Proof. Arguing indirectly we assume that for all $G, G' \in \mathcal{G}$ either $|G \cap G'| > k - t$ or $|G \cap G'| < t$ holds. To get started let us find $B_1, B_2 \in \mathcal{G}$ with $|B_1 \cap B_2| < t$.

To this effect fix an arbitrary $B_1 \in \mathcal{G}$ and a subset $T \subset B_1$, $|T| = t$. Since even for $s = 3$ we have $\tau(\mathcal{G}) > t$, there exists $B_2 \in \mathcal{G}$ with $B_2 \cap T = \emptyset$. This implies $|B_1 \cap B_2| \leq k - t$ and therefore $|B_1 \cap B_2| < t$, as desired.

Now suppose that we have found a t-broom $\{B_1, \dots, B_p\} \subset \mathcal{G}$ of size p, $2 \leq p \leq s$. To conclude the proof we show that it can be extended to a larger t-broom.

Define $Y = \bigcup\limits_{1 \leq i < j \leq p} B_i \cap B_j$. Note that

$$|Y| \leq \binom{p}{2}(t-1) < \binom{s}{2}t \leq \tau(\mathcal{G}).$$

Define $R_i = Y \cap B_i$ and $E_i = B_i \setminus Y$. Next we define a subset S_i of E_i. If $|R_i| \geq t$ we let $S_i = \emptyset$. If $|R_i| < t$ then we let S_i be an arbitrary $(t - |R_i|)$-subset of E_i $(1 \leq i \leq p)$. Let us show

$$(*) \qquad |Y| + \sum_{1 \leq i \leq p} |S_i| < \binom{s}{2}t.$$

If $p = 2$ then $B_1 \cap B_2 = Y = R_1 = R_2$. Now $|R_i| + |S_i| \leq t$ and $R_i \neq \emptyset$ imply $|R_1| + |S_1| + |S_2| \leq 2t - 1 \leq \binom{s}{2}t$. For the case $p \geq 3$ let us use

$$|B_i \cap B_{i+1}| + |S_i| \leq t, \text{ valid for all } i < p, \text{ along with } |B_p \cap B_1| + |S_p| \leq t.$$

Adding these p inequalities together with the simpler $|B_i \cap B_j| < t$ for the remaining $\binom{p}{2} - p$ choices of $\{i, j\}$ gives $(*)$.

Set $Z = Y \cup S_1 \cup \ldots \cup S_p$. Now $\tau(\mathcal{G}) > \binom{s}{2}t$ implies the existence of $G \in \mathcal{G}$ satisfying $G \cap Z = \emptyset$. The careful choice of S_i entails $|G \cap B_i| \le k - t$. Consequently, $|G \cap B_i| < t$ and $\mathcal{B} \cup \{G\}$ is a t-broom of size $p + 1$. □

3 Constructing slim cuts

Let us fix $s = \lfloor k^{1/4} \rfloor$. This implies $s\binom{s}{2} < \frac{k}{2}$, a fact that we shall use without further reference.

Definition 3.1. Given a family of sets $\mathcal{A}$, the ℓ-graph $\mathcal{D}$ is called an ℓ-cut for $\mathcal{A}$ if for all $A \in \mathcal{A}$ there exists $D \in \mathcal{D}$ such that $D \subset A$.

Note that if the families $\mathcal{A}$ and $\mathcal{G}$ are cross-intersecting then every $G \in \mathcal{G}$ is a 1-cut for $\mathcal{A}$.

Our proof of the main theorem is based on suitable, relatively slim ℓ-cuts, for the family $\mathcal{F}(x)$ where $\mathcal{F}$ is a maximal intersecting k-graph. However, we prefer to proceed in the more general setting of pairs of cross-intersecting families.

Lemma 3.2. *Suppose that $\mathcal{A}$ and $\mathcal{G}$ are cross-intersecting, $\mathcal{G}$ is a k-graph with $\tau(\mathcal{G}) > \ell$. Then for every vertex y there exists an ℓ-cut $\mathcal{D}_y$ for $\mathcal{A}$ consisting entirely of sets not containing y and satisfying $|\mathcal{D}_y| \le k^\ell$.*

Proof. Let $G_1 \in \mathcal{G}$ satisfy $y \notin G_1$. Then the k elements of G_1 form a desired 1-cut proving the case $\ell = 1$. Now we apply induction. Suppose that for some p we have constructed a p-cut $\mathcal{D}_y$ for $\mathcal{A}$, $|\mathcal{D}_y| \le k^p$ and $y \notin D$ for all $D \in \mathcal{D}_y$. If $p < \ell$ then $|D \cup \{y\}| \le \ell$. Thus there exists a set $G(D, y) \in \mathcal{G}$ satisfying $G(D, y) \cap (D \cup \{y\}) = \emptyset$. Then the $(\ell+1)$-graph $\bigcup\limits_{D \in \mathcal{D}_y} \{D \cup (z) : z \in G(D, y)\}$ will be a $(\ell + 1)$-cut for $\mathcal{A}$, as desired. □

Proposition 3.3. *Suppose that $\mathcal{A}$ and $\mathcal{G}$ are cross-intersecting, $\mathcal{G}$ is an intersecting k-graph with $\tau(\mathcal{G}) > s\binom{s}{2}$, $s \ge 5$, $k \ge s^4 \ge 81$. Then there exists a $(s + 1)$-cut $\mathcal{D}$ for $\mathcal{A}$ satisfying*

$$|\mathcal{D}| < \left(1 - \frac{s+1}{3k}\right)^{s+1} k^{s+1} \tag{3.1}$$

or a 2-cut $\mathcal{D}'$ with

$$|\mathcal{D}'| < \left(1 - \frac{s+1}{3k}\right)^2 k^2. \tag{3.2}$$

Proof. Let us start with the harder case. We suppose

$$|G \cap G'| < s \quad \text{or} \quad |G \cap G'| > k - s \quad \text{for all} \quad G, G' \in \mathcal{G} \tag{3.3}$$

and prove the existence of a slim $(s+1)$-cut.

In view of Proposition 2.2 there exists a t-broom $\mathcal{B} = \{B_1, \dots, B_{s+1}\}$ of size $s+1$, $\mathcal{B} \subset \mathcal{G}$. We set again $Y = \bigcup\limits_{1 \le i < j \le s+1} B_i \cap B_j$, $E_i = B_i \setminus Y$. For each $y \in Y$ let $\mathcal{D}_y$ be an s-cut for $\mathcal{A}$, $|\mathcal{D}_y| \le k^s$. Set $\mathcal{E}_y = \{D \cup \{y\} : D \in \mathcal{D}_y\}$.

Define $\mathcal{E} = \big\{\{x_1, \dots, x_{s+1}\} : x_i \in E_i\big\}$. Since the E_i are pairwise disjoint, $\mathcal{E}$ is a $(s+1)$-graph with

$$|\mathcal{E}| = |E_1| \cdot \ldots \cdot |E_{s+1}|.$$

We claim that $\left(\bigcup\limits_{y \in Y} \mathcal{E}_y\right) \cup \mathcal{E} \overset{\text{def}}{=} \mathcal{D}$ is a $(s+1)$-cut for $\mathcal{A}$.

Let $A \in \mathcal{A}$. If $A \cap Y \ne \emptyset$ then choose $y \in A \cap Y$. Since $\mathcal{D}_y$ is an s-cut for A we can pick $D \in \mathcal{D}_y$ satisfying $D \subset A$. Thus $\{y\} \cup D$ is a $(s+1)$-set contained in A.

If $A \cap Y = \emptyset$ then the cross-intersecting property implies $A \cap E_i \ne \emptyset$ for $1 \le i \le s+1$. Picking $x_i \in A \cap E_i$ the $(s+1)$-set $\{x_1, \dots, x_{s+1}\}$ is a subset of A finishing the proof of the claim.

To estimate the size of this $(s+1)$-cut note that

$$|E_1| + \ldots + |E_{s+1}| = (s+1)k - 2|Y| \quad \text{and} \quad |Y| \ge \binom{s+1}{2}.$$

Invoking the inequality between arithmetic and geometric mean we infer

$$|\mathcal{E}| \le \left(k - \frac{2|Y|}{s+1}\right)^{s+1} \le k^{s+1} - 2|Y|k^s + s|Y|^2 \cdot k^{s-1}.$$

Consequently,

$$|\mathcal{D}| \le k^{s+1}\left(1 - \frac{|Y|}{k} + \frac{s|Y|^2}{k^2}\right). \tag{3.4}$$

For $2s|Y| < k$ the term in the bracket is a decreasing function of $|Y|$. Using $|Y| \le (s-1)\binom{s+1}{2}$, $2s|Y| \le s^2(s^2-1) < s^4$. Setting $s = \lfloor k^{1/4} \rfloor$ is sufficient. In this case the maximum of the RHS is attained if $|Y|$ is minimal, that is, $|Y| = \binom{s+1}{2}$.

For this value

$$\frac{|Y|}{k} - \frac{s|Y|^2}{k^2} = \frac{(s+1)s}{2k}\left[1 - \frac{s^2(s+1)}{2k}\right]. \tag{3.5}$$

Using $k \ge s^4$, the quantity in [ ] is at most $1 - \frac{1}{2s} - \frac{1}{2s^2} \ge 1 - \frac{1}{s}$ for $s \ge 1$. For $s \ge 5$ one has $\frac{s-1}{2} \ge \frac{s+1}{3}$ implying that the value of (3.4) is at least $\frac{(s+1)^2}{3k}$.

Using (3.4) we obtain

$$|\mathcal{D}| < k^{s+1}\left(1 - \frac{s+1}{3k}(s+1)\right) < \left(k\left(1 - \frac{s+1}{3k}\right)\right)^{s+1} \quad \text{as desired.}$$

Consider next the case that we can find G_1, G_2 satisfying

$$s \le |G_1 \cap G_2| \le k - s.$$

Set $Y = G_1 \cap G_2$, $E_i = G_i \setminus Y$, $i = 1, 2$.

Define $\mathcal{E} = \big\{\{x_1, x_2\} : \ x_1 \in E_1,\, x_2 \in E_2\big\}$, $|\mathcal{E}| = (k - |Y|)^2$. For each $y \in Y$ fix $G(y) \in \mathcal{G}$ with $y \notin G(y)$. Finally set

$$\mathcal{D}' = \mathcal{E} \cup \left(\bigcup_{y \in Y} \big\{\{y, u\} : u \in G(y)\big\}\right)$$

It is easy to verify that $\mathcal{D}'$ is a 2-cut.

$$|\mathcal{D}'| = (k - |Y|)^2 + |Y| \cdot k = k^2 - |Y|k + |Y|^2.$$

In the range $s \le |Y| \le k - s$ the maximum of the RHS is attained for $|Y| = s$ and $|Y| = k - s$. It is equal to

$$k^2 - sk + s^2 = k^2\left(1 - \frac{s}{k} + \frac{s^2}{k^2}\right) < k^2\left(1 - \frac{s+1}{3k}\right)^2$$

for $s \ge 3$ and thus for $k \ge 81$. □

4 The proof of Theorem 1.5

Recall Observation 1.4. This enables us to apply Proposition 3.3 with $\mathcal{A} = \mathcal{F}(x), \mathcal{G} = \mathcal{F}(\overline{x})$. However, one application is not sufficient, we need to repeat it. For a set D recall the definitions

$$\mathcal{A}(D) = \{A \setminus D : D \subset A \in \mathcal{A}\}, \quad \mathcal{G}(\overline{D}) = \{G \in \mathcal{G} : G \cap D = \emptyset\}.$$

Note also $\tau(\mathcal{G}(\overline{D})) \geq \tau(\mathcal{G}) - |D|$ and the fact that if $\mathcal{A}, \mathcal{G}$ are cross-intersecting then $\mathcal{A}(D)$ and $\mathcal{G}(\overline{D})$ are cross-intersecting as well.

Now we can describe the process. Set $\mathcal{A}_0 = \mathcal{A}$, $\mathcal{G}_0 = \mathcal{G}$, $D_0 = \emptyset$.

Suppose that we have defined already $\mathcal{A}_i$, $\mathcal{G}_i$ and D_i where $|D_i| > D_{i-1}|$, $\mathcal{A}_i = \mathcal{A}(D_i)$, $\mathcal{G}_i = \mathcal{G}(\overline{D}_i)$ and

$$|\mathcal{A}| < \left(1 - \frac{s+1}{3k}\right)^{|D_i|} |\mathcal{A}(D_i)| k^{|D_i|}. \tag{4.1}$$

Suppose that we can apply Proposition 3.3 to $\mathcal{A}_i$ and $\mathcal{G}_i$. Then either (3.1) or (3.2) holds.

In case of (3.1) we choose $D \in \mathcal{D}$ to maximize $|\mathcal{A}_i(D)|$ and set $D_{i+s+1} = D_i \cup D$, $\mathcal{A}_{i+1} = \mathcal{A}(D_i \cup D)$, $\mathcal{G}_{i+1} = \mathcal{G}(\overline{D_i \cup D})$. In view of (3.1) and $|D_{i+1}| = |D_i| + s + 1$ the inequality (4.1) holds with i replaced by $i+1$.

In case of (3.2) we proceed in absolutely the same way. The only difference is that $D' \in \mathcal{D}$ satisfies $|D'| = 2$. Therefore $|D_{i+1}| = |D_i| + 2$.

As long as $|D_i| < k/2$ we have

$$\tau(\mathcal{G}(\overline{D}_i)) > k - 1 - \frac{k-1}{2} \geq \lfloor k/2 \rfloor.$$

Thus we can proceed. Once we have $|D_i| \geq k/2$ we stop. In view of Observation 1.4, Lemma 3.2 implies

$$|\mathcal{A}(D_i)| \leq k^{k-1-|D_i|}.$$

Combining with (4.1) we infer

$$|\mathcal{A}| < k^{k-1} \left(1 - \frac{s+1}{3k}\right)^{k/2} < k^{k-1} \cdot e^{-\frac{1}{6}k^{1/4}}.$$

□ □ □

5 Concluding remarks

First note that every intersecting k-graph $\mathcal{F}$ with $\tau(\mathcal{F}) = k$ can be extended to a maximal intersecting k-graph on the same vertex set. Therefore the upper bound (1.1) is valid for such $\mathcal{F}$ as well. Let us remark that Gyárfás [G] proved that $\big|\{G : G \cap F \neq \emptyset \text{ for all } F \in \mathcal{F}, |G| = t\}\big| \leq k^t$ for all k-graphs with $\tau(\mathcal{F}) = t$, that is, without the intersection property. Equality holds if $\mathcal{F}$ consists of t pairwise disjoint edges.

Finally we remark that our methods can be refined to yield $m(k) < k^k \cdot e^{-ck^{1/3}}$. We preferred to prove the present bound keeping the argument and calculations simpler.

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