Antichains of fixed diameter

by Peter Frankl, Rényi Institute, Budapest, Hungary

Abstract

The main result of the paper is to prove for every $r \geq 0$ and $n > n_0(r)$ that every antichain on n vertices and consisting of more than $\binom{n}{r}$ sets contains two members whose symmetric difference is at least $2r+2$. The bound is best possible and all the extremal families are determined. For $r \geq 3$ we show that $n > 6(r+1)^2$ is sufficient.

1 Introduction

Let X be an n-element set. Usually we consider X to be $[n]$, the set of integers $1, 2, \dots, n$. The power set of X is denoted by 2^X and $\binom{X}{k}$ is the collection of all k-subsets of X.

Subsets of 2^X are called *families*. A family $\mathcal{F}$ is called an *antichain* if $F \not\subset F'$ for all distinct $F, F' \in \mathcal{F}$.

Sperner Theorem ([S])**.** *If $\mathcal{F} \subset 2^X$ is an antichain then*

$$|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}. \tag{1.1}$$

Moreover, equality is achieved only for $\mathcal{F} = \binom{n}{\lfloor n/2 \rfloor}$ and $\mathcal{F} = \binom{n}{\lceil n/2 \rceil}$ (for n even, they coincide).

This result was the starting point of a lot of research which by now is an important area inside combinatorics, called extremal set theory. Much of this research was motivated by problems of Paul Erdős. Let us give just two examples, related also to our results.

This research was supported by the National Research, Development and Innovation Office – NKFIH Fund No. K116769.

Definition 1.1. For a fixed integer d, $0 \le d < n$, the family $\mathcal{F} \subset 2^X$ is called *d-union* if

$$|F \cup F'| \le d \quad \text{for all} \quad F, F' \in \mathcal{F}. \tag{1.2}$$

If $d = 2r$ is even the family

$$\mathcal{K}(n,d) \stackrel{\text{def}}{=} \{K \subset X : |K| \le r\}$$

is obviously d-union. For $d = 2r+1$ and $x \in X$ fixed, let us define

$$\mathcal{K}_x(n,d) = \big\{K \subset X : |K \cap (X - \{x\})| \le r\big\}.$$

It is easy to check that $\mathcal{K}_x(n,d)$ is $2r+1$-union.

Katona Theorem ([Ka]). *Suppose that $\mathcal{F} \subset 2^X$ is d-union, $0 \le d < n$. Then* (i) *or* (ii) *holds.*

(i) *$d = 2r$ and $|\mathcal{F}| \le |\mathcal{K}(n,d)|$,*
(ii) *$d = 2r+1$ and $|\mathcal{F}| \le |\mathcal{K}_x(n,d)|$.*

For two sets A, B let $A+B$ denote their symmetric difference, i.e., $A+B = (A \setminus B) \cup (B \setminus A)$.

Let us define the *diameter* $\Delta(\mathcal{F})$ of a family by $\Delta(\mathcal{F}) = \max\big\{|A+B| : A, B \in \mathcal{F}\big\}$.

Kleitman Diameter Theorem ([Kl]). *Suppose that $\mathcal{F} \subset 2^X$ has diameter at most d, $0 \le d < n$. Then* (i) *or* (ii) *holds.*

(i) *$d = 2r$ and $|\mathcal{F}| \le |\mathcal{K}(n,d)|$,*
(ii) *$d = 2r+1$ and $|\mathcal{F}| \le |\mathcal{K}_x(n,d)|$.*

Since $|A+B| \le |A \cup B|$ for all A, B, Kleitman's Theorem is a strengthening of Katona's Theorem.

However, while Katona proved that except for the case $n = d+1$, $\mathcal{K}(n,d)$ and $\mathcal{K}_x(n,d)$ provide *all* the optimal families, until recently for Kleitman's Theorem no such characterisation was known.

Let us recall the following classical result.

Milner Theorem ([Mi]). *Suppose that $\mathcal{F} \subset 2^X$ is an antichain and it is d-union. Then*

$$|\mathcal{F}| \le \binom{n}{\lfloor d/2 \rfloor} \quad \textit{holds.} \tag{1.3}$$

The following very natural problem arose at the seminar of Katona in Budapest. It was communicated to the author by Füredi in 2014.

Problem. Suppose that $\mathcal{F} \subset 2^X$ is an antichain that has diameter d, $0 \leq d \leq n$, d is fixed. What is the maximum of $|\mathcal{F}|$?

Let us denote this maximum value by $m(n,d)$. If A, B are two *distinct* members of an antichain then both $A \setminus B$ and $B \setminus A$ are non-empty, forcing $|A + B| \geq 2$. This implies

$$m(n,0) = m(n,1) = 1. \tag{1.4}$$

Let us also mention that $m(n,n-1) = \binom{n}{\lfloor \frac{n-1}{2} \rfloor}$ is equivalent to an old result of Purdy [Pu]. Namely, requiring $\Delta(\mathcal{F}) \leq n-1$ is the same as saying that $\mathcal{F}$ contains no pair of complementary members.

For a family $\mathcal{F} \subset 2^X$ let $\mathcal{F}^c$ denote the family of the complements of sets from $\mathcal{F}$, i.e., $\mathcal{F}^c = \{X - F : F \in \mathcal{F}\}$.

Example 1.1. (i) $d = 2r$, $d \leq n$,

$$\mathcal{R}(n,d) = \binom{X}{r}.$$

(ii) $d = 2r+1 < n$ and $Q \subset X$, $|Q| = 2r+1$, fixed

$$\mathcal{R}_Q(n,d) = \binom{Q}{r+1} \cup \binom{X}{r} - \binom{Q}{r}.$$

It is easy to check that $\Delta(R(n,d)) = 2r$ and $\Delta\big(\mathcal{R}_Q(n,d)\big) = 2r+1$ hold. Note also that $|\mathcal{R}_Q(n,d)| = \binom{n}{r}$.

Our main result is the following.

Main Theorem. *Suppose that $d \geq 2$ is fixed and $n > n_0(d)$. Then*

$$m(n,d) = \binom{n}{\lfloor d/2 \rfloor}. \tag{1.5}$$

Moreover, $\mathcal{R}(n,d)$ and $\mathcal{R}_Q(n,d)$ and their complements are the only families achieving equality.

The proof of (1.5) is rather involved. The proof gives $n_0(d) < 3(d+1)^2/2$ for $d > 5$.

In the next section we are going to give a simple proof for the inequality $m(n,d) = \big(1+o(1)\big)\binom{n}{\lfloor d/2 \rfloor}$, where d is fixed and $n \to \infty$. Let us mention that this was proved independently by Füredi using a completely different argument.

Both Kleitman's and Milner's proofs are short and simple. Kleitman reduces the diameter problem to the union problem of Katona by introducing a very useful operation on families, called the *down-push.* Milner's proof, just as Sperner's proof depends on another operation on families. The problem is that neither of these operations maintains *simultaneously* the diameter and the property of being an antichain.

The paper is organized as follows. In Section 2 first we present some bounds for $m(n,d)$ that are somewhat weaker than (1.5) but which are much easier to prove. The proofs are based on little but useful tricks and in the case of Theorem 2.2 on linear independence.

The second part of this section reviews numerous tools that we need to establish (1.5). The most important among these results is Theorem 2.3, a stability theorem for the Kleitman Diameter Theorem. For the case of even diameter, $d = 2r$, it shows that every family of diameter d is either contained in a Hamming sphere of radius r or its size is much smaller than Kleitman's bound. For the odd case we need some slightly more complicated object, called double-spheres (cf. Definition 2.2).

Section 3 is dealing with the estimation of the maximum size of antichains *inside* spheres and double-spheres. Interestingly the answer depends on the location of the center. The proofs depend on a classical result of Kleitman, Edelberg and Lubell and the poset structure of the orbits (under the action of the automorphism group of these objects). The center Y is a subset of X. Theorem 3.3 shows that unless $Y = \emptyset$ or $Y = X$ for $d = 2r$, the largest antichain has size less than $\binom{n}{r}$.

To prove (1.5) for $d = 2r+1$ we need some stronger estimates. These are provided by Proposition 3.2 and Claim 3.3.

With all these preparations the proof of (1.5) for even diameter is very short. It is not even half a page. Along with this proof Section 4 contains Proposition 4.1. Noting that in any family at least half of the sets have the same parity, in a family $\mathcal{A}$ with $\Delta(\mathcal{A}) = 2r+1$ we can always find a subfamily $\mathcal{A}' \subset \mathcal{A}$ with $|\mathcal{A}'| \geq \frac{1}{2}|\mathcal{A}|$ and $\Delta(\mathcal{A}') \leq 2r$.

In Section 5 we are going to apply Proposition 4.1 in such a context to maintain a strong grip on the structure of the remaining sets, i.e., $\mathcal{A} \setminus \mathcal{A}'$. In particular, by Theorem 2.3, $\mathcal{A}'$ will be contained in a sphere $\mathcal{S}(Y, r)$ and $\mathcal{A} \setminus \mathcal{A}'$ in $\mathcal{S}(Y, r+1)$. The case $Y = \emptyset$ is handled swiftly.

In view of the results of Section 3, the larger $|Y|$ is, the stronger the bounds for the size of a largest antichain in $\mathcal{S}(Y, r)$ are. Therefore after treating the cases $|Y| = 1$ and $r = 1$, we can combine those results with Theorem 2.4 and conclude the proof of (1.5).

In Section 6 we prove Theorem 2.4. Improvements on this result would lead to lowering our requirements, $n > 6(r+1)^2$ to guarantee (1.5).

In Section 7 some conjectures and further results are stated.

2 Simple results and tools of proofs

Let $\mathcal{F} \subset 2^X$ be an antichain with $\Delta(\mathcal{F}) \leq d$.

If $d = 2r$ is even then the Kleitman Diameter Theorem implies

$$|\mathcal{F}| \leq \binom{n}{r} + \binom{n}{r-1} + \ldots + \binom{n}{0}. \tag{2.1}$$

One can improve on (2.1) by a simple trick.

Proposition 2.1.

$$m(n, 2r) \leq \binom{n}{r} + \binom{n}{r-2} + \ldots + \binom{n}{r - 2\lfloor r/2 \rfloor}.$$

Proof. Let $\mathcal{F} \subset 2^X$ be an antichain. For an arbitrary but fixed element $x \in X$ consider $\widetilde{\mathcal{F}} = \left\{F \cap (X - \{x\}) : F \in \mathcal{F}\right\}$. The point is that $\left|\widetilde{\mathcal{F}}\right| = |\mathcal{F}|$ because $\mathcal{F}$ is an antichain.

If $\Delta(\mathcal{F}) \leq 2r$ then the same holds for $\widetilde{\mathcal{F}}$ and applying the Kleitman Diameter Theorem yields

$$|\mathcal{F}| \leq \binom{n-1}{r} + \binom{n-1}{r-1} + \ldots = \binom{n}{r} + \binom{n}{r-2} + \ldots + \binom{n}{r - 2\lfloor r/2 \rfloor}. \qquad \square$$

Corollary 2.1. (i) $m(n, 2) = n$,
(ii) $m(n, 4) \leq \binom{n}{2} + 1$.

One can go one step further.

Theorem 2.1. *For $n > 2r > 0$ one has*

$$m(n, 2r) \leq \binom{n}{r} + \binom{n}{r-3} + \ldots + \binom{n}{r - 3\lfloor n/3 \rfloor}. \tag{2.2}$$

Proof. Suppose that $\mathcal{F} \subset 2^X$ is an antichain satisfying $\Delta(\mathcal{F}) \leq 2r$. Let $X = Z \dot\cup \{x, y\}$, $|Z| = n-2$ and define the two families $\mathcal{F}_0, \mathcal{F}_1 \subset 2^Z$ as follows.

$$\mathcal{F}_0 = \{F \cap Z : F \in \mathcal{F}\},$$
$$\mathcal{F}_1 = \{G \subset Z : \exists \text{ distinct } F, F' \in \mathcal{F} \text{ such that } F \cap Z = G = F' \cap Z\}.$$

Since $\Delta(\mathcal{F}_0) \leq \Delta(\mathcal{F})$,

$$|\mathcal{F}_0| \leq \binom{n-2}{r} + \binom{n-2}{r-1} + \ldots + \binom{n-2}{0} \tag{2.3}$$

follows from the Kleitman Diameter Theorem. In order to bound $|\mathcal{F}_1|$ let us prove:

Claim 2.1. *$\mathcal{F}_1$ is an antichain with $\Delta(\mathcal{F}_1) \leq 2r - 2$.*

Proof of the claim. Let $F, F' \in \mathcal{F}$ satisfy $F \cap Z = F' \cap Z$. Since $\mathcal{F}$ is an antichain, the only possibility for $F \setminus Z$, $F' \setminus Z$ are the two 1-element sets $\{x\}$, $\{y\}$. Thus for every $G \in \mathcal{F}_1$, $G \cup \{x\}$, $G \cup \{y\} \in \mathcal{F}$ holds. This implies that $\mathcal{F}_1$ is an antichain. Also, if $G, G' \in \mathcal{F}_1$ then

$$|G + G'| + 2 = \big|(G \cup \{x\}) + (G' \cup \{y\})\big| \leq 2r \quad \text{entails} \quad \Delta(\mathcal{F}_1) \leq 2r - 2. \qquad \square$$

Through the above proof the identity

$$|\mathcal{F}| = |\mathcal{F}_0| + |\mathcal{F}_1| \tag{2.4}$$

should be evident as well.

In order to prove (2.2) we apply induction on r. The base case, $r = 1$ is covered by Corollary 2.1 (i). Using $\binom{n}{i} = \binom{n-2}{i} + 2\binom{n-2}{i-1} + \binom{n-2}{i-2}$ along with (2.3) and (2.4) and applying the induction hypothesis to $\mathcal{F}_1$ gives

$$\begin{aligned} |\mathcal{F}| &\leq \binom{n-2}{r} + \binom{n-2}{r-1} + \binom{n-2}{r-2} + \ldots + \binom{n-2}{0} + \binom{n-2}{r-1} + \binom{n-2}{r-4} + \ldots \\ &= \binom{n}{r} + \binom{n}{r-3} + \ldots + \binom{n}{r - 3\lfloor r/3 \rfloor}. \end{aligned} \qquad \square$$

Corollary 2.2. (i) $m(n,4) = \binom{n}{2}$,

(ii) $m(n,6) \le \binom{n}{3} + 1$. $\square$

Let us note that either of (2.1) and (2.2) implies $m(n,d) = (1+o(1))\binom{n}{\lfloor d/2 \rfloor}$ for $d = 2r$, even. To prove the same for $d = 2r + 1$ let us use a linear independence argument from [F1]. The reason that we include this result is two-fold. First, the proof is much simpler than that of the Main Theorem. Second, it holds in a much wider range.

Theorem 2.2. *Let $\mathcal{F} \subset 2^{[n]}$ be an antichain satisfying $\Delta(\mathcal{F}) \le 2r+1$. Then for all $n \ge r \ge 0$,*

$$|\mathcal{F}| \le \binom{n}{r} + \binom{n}{r-1} + \ldots + \binom{n}{0} \quad \textit{holds.} \tag{2.5}$$

Proof. Let us order the members of $\mathcal{F}$ in increasing order of their size: $\mathcal{F} = (F_1, F_2, \ldots, F_m)$ where $|\mathcal{F}| = m$ and $|F_1| \le |F_2| \le \ldots \le |F_m|$ hold. For $0 \le i \le r$ define two $(0-1)$-matrices $C(i,\mathcal{F})$ and $D(i,\mathcal{F})$, both of size $\binom{n}{i} \times m$ where the rows are indexed by $A \in \binom{[n]}{i}$, the columns by $F_1, \ldots, F_m$, the general entry for the first is

$$c_i(A,j) = \begin{cases} 1 & \text{if } A \subset F_j, \\ 0 & \text{otherwise.} \end{cases}$$

For the second, the general entry is

$$d_i(A,j) = \begin{cases} 1 & \text{if } A \cap F_j = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $c_i(A,j)d_i(A,k) = 1$ if and only if the i-element set A is contained in $F_j - F_k$. This implies that the general entry $p(j,k)$ of the m by m matrix $P(i) = C(i,\mathcal{F})^T D(i,\mathcal{F})$ is $\binom{|F_j - F_k|}{i}$.

Consider the (composition) identity

$$\binom{x-1}{r} = \binom{x}{r} - \binom{x}{r-1} + \binom{x}{r-2} - \ldots + (-1)^r \binom{x}{0}. \tag{2.6}$$

Then (2.6) implies that the general entry of the m by m matrix $P = \sum_{0 \le i \le r} (-1)^{r-i} P(i)$ is $\binom{|F_j - F_k| - 1}{r}$.

Claim 2.2. *rank $P = m$.*

Proof of the claim. Note that $F_j - F_j = \emptyset$ implies that the diagonal entry is $\binom{-1}{r} = (-1)^r \neq 0$.

On the other hand,

$$1 \leq |F_j - F_k| \leq r \quad \text{for} \quad 1 \leq j < k \leq m. \tag{2.7}$$

Indeed, $F_j \not\subset F_k$ implies $|F_j - F_k| \geq 1$. Should $|F_j - F_k| \geq r + 1$ hold, $|F_k| \geq |F_j|$ would imply $|F_k - F_j| \geq |F_j - F_k| \geq r + 1$, i.e.,

$$|F_j + F_k| \geq 2r + 2$$

contradicting $\Delta(\mathcal{F}) \leq 2r + 1$. Now (2.7) implies $\binom{|F_j - F_k| - 1}{r} = 0$. That is, P is a lower triangular matrix with non-zero diagonal, proving the claim. $\square$

Now (2.5) follows from the obvious inequality: rank $C(i, \mathcal{F}) \leq \binom{n}{i}$. $\square$

Let us mention that (2.5) can be deduced from some results of Füredi and Sudakov [FS] as well.

Let us now present the main tools that we need for proving the main result, $m(n, d) = \binom{n}{\lfloor d/2 \rfloor}$ for $n > n_0(d)$.

Definition 2.1. For $0 \leq r < n$ and a subset $Y \subset X$ let $\mathcal{S}(Y, r) = \{A \subset X : |Y + A| \leq r\}$. The family $\mathcal{S}(Y, r)$ is usually called a *(Hamming) sphere* with *centre* Y and *radius* r.

Note that $|\mathcal{S}(Y, r)| = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{r}$.

By the triangle inequality, $\Delta(\mathcal{S}(Y, r)) \leq 2r$ holds with equality if $2r \leq n$.

Let X be the disjoint union of Y, Z and the one-element set $\{x\}$.

Definition 2.2. For $0 \leq r < n$ let us define the *double-sphere* $\mathcal{S}_x(Y, r) = \big\{A \subset X : \big|(Y + A) \setminus \{x\}\big| \leq r\big\}$.

The name double-sphere is justified by

$$\mathcal{S}_x(Y, r) = \mathcal{S}(Y, r) \cup \mathcal{S}(Y \cup \{x\}, r). \tag{2.8}$$

The diameter of $\mathcal{S}_x(Y, r)$ is easily seen to be equal to the minimum of $2r + 1$ and n.

In the next section we shall consider the problem of determining or estimating the maximum size of an antichain in spheres and double-spheres. This problem is of some independent interest (cf. Theorem 3.2, Proposition 3.1).

The reason why these objects play a major role in proving the main theorem is explained by the following very recent stability result for the Kleitman Diameter Theorem.

Theorem 2.3 ([F4])**.** *For $d < n-1$ a fixed integer let $\mathcal{G} \subset 2^X$ be a family of diameter at most d. Then either* (i) *or* (ii) *holds.*

(i) $d = 2r$. *If $\mathcal{G}$ is not contained in any sphere $\mathcal{S}(Y, r)$ then*

$$|\mathcal{G}| \leq \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{r} - \binom{n-r-1}{r} + 1. \tag{2.9}$$

(ii) $d = 2r+1$. *If $\mathcal{G}$ is not contained in any double-sphere $\mathcal{S}_x(Y, r)$ then*

$$|\mathcal{G}| \leq 2\left(\binom{n-1}{0} + \binom{n-1}{1} + \ldots + \binom{n-1}{r}\right) - \binom{n-r-2}{r} + 1. \tag{2.10}$$

Let us note that for $n \geq d+2$ (i) and (ii) imply that the spheres and double-spheres are the only families achieving equality in the Kleitman Diameter Theorem (for $d = n-1$ there are many more optimal families and the RHS of (2.9) and (2.10) reduces to Kleitman's bound).

Let us also mention that for fixed r and $n \to \infty$ the RHS of (2.9) is $O(n^{r-1})$. For the odd case the RHS of (2.10) is $(1+o(1))\binom{n}{r}$ although it is somewhat larger than the bound given in Theorem 2.2. It is important to mention that in general the RHS of (2.10) is larger than $\binom{n}{r}$. This is the reason that our proof of the Main Theorem for d odd is much more involved than the case when d is even.

In the proof of (1.5) we shall use the following results as well.

Proposition 2.2 ([F2])**.** *Let $\mathcal{H} \subset \binom{[n]}{a}$, $a > 0$. Then for every positive integer s such that $n \geq sa$, the condition $|\mathcal{H}| > s\binom{n-1}{a-1}$ implies the existence of $s+1$ pairwise disjoint sets in $\mathcal{H}$.*

If $F \cap F' \neq \emptyset$ for all $F, F' \in \mathcal{F}$ then $\mathcal{F}$ is called *intersecting.*

A family $\mathcal{H}$ is said to be *ℓ-uniform* if $|H| = \ell$ for all $H \in \mathcal{H}$.

Definition 2.3. The *immediate shadow* $\sigma(\mathcal{H})$ of an ℓ-uniform family $\mathcal{H}$ is defined as

$$\sigma(\mathcal{H}) \stackrel{\text{def}}{=} \{G : |G| = \ell - 1,\ \exists H \in \mathcal{H},\ \text{satisfying } G \subset H\} = \bigcup_{H \in \mathcal{H}} \binom{H}{\ell-1}.$$

Let us recall the following special case of a fundamental result of Katona.

Proposition 2.3 (Katona Intersection Shadow Theorem [Ka]). *Let $\mathcal{H}$ be ℓ-uniform and intersecting. Then $|\sigma(\mathcal{H})| \geq |\mathcal{H}|$. Moreover, equality holds iff $\mathcal{H} = \binom{Q}{\ell}$ for some $(2\ell-1)$-element set Q.*

Proposition 2.4 (LYM Inequality (see [En])). *If $\mathcal{G} \subset 2^{[n]}$ is an antichain then $\sum\limits_{G \in \mathcal{G}} \frac{1}{\binom{n}{|G|}} \leq 1$ holds.*

We only use the following immediate corollary.

Corollary 2.3. *Let n, r be positive integers, $n \geq 2r$. If $\mathcal{G} \subset 2^{[n]}$ is an antichain satisfying $|G| \leq r$ for all $G \in \mathcal{G}$ then*

$$|\mathcal{G}| \leq \binom{n}{r} \text{ with equality holding if and only if } \mathcal{G} = \binom{[n]}{r}. \tag{2.11}$$

For a family $\mathcal{F} \subset 2^{[n]}$ and an element $i \in [n]$ define $\mathcal{F}(\bar{i}) = \{F \in \mathcal{F} : i \notin \mathcal{F}\}$.

Theorem 2.4. *If $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting and $n \geq 6k^2$, then there exists an element $x \in [n]$ such that*

$$\big|\mathcal{F}(\bar{x})\big| \leq \binom{n-3}{k-2} \quad \textit{holds.} \tag{2.12}$$

This theorem tells us that by removing at most $\binom{n-3}{k-2}$ members of an intersecting family, we can make it trivial. Indeed, $x \in F$ holds for all F in $\mathcal{F} \setminus \mathcal{F}(\bar{x})$.

Consider the following family: $\mathcal{T} = \Big\{T \in \binom{[n]}{k} : \big|T \cap \{1,2,3\}\big| \geq 2\Big\}$. Then for $x \in \{1,2,3\}$ one has $|\mathcal{T}(\bar{x})| = \binom{n-3}{k-2}$ showing that (2.12) is best possible.

Note that $|\mathcal{T}| = 3\binom{n-3}{k-2} + \binom{n-3}{k-3} = \binom{n-2}{k-2} + 2\binom{n-3}{k-3}$.

In [F3] (2.12) was proved for $n > 2k$ under the additional assumption $|\mathcal{F}| \geq |\mathcal{T}|$. Using this (2.12) follows for $n > n_0(k)$. Lemons and Palmer [LP] established (2.12) for $n > 6k^3$ using a different approach. The bound $n \geq 6k^2$ that we prove in Section 6 is stronger but far from optimal. We believe that (2.12) holds for $n > 3k$.

3 Antichains in spheres and double-spheres

For a set Q let $\Sigma(Q)$ denote the full symmetric group acting on Q, $|\Sigma(Q)| = |Q|!$

We view the Hamming sphere $\mathcal{S}(Y, r)$ as a partially ordered set (poset for short). The partial order $\prec$ is defined by $A \prec B$ iff $A \subset B$ holds.

Given a partition $X = Y \dot{\cup} Z$ and an integer $1 \le r \le n$ the direct sum of $\Sigma(Y)$ and $\Sigma(Z)$ is acting naturally on the sphere $\mathcal{S}(Y, r)$. For $0 \le a \le |Y|$, $0 \le b \le |Z|$ and $a + b \le r$ let us define the set

$$\mathcal{D}(a, b) = \left\{ A \cup B : A \in \binom{Y}{|Y| - a},\ B \in \binom{Z}{b} \right\}.$$

It is easily seen that the $\mathcal{D}(a, b)$ are the orbits of $\mathcal{S}(Y, r)$ under the action of $\Sigma(Y) \oplus \Sigma(Z)$.

Let us recall the following classical result.

Theorem 3.1 (Kleitman, Edelberg, Lubell [KEL])**.** *Let P be a partially ordered set and Γ a group of order-preserving automorphisms acting on P. Then there exists an antichain of maximum size in P that is the union of full orbits of Γ.*

Let us use this result to estimate the size of maximal antichains in $\mathcal{S}(Y, r)$. Fix an antichain $\mathcal{A} \subset \mathcal{S}(Y, r)$ which is the union of orbits.

First we define a poset $P = P(Y, r)$ whose vertices are the orbits $\mathcal{D}(a, b)$. The relation $\prec$ is defined by $\mathcal{D}(a, b) \prec \mathcal{D}(a', b')$ iff there exist $D \in \mathcal{D}(a, b)$ and $D' \in \mathcal{D}(a', b')$ such that $D \subseteq D'$ holds. It is easy to see that this is equivalent to $a \ge a'$ and $b \le b'$ holding simultaneously.

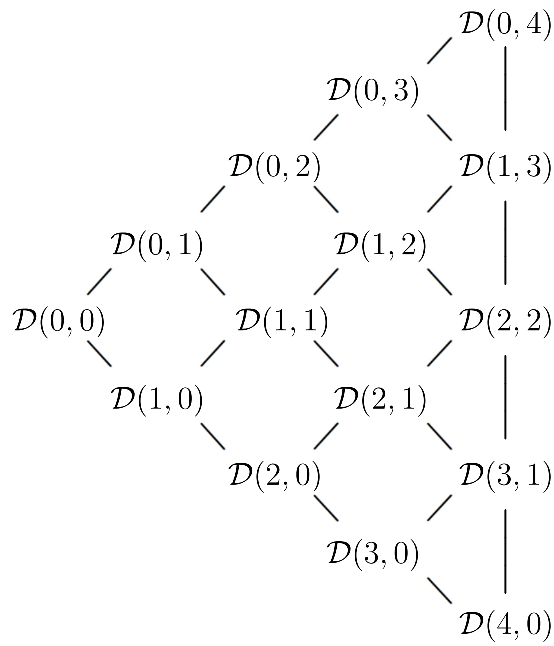

Figure 1. The Hasse diagram of P for $r = 4$

Observation 3.1. *If $\big|(a+b) - (a'+b')\big| \leq 1$ then $\mathcal{D}(a,b)$ and $\mathcal{D}(a',b')$ are comparable in P.*

Proof. If $a = a'$ and $b = b'$, the statement is trivial. By symmetry suppose $a > a'$ that is $a \geq a' + 1$. If $b \leq b'$, we are done. On the other hand, if $b \geq b' + 1$ then $(a+b) - (a'+b') \geq 1 + 1 = 2$, a contradiction. $\square$

Note that the sizes of the orbits $\mathcal{D}(a,b)$ satisfy

$$|\mathcal{D}(a,b)| = \binom{|Y|}{a}\binom{|Z|}{b}, \tag{3.1}$$

$$\sum_{a+b=\ell} |\mathcal{D}(a,b)| = \binom{n}{\ell} \quad \text{for all} \quad 0 \leq \ell \leq r. \tag{3.2}$$

Theorem 3.2. *Let $[n] = Y \dot\cup Z$ with $n > 2r$ and $\mathcal{A} \subset \mathcal{S}(Y,r)$ be an antichain of maximum size that is the union of orbits. Then there are two decreasing sequences of integers $a_1 > \ldots > a_\ell \geq 0$, $b_1 > \ldots > b_\ell \geq 0$ such that $\mathcal{A} = \mathcal{D}(a_1,b_1) \cup \ldots \cup \mathcal{D}(a_\ell,b_\ell)$ and for every $1 \leq i < \ell$ at least one of $a_i - a_{i+1} = 1$ and $b_i - b_{i+1} = 1$ holds. Also $a_1 + b_1 = r$.*

Proof. Suppose that $\mathcal{A} = \mathcal{D}(a_1, b_1) \cup \ldots \cup \mathcal{D}(a_\ell, b_\ell)$. Without loss of generality we may assume that $a_1 \geq \ldots \geq a_\ell$ holds. Using that the $\mathcal{D}(a_i, b_i)$ are not in containment relation we infer that the inequalities are strict. For the same reason $b_1 > \ldots > b_\ell$ must hold.

If $a_i > a_{i+1} + 1$ and $b_i > b_{i+1} + 1$ then one can add $\mathcal{D}(a_i - 1, b_i - 1)$ to $\mathcal{A}$, contradicting the maximality of $|\mathcal{A}|$. Suppose now that $|\mathcal{A}|$ is maximal but $a_1 + b_1 < r$. Since $n = |Y| + |Z| > 2r$ either $|Y| > 2a_1 + 1$ or $|Z| > 2b_1 + 1$. Indeed, the opposite leads to

$$n = |Y| + |Z| \leq 2a_1 + 1 + 2b_1 + 1 = 2(a_1 + b_1 + 1) \leq 2r,$$

contradicting $n > 2r$.

Suppose by symmetry that $|Z| > 2b_1 + 1$. Then $\binom{|Z|}{b_1+1} \Big/ \binom{|Z|}{b_1} = \frac{|Z|-b_1}{b_1+1} > 1$ and thus $\left|\mathcal{D}(a_1, b_1 + 1)\right| > \left|\mathcal{D}(a_1, b_1)\right|$ follows.

Replacing $\mathcal{D}(a_1, b_1)$ by $\mathcal{D}(a_1, b_1+1)$ contradicts the maximality of $|\mathcal{A}|$. $\square$

Suppose that $|Z| \geq |Y|$ and note the obvious inequality $\binom{q}{1} \geq \binom{q}{0}$ for $q \geq 1$. Then one can look at Figure 1 and use Theorem 3.2 to prove the following:

Corollary 3.1. *Let $|Z| > |Y|$. For $r = 2, 3, 4$ the only candidates for maximal antichains are the following.*

(i) $r = 2$, $\mathcal{D}(0, 2)$ *and* $\mathcal{D}(1, 1) \cup \mathcal{D}(0, 0)$*;*
(ii) $r = 3$, $\mathcal{D}(0, 3)$ *and* $\mathcal{D}(1, 2) \cup \mathcal{D}(0, 1)$*;*
(iii) $r = 4$, $\mathcal{D}(0, 4)$*;* $\mathcal{D}(1, 3) \cup \mathcal{D}(0, 2)$ *and* $\mathcal{D}(2, 2) \cup \mathcal{D}(1, 1) \cup \mathcal{D}(0, 0)$*.*

Assuming $|Z| \geq |Y| \geq 2r$ one can strengthen Theorem 3.2.

Proposition 3.1. *Suppose that $|Z| \geq |Y| \geq 2r$ and $\mathcal{A} \subset \mathcal{S}(Y, r)$ is a maximal antichain which is the union of orbits. Then there exist $0 \leq a \leq b \leq r$ such that $a + b = r$ and*

$$\mathcal{A} = \mathcal{D}(a, b) \cup \mathcal{D}(a-1, b-1) \cup \ldots \cup \mathcal{D}(0, b-a) \quad \textit{hold.}$$

Proof. In view of Theorem 3.2 we only have to show that for every $1 \leq i < \ell$ one has $a_{i+1} = a_i - 1$, $b_{i+1} = b_i - 1$. Note that for $m \geq 2r$ the inequality $\binom{m}{r-1} > \binom{m}{r-2} > \ldots > \binom{m}{0}$ holds. If, say, $a_{i+1} < a_i - 1$, then we can replace $\mathcal{A}$ by $\big(\mathcal{A} - \mathcal{D}(a_{i+1}, b_{i+1})\big) \cup \mathcal{D}(a_i - 1, b_{i+1})$ and increase $|\mathcal{A}|$, a contradiction. $\square$

Let us use Theorem 3.2 to show that $\binom{n}{r}$ is an upper bound for $|\mathcal{A}|$ for all antichains $\mathcal{A} \subset \mathcal{S}(Y, r)$ with equality possible only for $Y = \emptyset$, $\mathcal{A} = \binom{[n]}{r}$.

Theorem 3.3. *Let $[n] = Y \dot\cup Z$ be a partition and let r be a positive integer such that $n > 2r$. Suppose that $\mathcal{A} \subset \mathcal{S}(Y,r)$ is an antichain. Then $|\mathcal{A}| \le \binom{n}{r}$ holds with equality if and only if $Y = \emptyset$ and $\mathcal{A} = \binom{[n]}{r}$ or $Z = \emptyset$ and $\mathcal{A} = \binom{[n]}{n-r}$.*

Proof. (The reader is kindly advised to look at Figure 1.) Using Theorem 3.2 suppose that $\mathcal{A} = \mathcal{D}(a_1,b_1) \cup \ldots \cup \mathcal{D}(a_j,b_j)$, $a_1 + b_1 = r$, $a_1 > \ldots > a_j$ and $b_1 > \ldots > b_j$.

For $2 \le i \le j$ consider the two orbits $\mathcal{D}(a_i, r-a_i)$ and $\mathcal{D}(r-b_i, b_i)$. We claim that together with $\mathcal{D}(a_1,b_1)$ these $2\ell - 1$ orbits are all *distinct*. Using that the a_i and the b_i are all distinct, the only possibility for coincidence is $\mathcal{D}(a_i, r-a_i) = \mathcal{D}(r-b_{i'}, b_{i'})$. However, $a_i = r - b_{i'}$ implies $r = a_i + b_{i'} < a_1 + b_1 = r$, a contradiction. Note the inequality $\binom{q}{a} < \binom{q}{b}$, valid for all nonnegative integers a, b, q satisfying $a < b$ and $a + b < q$.

We claim that for every $2 \le i \le j$ either $a_i + r - b_i < |Y|$ or $r - a_i + b_i < |Z|$ holds. Indeed, the opposite would lead to $2r = r - a_i + b_i + a_i + r - b_i \ge |Y| + |Z| > 2r$, a contradiction.

Suppose by symmetry that $|Z| > r - a_i + b_i$ holds. Then $a_i + b_i < r$ implies $r - a_i > b_i$ and thus $\binom{|Z|}{r-a_i} > \binom{|Z|}{b_i}$. Consequently, $\big|\mathcal{D}(a_i, r-a_i)\big| > \big|\mathcal{D}(a_i,b_i)\big|$ follows.

Using (3.2) we infer

$$\binom{n}{r} = \sum_{a+b=r} |\mathcal{D}(a,b)| \ge \sum_{1\le i\le j} \big|\mathcal{D}(a_i,b_i)\big| = |\mathcal{A}|, \quad \text{as desired.}$$

In case of equality, $j = 1$ and $\big|\mathcal{D}(a_1,b_1)\big|$ is the only term in (3.2). Thus $Y = \emptyset$ or $Z = \emptyset$. In the first case $\mathcal{A} = \binom{[n]}{r}$, in the second $\mathcal{A} = \binom{[n]}{n-r}$ follow from (2.11). $\square$

Using (3.1) an easy calculation shows that among the $\mathcal{D}(a, r-a)$, $0 \le a \le r$,

$\mathcal{D}(r,0)$ has the largest size if $n > (r+1)(|Z|+1)$,

$\mathcal{D}(0,r)$ has the largest size if $n > (r+1)(|Y|+1)$.

Let us prove a general estimate.

Proposition 3.2. *Let $0 < a < r$. Then the following holds.*

(i) $r = 2$ $\quad \big|\mathcal{D}(1,1)\big| \le \frac{n^2}{4}$;

(ii) $r = 3$ $\quad \big|\mathcal{D}(a, r-a)\big| \le \frac{4}{9} \cdot \frac{n^3}{6}$;

(iii) $r \ge 4$ $\quad \big|\mathcal{D}(a, r-a)\big| \le \left(\frac{r-1}{r}\right)^{r-1} \cdot \frac{n^r}{r!} \le \frac{27}{64} \cdot \frac{n^r}{r!}$.

Proof. Note that (i) is trivial from $|Y|+|Z|=n$ and $|Y|\cdot|Z| \le \left(\frac{|Y|+|Z|}{2}\right)^2$.

For (ii) and (iii) assume by symmetry that $|Y| \le |Z|$. Then $\big|\mathcal{D}(a,r-a)\big| \ge \big|\mathcal{D}(r-a,a)\big|$ holds for $a \le r-a$. Therefore we assume $a \le r/2$.

$$\big|\mathcal{D}(1,2)\big| = |Y|\binom{n-|Y|}{2} < \frac{|Y||n-|Y||^2}{2} = \frac{2|Y|\times(n-|Y|)\times(n-|Y|)}{4}.$$

By the inequality between arithmetic and geometric mean the enumerator is at most $\left(\frac{2n}{3}\right)^3$ yielding

$$\big|\mathcal{D}(1,2)\big| \le \frac{8}{27}n^3/4 = \frac{4}{9}\cdot\frac{n^3}{6} \quad \text{proving (ii).}$$

(iii) $\big|\mathcal{D}(1,r-1)\big| \le \left(\frac{r-1}{r}\right)^{r-1}\frac{n^r}{r!}$ is proved in the same way as (ii).

In the general case we have in basically the same way $\big|\mathcal{D}(a,r-a)\big| < \frac{|Y|^a|n-|Y||^{r-a}}{a!(r-a)!}$ and the RHS is maximal for $|Y| = \frac{a}{r}\cdot n$. Plugging in this value we obtain

$$\frac{n^r}{r^r}\left(\frac{a^a\cdot(r-a)^{r-a}}{a!(r-a)!}\right).$$

Let us denote the term in the bracket by $f(a)$. We are going to conclude the proof by showing that $f(a-1) > f(a)$. Indeed, once we know this we have the chain of inequalities $f(1) > f(2) > \ldots > f(a)$ and (iii) is reduced to the case of $\big|\mathcal{D}(1,r-1)\big|$ which we showed above.

Let us recall the well-known fact that $\left(1-\frac{1}{d+1}\right)^d$ is a monotone decreasing function of the positive integer d.

$$\frac{a^a(r-a)^{r-a}}{a!(r-a)!} < \frac{(a-1)^{a-1}(r-a+1)^{r-a+1}}{(a-1)!(r-a+1)!}$$

is equivalent to $\left(1-\frac{1}{r-a+1}\right)^{r-a} < \left(1-\frac{1}{a}\right)^{a-1}$ which follows from $a \le \frac{r}{2}$. $\square$

Let now $X = Y\dot\cup Z\dot\cup\{x\}$ and consider the double-sphere $\mathcal{S}_x(Y,r)$. Along with every orbit $\mathcal{D}(a,b)$ we get also a new orbit $\mathcal{D}_x(a,b)$:

$$\mathcal{D}_x(a,b) = \big\{D\cup\{x\} : D\in\mathcal{D}(a,b)\big\}.$$

Now $\mathcal{D}(a,b) \prec \mathcal{D}_x(a',b')$ iff $\mathcal{D}(a,b) \prec \mathcal{D}(a',b')$ holds. Let $\mathcal{B} \subset \mathcal{S}_x(Y,r)$ be a maximal antichain that is the union of full orbits. For the case $r=1$ the

Hasse diagram consists of two chains (cf. Figure 2).

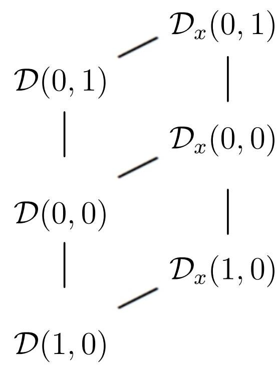

Figure 2

Since $|\mathcal{D}(0,1)| = |Z|$, $|\mathcal{D}(1,0)| = |Y|$ and $n = |Y| + |Z| + 1$, one easily deduces the next statement.

Claim 3.1. *Let $\mathcal{B} \subset \mathcal{S}_x(Y,1)$ be an antichain. For $Y \neq \emptyset \neq Z$, $|\mathcal{B}| \leq n-1$ holds.*

Let us prove also the following.

Proposition 3.3. *Let $\mathcal{B} \subset \mathcal{S}_x(Y,r)$ be an antichain. If $Y = \emptyset$ then*

$$|\mathcal{B}| \leq \binom{n}{r} \tag{3.3}$$

with equality holding only if $\mathcal{B} = \binom{X}{r}$.

Proof. Note that $\mathcal{S}_x(\emptyset, r) \setminus \mathcal{D}_x(0,r)$ has rank r, i.e., it contains no members of size exceeding r. Thus the statement is evident both by Sperner's original argument and the LYM inequality (2.11) if $\mathcal{B} \cap \mathcal{D}_x(0,r) = \emptyset$.

Suppose next that $\mathcal{B}_x \stackrel{\text{def}}{=} \mathcal{B} \cap \mathcal{D}_x(0,r)$ is non-empty. Setting $\mathcal{B}_0 = \big\{B - \{x\} : B \in \mathcal{B}_x\big\}$, it satisfies $|\mathcal{B}_0| = |\mathcal{B}_x|$. Define the immediate shadow:

$$\mathcal{C} = \left\{ C \in \binom{X}{r} : \exists B \in \mathcal{B}_x,\ C \subset B \right\}.$$

Then $\mathcal{B}_0$ is a proper subset of $\mathcal{C}$. Consequently, the antichain $(\mathcal{B} \setminus \mathcal{B}_x) \cup \mathcal{C}$ has size more than $\mathcal{B}$ and it is contained in $\mathcal{S}_x(\emptyset, r) \setminus \mathcal{D}_x(0,r)$. Now the statement follows. $\square$

Let us recall the notation $\mathcal{F}^c = \{X - F : F \in \mathcal{F}\}$. Then $\mathcal{S}_x(Y,r)^c = \mathcal{S}_x(Z,r)$ holds for $X = Y \dot\cup Z \dot\cup \{x\}$. Thus Proposition 3.3 holds for $Z = \emptyset$ as well.

Let us use an easy probabilistic argument to show the following estimate.

Lemma 3.1. *If $n-1 > 4r \cdot |Y|$ then for every $0 \le i \le r$*

$$|\mathcal{D}(0,i)| > \frac{3}{4}\binom{n-1}{i} \quad \textit{holds.} \tag{3.4}$$

Proof. Let v be a random element of $X - \{x\}$, chosen with uniform distribution. Then $P(v \in Y)$ is $|Y|/(n-1)$. This shows that for a random i-subset $R \subset X - \{x\}$ the expected size of $R \cap Y$ is less than $1/4$ for $i \le r$. Since this expected number is equal to $\sum\limits_{1 \le j \le i} j|\mathcal{D}(j, i-j)|/\binom{n-1}{i}$, (3.4) follows from (3.2) and Markov's inequality. □

Corollary 3.2. *If $n - 1 > 4r|Y|$ and $\mathcal{B} \subset \mathcal{S}_x(Y,r)$ is an antichain then*

$$|\mathcal{B}| \le \binom{n-1-|Y|}{r} + |Y| \cdot \binom{n-1-|Y|}{r-1} + \binom{n-1-|Y|}{r-2}. \tag{3.5}$$

Proof. By Theorem 3.1 we may assume that $\mathcal{B}$ is the union of some of the orbits $\mathcal{D}(a,b)$ and $\mathcal{D}_x(a',b')$. In view of (3.4) for every i the orbits $\mathcal{D}(0,i)$ and $\mathcal{D}_x(0,i)$ have equal size and dominate completely the remaining $\mathcal{D}(a,b)$, $\mathcal{D}_x(a,b)$ with $a+b=i$.

Consequently, to maximize $|\mathcal{B}|$ it has to contain either $\mathcal{D}(0,r)$ or $\mathcal{D}_x(0,r)$. In case of the latter $\mathcal{B} = \mathcal{D}_x(0,r)$. On the other hand, if $\mathcal{D}(0,r) \subset \mathcal{B}$ then to maximize $|\mathcal{B}|$ we have two choices. Namely, $\mathcal{D}_x(0,r-1)$ and $\mathcal{D}_x(1,r-1)$. The size of the latter is larger by a factor of $|Y|$. Finally, one can add $\mathcal{D}_x(0,r-2)$. □

Claim 3.2. *Suppose that $|Y| \ge 2$. Then the RHS of* (3.5) *is at most* $\binom{n-1}{r}$.

Proof. Indeed

$$\binom{n-1}{r} - \binom{n-1-|Y|}{r} = \binom{n-2}{r-1} + \ldots + \binom{n-|Y|}{r-1} + \binom{n-|Y|-1}{r-1}.$$

There are $|Y|$ terms on the RHS, $\binom{n-|Y|-1}{r-1}$ being the smallest and $\binom{n-|Y|}{r-1} = \binom{n-|Y|-1}{r-1} + \binom{n-|Y|-1}{r-2}$. □

Let us close this section with a simple result.

Claim 3.3. *Let* $[n] = Y\dot{\cup}Z\dot{\cup}\{x\}$*,* $|Y| \le |Z|$ *and let* $\mathcal{G} \subset \mathcal{S}_x(Y,2)$ *be an antichain. Then*

$$|\mathcal{G}| \le \big|\mathcal{D}(0,2)\big| + \big|\mathcal{D}_x(1,1)\big| + \big|\mathcal{D}_x(0,0)\big| = \binom{n-1}{2} - \binom{|Y|}{2} + 1. \tag{3.6}$$

Proof. Because $|Y| \le |Z|$, $\big|\mathcal{D}(2,0)\big|$ is the smallest among the three orbits $\mathcal{D}(a,b)$ with $a+b=2$. Thus to maximize $|\mathcal{G}|$ we should choose the other two. Taking into consideration $\mathcal{D}(1,1) \prec \mathcal{D}_x(0,2)$, the best choice is $\mathcal{D}_x(1,1)$ and $\mathcal{D}(0,2)$. In view of Corollary 3.1 (i) to these we can only add $\mathcal{D}_x(0,0)$ proving (3.6). □

4 The even case and some preparation

Suppose that $\mathcal{A} \subset 2^{[n]}$ is an antichain satisfying $\Delta(\mathcal{A}) \le 2r$. Let us apply Theorem 2.3 to $\mathcal{A}$. If $\mathcal{A}$ is not contained in a sphere $\mathcal{S}(Y,r)$ then

$$|\mathcal{A}| \le \sum_{0\le i\le r}\binom{n}{i} - \binom{n-r-1}{r} + 1.$$

We can rewrite the RHS as

$$\binom{n}{0} + \ldots + \binom{n}{r-1} + \binom{n-1}{r-1} + \binom{n-2}{r-1} + \ldots + \binom{n-r-1}{r-1} + 1.$$

For, e.g., $n > 3r$, $r \ge 2$ it is smaller than $(r+2)\binom{n-1}{r-1}$.

Using $\binom{n-1}{r-1} = \frac{r}{n}\binom{n}{r}$, $|\mathcal{A}| < \binom{n}{r}$ follows for $n > r(r+2)$, as desired.

Thus we may assume that $\mathcal{A} \subset \mathcal{S}(Y,r)$ holds for some $Y \subset [n]$.

Noting that $\mathcal{A}^c = \{[n] - A : A \in \mathcal{A}\}$ is also an antichain, $\mathcal{A}^c \subset \mathcal{S}([n] - Y, r)$, we may assume that $|Y| \le n/2$.

Let us suppose that $n \ge 4r$ and apply Theorem 3.3. Then $|\mathcal{A}| < \binom{n}{r}$ follows unless $Y = \emptyset$. Moreover, in case of equality $\mathcal{A} = \binom{[n]}{r}$ holds. Thus we have proved $m(n,2r) = \binom{n}{r}$ for $n > n_0(r) = r(r+2)$.

To prove $m(n,2r+1) = \binom{n}{r}$ with a somewhat larger $n_0(r)$ we need some preparation.

Proposition 4.1. *Suppose that* $\mathcal{A} \subset \mathcal{S}(Y,r)$ *for some* $Y \subset [n]$*,* $|\mathcal{A}| \ge \frac{1}{2}\binom{n}{r}$ *and* $n > 2r(r+5)$*, then there exist* $B_0, \ldots, B_{r+3} \in \mathcal{A}$ *such that* $F_i = B_i + Y$*,* $0 \le i \le r+3$*, are pairwise disjoint* r*-element sets.*

Proof. There are $\binom{n}{r-1}+\binom{n}{r-2}+\ldots+\binom{n}{0}$ sets B in $\mathcal{S}(Y,r)$ such that $|B+Y|<r$. Let us omit these sets from $\mathcal{A}$. That is, we consider $\mathcal{A}\setminus\mathcal{S}(Y,r-1)$. Let us remark that for $n\geq 4r$ one has $\binom{n}{r-1}<\frac{4}{3}\binom{n-1}{r-1}$ and $\binom{n}{r-i}/\binom{n}{r-1}<3^{-i+1}$ for $i=2,\ldots,r$ implying $|\mathcal{S}(Y,r-1)|<\frac{3}{2}\binom{n}{r-1}<2\binom{n-1}{r-1}$. Thus

$$\big|\mathcal{A}\setminus\mathcal{S}(Y,r-1)\big|\geq\frac{1}{2}\binom{n}{r}-2\binom{n-1}{r-1}. \tag{4.1}$$

Define $\mathcal{F}=\big\{B+Y : B\in\mathcal{A}\setminus\mathcal{S}(Y,r-1)\big\}\subset\binom{[n]}{r}$. In view of Proposition 2.2, $|\mathcal{F}|>(r+3)\binom{n-1}{r-1}$ is sufficient to guarantee the existence of $r+4$ pairwise disjoint members in $\mathcal{F}$. On the other hand the opposite and (4.1) would imply

$$|\mathcal{A}|\leq(r+5)\binom{n-1}{r-1}=\frac{(r+5)r}{n}\binom{n}{r}$$

which contradicts

$$|\mathcal{A}|\geq\frac{1}{2}\binom{n}{r}\quad\text{for}\quad n>2r(r+5).$$

□

Let us show also that $|\mathcal{A}|\geq\frac{1}{2}\binom{n}{r}$ is sufficient in the above range to imply via Theorem 2.3 that $\mathcal{A}\subset\mathcal{S}(Y,r)$ for some Y. That is, we need to prove the following inequality.

Claim 4.1. *If $n>2r(r+5)$ then*

$$\frac{1}{2}\binom{n}{r}>\left(\binom{n}{0}+\ldots+\binom{n}{r-1}\right)+\left(\binom{n}{r}-\binom{n-r-1}{r}\right)+1. \tag{4.2}$$

Proof. The term in the first bracket is less than $2\binom{n-1}{r-1}$ as we showed above (just before (4.1)). Let us rewrite and estimate the expression in the second bracket:

$$\binom{n}{r}-\binom{n-r-1}{r}=\binom{n-1}{r-1}+\binom{n-2}{r-1}+\ldots+\binom{n-r-1}{r-1}\leq(r+1)\binom{n-1}{r-1}.$$

Using generously the inequality $1\leq\binom{n-1}{r-1}$ we still obtain that the RHS is less than $(r+4)\binom{n-1}{r-1}=\frac{r(r+4)}{n}\binom{n}{r}$ and (4.2) follows. □

5 Proof of the Main Theorem

Let $\mathcal{A} \subset 2^{[n]}$ be an antichain with $\Delta(\mathcal{A}) = 2r+1$. We want to prove $|\mathcal{A}| \leq \binom{n}{r}$ and determine all optimal families. Throughout the whole proof we are going to assume that $|\mathcal{A}| \geq \binom{n}{r}$. Let us also assume $n > 2r(r+5)$ in order to be able to use Proposition 4.1 and Claim 4.1.

Let us first partition $\mathcal{A}$ into $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$ where $\mathcal{A}_i = \big\{A \in \mathcal{A} : |A| \equiv i \pmod 2\big\}$.

Note that $|A + A'|$ is always *even* for $A, A' \in \mathcal{A}_i$ forcing $\Delta(\mathcal{A}_i) \leq 2r$. Thus we may apply Theorem 2.3 separately to both $\mathcal{A}_0$ and $\mathcal{A}_1$.

Since $|\mathcal{A}| = |\mathcal{A}_0| + |\mathcal{A}_1| \geq \binom{n}{r}$ holds, we can choose the larger one, say $\mathcal{A}_\ell$ to satisfy $|\mathcal{A}_\ell| \geq \frac{1}{2}\binom{n}{r}$. By Theorem 2.3 and Claim 4.1 we infer $\mathcal{A}_\ell \subset \mathcal{S}(Y, r)$ for some $Y \subset [n]$.

Replacing if necessary $\mathcal{A}$ by the family of the complements, $\mathcal{A}^c$ we can assume that $|Y| \leq n/2$ holds. Using Proposition 4.1, we find that there exist $B_0, \ldots, B_{r+3} \in \mathcal{A}_\ell$ so that setting $A_i = B_i + Y$ the sets $A_0, A_1, \ldots, A_{r+3}$ are pairwise disjoint r-element sets.

Claim 5.1. *$\mathcal{A}_{1-\ell} \subset \mathcal{S}(Y, r+1)$ holds.*

Proof. Note that for any sets A, B, C, $(A+C) + (B+C) = A+B$. Take a member $B \in \mathcal{A}_{1-\ell}$ and define $A = B + Y$. We have to prove $|A| \leq r+1$. Suppose the contrary. Then by the parity condition $|A| \geq r+3$.

Now $|A + A_j| = |B + B_j| \leq 2r+1$ implies that $A \cap A_j \neq \emptyset$. Choose an element $x_j \in A \cap A_j$ for $j = 0, 1, \ldots, r+3$. Since the A_j are pairwise disjoint, $|A \setminus A_j| \geq r+4-1 = r+3$ follows. From $|A + A_j| \leq 2r+1$, we infer $|A_j \setminus A| \leq r-2$. That is, $|A \cap A_j| \geq 2$ for all j. However, this implies $|A \setminus A_0| \geq \sum\limits_{1 \leq j \leq r+3} |A \cap A_j| \geq 2r+6$, contradicting $|A + A_0| \leq 2r+1$. □

Claim 5.2. *If $B \in \mathcal{A}_{1-\ell}$ satisfies $|B \setminus Y| = r+1$ then $Y \subset B$ holds.*

Proof. This is evident from $|B + Y| \leq r+1$. □

Claim 5.3. *If B and $B' \in \mathcal{A}$ satisfy $|B + Y| = r+1$, $|B' + Y| = r+1$ then*

$$(B \setminus Y) \cap (B' \setminus Y) \neq \emptyset \quad \textit{holds.} \tag{5.1}$$

Proof. The opposite would mean

$$|B+B'| = |(B+Y)+(B'+Y)| = |B+Y| + |B'+Y| = 2(r+1) > 2r+1. \quad \square$$

Now it is easy to finish the proof for the case $Y = \emptyset$.

Note that $\mathcal{A} \subset \mathcal{S}(\emptyset, r+1)$ implies that $|A| \le r+1$ for all $A \in \mathcal{A}$. Define $\mathcal{A}^{(r+1)} = \{A \in \mathcal{A} : |A| = r+1\}$. If $\mathcal{A}^{(r+1)} = \emptyset$, we are done by (2.11). In view of Claim 5.3 the family $\mathcal{A}^{(r+1)}$ is intersecting.

Recall Definition 2.3 and set $\mathcal{B} = \sigma(\mathcal{A}^{(r+1)})$. Since $\mathcal{A}^{(r+1)}$ consists of the elements of maximum size of the antichain $\mathcal{A}$, the family $\mathcal{A}^* \overset{\text{def}}{=} (\mathcal{A} \setminus \mathcal{A}^{(r+1)}) \cup \mathcal{B}$ is an antichain. In view of (2.11), $|\mathcal{A}^*| \le \binom{n}{r}$ holds with equality iff $\mathcal{A}^* = \binom{[n]}{r}$. On the other hand, by Proposition 2.3 one has $|\mathcal{B}| \ge |\mathcal{A}^{(r+1)}|$ with equality iff for some $Q \in \binom{[n]}{2r+1}$ one has $\mathcal{A}^{(r+1)} = \binom{Q}{r+1}$. That shows that $|\mathcal{A}| \le |\mathcal{A}^*| \le \binom{n}{r}$ and in case of equality either $\mathcal{A}^{(r+1)} = \emptyset$, $\mathcal{A} = \binom{[n]}{r}$ or $\mathcal{A} = \binom{Q}{r+1} \cup \binom{[n]}{r} - \binom{Q}{r}$ as desired.

From now on let $|Y| \ne \emptyset$ hold.

Proposition 5.1. *Assume that $|\mathcal{A}| \ge \binom{n}{r}$ and $|\mathcal{A}|$ is maximal. Then $\mathcal{A} \cap \mathcal{D}(0, r+1) = \emptyset$.*

Proof. Note that $\mathcal{D}(0, r+1)$ consists of sets of the form $Y \cup D$ where $D \subset Z = [n] - Y$, $|D| = r+1$. These are the sets of largest size in $\mathcal{S}(Y, r+1)$.

Define $\mathcal{D} = \{D \in \binom{Z}{r+1} : Y \cup D \in \mathcal{A}\}$ and suppose that $\mathcal{D} \ne \emptyset$. We know that $\mathcal{D}$ is intersecting. Let $\mathcal{E}$ be the immediate shadow of $\mathcal{D}$, that is

$$\mathcal{E} = \left\{E \in \binom{Z}{r} : \exists\, D \in \mathcal{D},\ E \subset D\right\}.$$

Define now $\mathcal{E}^* = \{Y \cup E : E \in \mathcal{E}\}$. We claim that $\mathcal{E}^* \cup \mathcal{A} \setminus \mathcal{D}(0, r+1)$ is an antichain of diameter at most $2r+1$. Indeed it is an antichain because we replaced the sets of largest size in $\mathcal{A}$ by some of their subsets of size just 1 smaller.

The sets in $\mathcal{E}^*$ and $\mathcal{A}_\ell$ are in $\mathcal{S}(Y, r)$ which guarantees that there is no diameter problem there. The only pairs that we have to check are $E \cup Y$ for $E \in \mathcal{E}$ and $A \in \mathcal{A}_{1-\ell}$. Take $D \in \mathcal{D}$ with $E \subset D$. Then $D \cup Y \in \mathcal{A}_{1-\ell}$ forcing $|A + (D \cup Y)| \le 2r$. As $|D + E| = 1$, $|A + (E \cup Y)| \le 2r + 1$ follows.

If $|\mathcal{E}| > |\mathcal{D}|$ then we get a contradiction with the maximality of $|\mathcal{A}|$. Applying Proposition 2.3 we infer that $|\mathcal{D}| = |\mathcal{E}|$ and $\mathcal{D} = \binom{Q}{r+1}$ for some $Q \in \binom{Z}{2r+1}$.

$$\mathcal{S}(Y, r+1) = \mathcal{S}(Y, r) \cup \mathcal{D}(0, r+1) \cup \ldots \cup \mathcal{D}(r+1, 0).$$

Next we claim that $\mathcal{A} \setminus \mathcal{D}(0, r+1) \subset \mathcal{S}(Y, r)$. Indeed, if $A \in \mathcal{D}(i, r+1-i)$ for some $i \ge 1$ then $|A \cap Z| = r+1-i \le r$ implies the existence of some

$D \in \binom{Q}{r+1}$ with $A \cap D = \emptyset$. However, this implies $|A + (D \cup Y)| = 2(r+1)$, in contradiction with $\Delta(\mathcal{A}) \le 2r+1$. Now we can get the contradiction with $|\mathcal{A}| \ge \binom{n}{r}$ because $\mathcal{E}^* \cup \mathcal{A} \setminus \mathcal{D}(0, r+1) \subset \mathcal{S}(Y, r)$ is an antichain of exactly the same size but $Y \ne \emptyset$ and this contradicts Theorem 3.3. $\square$

Let us now conclude the proof for the case $Y = \{y\}$ and treat the case $|Y| > 1$ later. Since $|Y| = 1$, $\mathcal{A} \setminus \mathcal{S}(Y, r) \subset \mathcal{D}(1, r)$ follows from Proposition 5.1. Note that the members of $\mathcal{D}(1, r)$ have size r. Thus the only sets of maximal size are the $(r+1)$-element sets in $\mathcal{D}(0, r)$. If $\mathcal{D}_0 \overset{\text{def}}{=} \mathcal{A} \cap \mathcal{D}(0, r)$ is non-empty then its immediate shadow is strictly larger. Indeed, every $D \in \mathcal{D}_0$ contains the unique element y of Y. Thus $D - \{y\} \in \sigma(\mathcal{D}_0)$. So far this gives $|\sigma(\mathcal{D}_0)| \ge |\mathcal{D}_0|$. However, omitting some other element of D shows that the inequality is strict. Since $(\mathcal{A} \setminus \mathcal{D}_0) \cup \sigma(\mathcal{D}_0)$ is an antichain with no sets of size more than r, we get a contradiction with the maximality of $|\mathcal{A}|$.

The only way to escape the contradiction is $\mathcal{D}_0 = \mathcal{A} \cap \mathcal{D}(0, r) = \emptyset$. However, as we just stated above, in this case $|A| \le r$ holds for all $A \in \mathcal{A}$ and $|\mathcal{A}| \le \binom{n}{r}$ follows (together with the uniqueness) from (2.11). Now the proof of the case $|Y| = 1$ is complete.

Assume that $|Y| > 1$. Let us define $\mathcal{B} = \mathcal{A} \setminus \mathcal{S}(Y, r)$, then by Claim 5.1 $\mathcal{B}$ consists of the sets at distance exactly $r+1$ from Y. Define further $\mathcal{F} = \{B + Y : B \in \mathcal{B}\}$, the translate of $\mathcal{B}$ by Y. Since $\Delta(\mathcal{F}) = \Delta(\mathcal{B}) \le 2r+1$, $2(r+1) > 2r+1$ implies that the $(r+1)$-uniform family $\mathcal{F}$ is intersecting.

Let us apply Theorem 2.4 to $\mathcal{F}$ with $k = r+1$. Let $x \in [n]$ be the element satisfying

$$|\mathcal{F}(\bar{x})| \le \binom{n-3}{r-1}. \tag{5.2}$$

Note that $\mathcal{F} \setminus \mathcal{F}(\bar{x}) \subset \mathcal{S}(\{x\}, r)$. Consequently $\mathcal{B}_0 \overset{\text{def}}{=} \big(\mathcal{F} \setminus \mathcal{F}(\bar{x})\big) + Y$, which is a subfamily of $\mathcal{B}$, is contained in the sphere $\mathcal{S}(\{x\} + Y, r)$.

Now $\mathcal{S}(Y, r) \cup \mathcal{S}(\{x\} + Y, r)$ is the double-sphere $\mathcal{S}_x\big(Y \cap (\{x\} + Y), r\big)$.

Define $Y_0 = Y \cap (\{x\} + Y)$ and $\mathcal{A}_0 = \mathcal{A} \cap \mathcal{S}_x(Y, r)$. Since $Y_0 = Y$ or $Y_0 = Y \setminus \{x\}$, $|Y_0| \ge |Y| - 1$. By (5.2) we have

$$|\mathcal{A}_0| \ge |\mathcal{A}| - \binom{n-3}{r-1}. \tag{5.3}$$

Actually, this case is impossible. Namely, if $\mathcal{A} = \mathcal{D}(1, r) \cup \mathcal{D}(0, r-1) = \binom{[n]}{r}$ then $\mathcal{A}_{1-\ell} = \mathcal{D}(1, r)$ is larger than $\mathcal{A}_\ell = \mathcal{D}(0, r-1)$ which is contrary to our initial assumption.

Let us first treat the simple case $r = 1$. The intersecting family $\mathcal{F}$ is 2-uniform. Therefore either $\mathcal{F}$ is a star (and $\mathcal{F}(\bar{x}) = \emptyset$) or $\mathcal{F}$ consists of the three edges $\{x, x'\}, \{x, x''\}, \{x', x''\}$ (and $|\mathcal{F}(\bar{x})| = 1$).

In the first case $|\mathcal{A}| < n$ follows from Claim 3.1. Even in the second case $|\mathcal{A}| \leq n$ follows. We want to show that equality is impossible. Using Proposition 5.1 along with the fact that $\mathcal{A}$ is an antichain, $\{x, x', x''\} \subset Y$ follows. Thus $Y_0 = Y \setminus \{x\}$ and being an antichain implies that both $\mathcal{A} \cap \mathcal{D}(0,1)$ and $\mathcal{A} \cap \mathcal{D}_x(0,1)$ are empty. Consequently, the restriction $\mathcal{A}_{|Y} \stackrel{\text{def}}{=} \{A \cap Y : A \in \mathcal{A}\}$ is also an antichain. I.e., we can forget about Z and deduce

$$|\mathcal{A}| \leq |Y| < n, \quad \text{a contradiction.}$$

From now on let $r \geq 2$.

If $n > 4(r+1)|Y| + 1$ then Corollary 3.2 and Claim 3.2 imply $|\mathcal{A}_0| \leq \binom{n-|Y_0|}{r} \leq \binom{n-1}{r}$. Using (5.3) we infer $|\mathcal{A}| \leq \binom{n-1}{r} + \binom{n-3}{r-1} < \binom{n}{r}$ for $r \geq 2$.

For the remaining subcase, i.e., $\frac{n-1}{4r} \leq |Y| \leq \frac{n}{2}$ we only have to notice that (unless $r = 2$) adding the largest two terms among $\mathcal{D}(a, b)$ with $a + b = r$ is by Proposition 3.2 less than $\frac{8}{9} \frac{n^r}{r!}$.

Invoking Proposition 3.1 and using (5.3) it is sufficient to prove the following.

Claim 5.4. *For $n \geq 5(r+1)r$ one has*

$$\frac{8}{9}\frac{n^r}{r!} + \binom{n-3}{r-1} + \binom{n}{r-2} + \ldots + \binom{n}{1} + \binom{n}{0} < \binom{n}{r}. \tag{5.4}$$

Proof of the claim. First note that for $n \geq 10r$ we have

$$\binom{n}{i} \Big/ \binom{n}{i+1} = \frac{i+1}{n-i} < \frac{1}{9} \quad \text{for} \quad 0 \leq i < r-2.$$

Thus $\binom{n}{r-2} + \ldots + \binom{n}{0} < \frac{9}{8}\binom{n}{r-2}$.

Also, for $n \geq 10r$ one has

$$\binom{n-3}{r-2} \Big/ \binom{n}{r-2} > \left(\frac{9}{10}\right)^3 = 0.729.$$

Using $\binom{n-3}{r-1} + 2\binom{n-3}{r-2} + \binom{n-3}{r-3} = \binom{n-1}{r-1}$ we infer that the LHS of (5.4) is less than $\frac{8}{9}\frac{n^r}{r!} + \binom{n-1}{r-1}$. Consequently (5.4) would follow from $\frac{8}{9}\frac{n^r}{r!} < \binom{n-1}{r}$.

Equivalently,

$$\prod_{1\le i\le r}\left(1-\frac{i}{n}\right)>\frac{8}{9}.$$

Using $(1-a)(1-b)>(1-a-b)$, $1-\frac{\binom{r+1}{2}}{n}$ is a lower bound for the LHS. Since $n>9\binom{r+1}{2}$, (5.4) is proven. $\square$

The final case is $r=2$ and

$$\frac{n-1}{8}\le |Y|\le \frac{n}{2}.$$

In view of Claim 3.3 we have

$$|\mathcal{A}_0|\le \binom{n}{2}-\binom{|Y|}{2}+1.$$

Using (5.3), all we need is to prove

$$\binom{\lceil (n-1)/8\rceil}{2}>n-3+1.$$

The above inequality holds easily for $n>130$.

6 The proof of Theorem 2.4

For an arbitrary family $\mathcal{F}\subset 2^{[n]}$ let us define its *diversity*, $\delta(\mathcal{F})$ by

$$\delta(\mathcal{F})=\min_{x\in[n]}\Big\{\big|\mathcal{F}(\bar{x})\big|\Big\}.$$

Obviously, $\delta(\mathcal{F})=0$ iff $\bigcap\limits_{F\in\mathcal{F}}F\ne\emptyset$. That is, $\delta(\mathcal{F})$ measures how far $\mathcal{F}$ is from the trivial intersecting family, often called a star. The following inequality is obvious by the definition.

(6.1) $$\delta(\mathcal{F})\le\delta(\mathcal{G})\ \text{ whenever }\ \mathcal{F}\subset\mathcal{G}.$$

In view of (6.1), while proving (2.12) for an intersecting family $\mathcal{F}\subset\binom{[n]}{k}$ we can keep on adding newer and newer k-element sets as long as the new family is intersecting.

Let us say that $\mathcal{F} \subset \binom{[n]}{k}$ is *saturated* if $\mathcal{F}$ is intersecting but $\mathcal{F} \cup \{G\}$ is not intersecting for any $G \in \binom{[n]}{k} \setminus \mathcal{F}$. In view of the above, while proving (2.12) we can always assume that $\mathcal{F}$ is saturated.

For a saturated intersecting family $\mathcal{F} \subset 2^{[n]}$ a set T is *transversal* if $T \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. Since $\mathcal{F}$ is intersecting, all $F \in \mathcal{F}$ are transversals.

Define $\mathcal{K}(\mathcal{F}) \subset 2^{[n]}$ as the family of *minimal* transversals of $\mathcal{F}$, that is,

$$\mathcal{K}(\mathcal{F}) = \big\{K \subset [n] : K \text{ is a transversal of } \mathcal{F}, \text{ no } K_0 \subsetneqq K \text{ is a transversal of } \mathcal{F}\big\}.$$

Claim 6.1. *Let $n \geq 2k$. Then we have*

(i) *$\mathcal{K}(\mathcal{F})$ is intersecting.*

(ii) *For every $F \in \mathcal{F}$, $\exists\, K \in \mathcal{K}(\mathcal{F})$, $K \subset F$.*

Proof. Suppose that $K, L \in \mathcal{K}(\mathcal{F})$ and $K \cap L = \emptyset$. Since $n \geq 2k$ we can choose $G \supset K$, $F \supset L$ with $|G| = |F| = k$, $G \cap F = \emptyset$ and $G \cup F \subset [n]$. By the definition of $\mathcal{K}(\mathcal{F})$, both G and F are in $\mathcal{F}$. Now $F \cap G = \emptyset$ contradicts the assumption that $\mathcal{F}$ is intersecting.

As to (ii), the only k-set containing $F \in \mathcal{F}$ is F itself. That is F itself was a candidate for being in $\mathcal{K}(\mathcal{F})$. If $F \notin \mathcal{K}(\mathcal{F})$ then there is some proper subset of F which is in $\mathcal{K}(\mathcal{F})$. Thus (ii) follows. $\square$

Let $\tau(\mathcal{F}) = \big\{\min\{|K| : K \in \mathcal{K}(\mathcal{F})\}\big\}$ be the transversal number of $\mathcal{F}$. With this notation $\delta(\mathcal{F}) = 0$ iff $\tau(\mathcal{F}) = 1$. We may assume that $\tau(\mathcal{F}) \geq 2$.

For $2 \leq i \leq k$ define $\mathcal{K}_i = \{K \in \mathcal{K}(\mathcal{F}) : |K| = i\}$ and define also $t = t(\mathcal{F})$ by

$$t = \min\big\{i : \tau(\mathcal{K}_2 \cup \ldots \cup \mathcal{K}_i) \geq 2\big\}.$$

Note that $2 \leq t \leq k$.

Let us first prove Theorem 2.4 in the case $t = 2$, that is, $\tau(\mathcal{K}_2) \geq 2$.

In view of Claim 6.1 the family $\mathcal{K}_2$ is intersecting. Now $\tau(\mathcal{K}_2) \geq 2$ forces that $\mathcal{K}_2$ is formed by the three edges of a triangle, say $(1,2)$, $(1,3)$ and $(2,3)$. Using Claim 6.1 again we infer that

$$\mathcal{F} \subset \mathcal{F}^* \stackrel{\text{def}}{=} \left\{F \in \binom{[n]}{k} : |F \cap [3]| \geq 2\right\}.$$

Since $\mathcal{F}$ is saturated, $\mathcal{F} = \mathcal{F}^*$ follows. Now $\big|\mathcal{F}^*(1)\big| = \binom{n-3}{k-2}$ implies (2.12).

From now on $t \geq 3$. We want to fix an element $x \in X$. If $\mathcal{K}_2 \cup \ldots \cup \mathcal{K}_{t-1} \neq \emptyset$ then by definition we may choose x to satisfy $x \in K$ for all $K \in \mathcal{K}_2 \cup \ldots \cup \mathcal{K}_{t-1}$. In this case we fix such an x together with a set $K_0 \in \mathcal{K}_2 \cup \ldots \cup \mathcal{K}_{t-1}$.

If $\mathcal{K}_2 \cup \ldots \cup \mathcal{K}_{t-1} = \emptyset$ we simply fix $K_0 \in \mathcal{K}_t$ and let x be an arbitrary element of K_0.

Note that in both cases $\mathcal{K}_i(\bar{x}) = \emptyset$ for $2 \le i < t$. For $t \le i \le k$ define $k(i) = \big|\mathcal{K}_i(\bar{x})\big|$. From Claim 6.1 it follows that

$$|\mathcal{F}(\bar{x})| \le \sum_{t \le i \le k} k(i) \binom{n-i-1}{k-i}. \tag{6.2}$$

Next we define a branching process in order to prove an inequality involving $k(i)$, $t \le i \le k$.

Proposition 6.1.

$$\sum_{t \le i \le k} k^{k-i} k(i) \le (t-1)t \cdot k^{k-2}. \tag{6.3}$$

Proof. Set $s = |K_0 \setminus \{x_0\}|$, $s \le t-1$. Let $K_0 \setminus \{x\} = \left\{v_1^{(1)}, \ldots, v_s^{(1)}\right\}$ and let us start with the s sequences $(v_1^{(1)}), \ldots, (v_s^{(1)})$ of length 1. For each $v_j^{(1)}$ fix a set $K_j \in \mathcal{K}_2 \cup \ldots \cup \mathcal{K}_t$ not containing $v_j^{(1)}$. It is possible in view of $\tau(\mathcal{K}_2 \cup \ldots \cup \mathcal{K}_t) \ge 2$.

Next we "branch" $(v_j^{(1)})$ into at most $|K_j|$ sequences $(v_j^{(1)}, v_j^{(2)})$ where $v_j^{(2)} \in K_j \setminus \{x\}$. Note that in this way we construct not more than $|K_1| + \ldots + |K_s| \le (t-1)t$ sequences of length two.

Suppose now that we have constructed in the above way a system of at most $t(t-1)k^{j-2}$ sequences $(w_1, w_2, \ldots, w_j)$ of length j, $w_i \in [n] \setminus \{x\}$, $j \ge 2$. If $j = k$ then we stop. For $2 \le j < k$ we first prepare a dummy set $R_{j+1} = \{u_1, \ldots, u_k\}$. If $\{w_1, \ldots, w_j\} \cap F \ne \emptyset$ for all $F \in \mathcal{F}$ then we extend $(w_1, \ldots, w_j)$ into $(w_1, \ldots, w_j, u_q)$ in all possible k ways, $1 \le q \le k$.

If $\{w_1, \ldots, w_j\} \cap F_j = \emptyset$ for some $F_j \in \mathcal{F}$ then let $\{y_1, \ldots, y_p\} = F_j \setminus \{x\}$, $p = k$ or $k-1$. Extend $(w_1, \ldots, w_j)$ in all possible ways to $(w_1, \ldots, w_j, y_i)$, $1 \le i \le p$. This way we produce at most $t(t-1)k^{j-1}$ sequences of length $j+1$.

Once we reach a transversal, say of size i, then we extend it into k^{k-i} sequences using the dummy sets. Thus in order to prove (6.3) it is sufficient to show that every $K \in \mathcal{K}(\bar{x})$ occurs as a sequence of length $|K|$ in the branching process.

Note that $x \notin K$ and $K \cap K_0 \ne \emptyset$ imply that for at least one of the initial sequences $v_i^{(1)} \in K$.

Let j be the maximal length of a sequence $(w_1, \dots, w_j)$ satisfying $w_i \in K$, $1 \le i \le j$. Suppose for contradiction $j < |K|$. Let $F_j \in \mathcal{F}$ be the corresponding set we defined above. $(w_1, \dots, w_j) \cap F_j = \emptyset$ and $K \cap F_j \neq \emptyset$ imply that for some $y_q \in K \setminus (w_1, \dots, w_j)$ the sequence $(w_1, \dots, w_j, y_q)$ is also among the sequences, contradicting the maximal choice of j. $\square$

We want to prove (2.12) using (6.2), i.e., in the form

$$\frac{|\mathcal{F}(\bar{x})|}{\binom{n-3}{k-2}} \le \sum_{t \le i \le k} k(i) \frac{\binom{n-i-1}{k-i}}{\binom{n-3}{k-2}} \le 1. \tag{6.4}$$

Let us rewrite (6.3) in an analogous form.

$$\sum_{t \le i \le k} k(i) \frac{1}{k^{i-2} t(t-1)} \le 1. \tag{6.5}$$

Now (6.5) implies (6.4) once we prove the following.

Lemma 6.1. *Let $n \ge 6k^2$. Then*

$$\frac{\binom{n-i-1}{k-i}}{\binom{n-3}{k-2}} \le \frac{1}{k^{i-2} t(t-1)} \quad \text{for all } \ t \ge 3 \ \text{ and } \ t \le i \le k. \tag{6.6}$$

Proof. First note

$$\frac{k^{i-2}\binom{n-i-3}{k-i}}{\binom{n-3}{k-2}} = \prod_{1 \le p \le i-2} \frac{k \cdot (k-1-p)}{n-2-p}. \tag{6.7}$$

Note that for $n \ge 6k^2$ each term on the right-hand side of (6.2) is less than $1/6$. Thus the right-hand side is less than $6^{-(i-2)}$. Consequently, if we can prove $6^{t-2} \ge t(t-1)$ then (6.6) follows for all $t < i \le k$ as well.

For $t = 3$, $6 = 3 \times 2$. To conclude the proof it is sufficient to note that $\frac{(t+1)t}{t(t-1)} = \frac{t+1}{t-1} \le 2 < 6$ for all $t \ge 3$. $\square$

7 Concluding remarks, open problems

In view of our results it is natural to make the following

Conjecture 7.1. *Let n, d be positive integers, $n > d$. Suppose that $\mathcal{F} \subset 2^{[n]}$ is an antichain with diameter $\Delta(\mathcal{F}) \le d$. Then*

$$|\mathcal{F}| \le \binom{n}{\lfloor d/2 \rfloor}. \tag{7.1}$$

In the paper we proved (7.1) for $n > n_0(d)$. Let us discuss a little bit the value of $n_0(d)$, given by the proof. If $d = 2r$ then we needed only $n \ge r(r+2)$.

If $d = 2r+1$ then we needed $n \ge 2r(r+5)$ in order to apply Theorem 2.3.

Then we used Theorem 2.4 which we proved for $n \ge 6(r+1)^2$. Since $3(r+1)^2 > r(r+5)$ is true for $r \ge 1$, the proof gives (7.1) for $n \ge 6(r+1)^2$ for $r \ge 3$.

Even if we can prove Theorem 2.4 for $n > cr$ we still get a quadratic bound because we need Theorem 2.3.

That is, to prove Conjecture 7.1 with $n_0(r) = o(r^2)$ seems to require completely new ideas.

Erdős [Er] generalised Sperner's theorem by relaxing the condition that $\mathcal{F}$ is an antichain to $\mathcal{F}$ not containing a chain of $t+1$ members (i.e., $F_0 \subset F_1 \subset \ldots \subset F_t$). The analogous statement might hold for the diameter condition.

Conjecture 7.2. *Let n, d, ℓ be positive integers, $n > d$, $d \ge \ell$. Suppose that $\mathcal{F} \subset 2^{[n]}$ does not contain $F_0 \subset F_1 \subset \ldots \subset F_\ell$ with $F_0, \ldots, F_\ell$ distinct members of $\mathcal{F}$. Moreover $\Delta(\mathcal{F}) \le d$. Then setting $s = \min\{\ell - 1, \lfloor d/2 \rfloor\}$ one has*

$$|\mathcal{F}| \le \sum_{\lfloor d/2 \rfloor \ge i \ge \lfloor d/2 \rfloor - s} \binom{n}{i}. \tag{7.2}$$

With the methods of the present paper one can prove (7.2) for $n > n_0(d, \ell)$. For $s = \lfloor d/2 \rfloor$ the bound (7.2) follows directly from Kleitman's Diameter Theorem.

Let us recall that a family $\mathcal{F} \subset \binom{X}{k}$ is called t-intersecting ($t \ge 1$) if $|F \cap F'| \ge t$ holds for all $F, F' \in \mathcal{F}$. One can extend Theorem 2.4 in two directions.

Theorem 7.1. *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is t-intersecting, $k > t > 0$ and $n > n_0(k, t)$. Then there exists an $x \in [n]$ such that*

$$\left|\mathcal{F}(\bar{x})\right| \le \binom{n-t-2}{k-t-1} \quad \textit{holds}. \tag{7.3}$$

The family $\mathcal{G} = \left\{G \in \binom{[n]}{k} : \left|G \cap [t+2]\right| \geq t+1\right\}$ shows that (7.3) is best possible.

For the other direction we need a definition. The independence number of a family $\mathcal{F} \subset \binom{[n]}{k}$ is the maximal integer q such that $\mathcal{F}$ contains q pairwise disjoint members. It is denoted by $\nu(\mathcal{F})$. For a subset $Q \subset [n]$ recall the notation $\mathcal{F}(\bar{Q}) = \{F \in \mathcal{F} : F \cap Q = \emptyset\}$.

Theorem 7.2. *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ satisfies $\nu(\mathcal{F}) = q$. Then for $n > n_0(k, q)$ there is some $Q \in \binom{[n]}{q}$ satisfying*

$$\left|\mathcal{F}(\bar{Q})\right| \leq \sum_{2 \leq i \leq q+1} \binom{q+1}{i}\binom{n-2q-1}{k-i}. \tag{7.4}$$

Note that for $q = 1$ the RHS is simply $\binom{n-3}{k-2}$ from (2.12). Again, (7.4) is best possible. Here is the example:

$$\mathcal{G} = \left\{G \in \binom{[n]}{k} : \left|G \cap [2q+1]\right| \geq 2\right\}.$$

Acknowledgement. The author is indebted to Zoltán Füredi for telling him the problem, many fruitful discussions and much needed encouragement in the long process of finding a proof that *really* works.

References

[En] K. Engel, *Sperner Theory*, Cambridge University Press, 1997.

[Er] P. Erdős, On a lemma of Littlewood and Offord, *Bull. Amer. Math. Soc.* **51** (1945), 898–902.

[F1] P. Frankl, Bounding the size of a family knowing the cardinality of differences, *Studia Sci. Math. Hungar.* **20** (1985), 33–36.

[F2] P. Frankl, The shifting technique in extremal set theory, in: *Surveys in combinatorics 1987*, 81–110, London Math. Soc. Lecture Note Ser. 123, Cambridge Univ. Press, Cambridge, 1987.

[F3] P. Frankl, Erdős–Ko–Rado theorem with conditions on the maximal degree, *J. Comb. Theory, Ser. A* **46**(2) (1987), 252–263.

[F4] P. Frankl, A stability result for families with fixed diameter, *Combinatorics, Probability and Computing* (2017), 1–11.

[FS] Z. Füredi and B. Sudakov, Extremal set-systems with restricted k-wise intersections, *J. Combinatorial Theory Ser. A* **105** (2004), 143–159.

[Ka] G. O. H. Katona, Intersection theorems for systems of finite sets, *Acta Math. Hungar.* **15** (1964), 329–337.

[Kl] D. J. Kleitman, On a combinatorial conjecture of Erdős, *J. Combinatorial Th.* **1** (1966), 209–214.

[KEL] D. J. Kleitman, M. Edelberg, D. Lubell, Maximal sized antichains in partial orders, *Discrete Math.* **1** (1971), 47-53.

[LP] N. Lemons and C. Palmer, The unbalance of set systems, *Graphs and Combinatorics* **24** (2008), 361–365.

[Mi] E. C. Milner, A combinatorial theorem on systems of sets, *J. London Math. Soc.* **43** (1968), 204–206.

[Pu] G. Purdy, A result on Sperner collections, *Utilitas Math.* **12** (1977), 95–99.

[Py] L. Pyber, A new generalization of the Erdős–Ko–Rado theorem, *J. Combinatorial Th., Ser. A* **43** (1986), 85–90.

[S] E. Sperner, Ein Satz über Untermengen einer endlichen Menge, *Math. Z.* **27** (1928), 544–548.