Best possible bounds concerning the set-wise union of families

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Abstract

For two families of sets $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$ we define their set-wise union, $\mathcal{F} \vee \mathcal{G} = \{F \cup G : F \in \mathcal{F}, G \in \mathcal{G}\}$ and establish several – hopefully useful – inequalities concerning $|\mathcal{F} \vee \mathcal{G}|$. Some applications are provided as well.

1 Introduction

For a non-negative integer n let $[n] = \{1, \ldots, n\}$ be the standard n-element set and $2^{[n]}$ its power set. A subset $\mathcal{F} \subset 2^{[n]}$ is called a *family.* If $G \subset F \in \mathcal{F}$ implies $G \in \mathcal{F}$ for all $G, F \subset [n]$ then $\mathcal{G}$ is called a *complex* (*down-set*). Let F^c denote the complement, $[n] \setminus F$ of F. Also let $\mathcal{F}^c = \{F^c : F \in \mathcal{F}\}$ be the *complementary* family. One of the earliest and no doubt the easiest result in extremal set theory, contained in the seminal paper of Erdős, Ko and Rado can be formulated as follows.

Theorem 0 ([EKR]). *Suppose that there are no $F, G \in \mathcal{F}$ satisfying $F \cup G = [n]$. Then*

$$2 \cdot |\mathcal{F}| \leq 2^n. \tag{1}$$

Proof. Just note that the condition implies $\mathcal{F} \cap \mathcal{F}^c = \emptyset$. □

This simple result was the starting point of a lot of research.

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Definition 1. For a positive integer t let us say that $\mathcal{F} \subset 2^{[n]}$ is *t-union* if $|F \cup G| \le n - t$ for all $F, G \in \mathcal{F}$.

An important result of Katona [Ka] was the determination of the maximum size of t-union families.

In the present paper we mostly deal with problems concerning several families.

Definition 2. For positive integers t and r, $r \ge 2$ and non-empty families $\mathcal{F}_1, \ldots, \mathcal{F}_r \subset 2^{[n]}$, we say that they are *cross t-union* if $|F_1 \cup \ldots \cup F_r| \le n - t$ for all $F_1 \in \mathcal{F}_1, \ldots, F_r \in \mathcal{F}_r$.

Definition 3. For families $\mathcal{F}, \mathcal{G}$ let $\mathcal{F} \vee \mathcal{G}$ denote their *set-wise union*,

$$\mathcal{F} \vee \mathcal{G} = \{F \cup G : F \in \mathcal{F},\ G \in \mathcal{G}\}.$$

To state our main results we need one more definition. A family $\mathcal{F} \subset 2^{[n]}$ is said to be *covering* if $\{i\} \in \mathcal{F}$ for all $i \in [n]$. If $\mathcal{F}$ is a complex, it is equivalent to saying that $\bigcup\limits_{F \in \mathcal{F}} F = [n]$.

Let us use the term *cross-union* for cross 1-union.

Theorem 1. *Suppose that $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$ are cross-union and covering complexes. Then*

$$|\mathcal{F} \vee \mathcal{G}| \ge \frac{7}{8}(|\mathcal{F}| + |\mathcal{G}|). \tag{2}$$

Example 1. Let $n \ge 3$ and define $\mathcal{A} = \big\{A \subset [n] : |A \cap [3]| \le 1\big\}$. Then $|\mathcal{A}| = 2^{n-1}$ and $|\mathcal{A} \vee \mathcal{A}| = \frac{7}{8}2^n$ hold.

The above example shows that (2) is best possible.

Theorem 2. *Suppose that $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$ are non-empty cross-union complexes and $\mathcal{F}$ is covering. Then*

$$|\mathcal{F} \vee \mathcal{G}| \ge \frac{3}{4}(|\mathcal{F}| + |\mathcal{G}|). \tag{3}$$

The bound (3) is best possible as shown by the next example.

Example 2. Let $n \ge 2$ and define $\mathcal{A} = \{A \subset [n] : |A \cap [2]| \le 1\}$, $\mathcal{B} = \{B \subset [n] : B \cap [2] = \emptyset\}$.

Theorem 3. *Suppose that $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$ are cross 2-union and covering complexes. Then*

$$|\mathcal{F} \vee \mathcal{G}| > |\mathcal{F}| + |\mathcal{G}|. \tag{4}$$

2 The proof of Theorems 1 and 2

Let us first note that if $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$ are cross-union then

$$|\mathcal{F}| + |\mathcal{G}| \leq 2^n. \tag{2.1}$$

Indeed the cross-union property guarantees $\mathcal{F} \cap \mathcal{G}^c = \emptyset$ and thereby $|\mathcal{F}| + |\mathcal{G}| = |\mathcal{F}| + |\mathcal{G}^c| \leq |2^{[n]}| = 2^n$.

In view of (2.1) the following statement easily implies Theorem 1.

Theorem 2.1. *Let $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$ be covering complexes. Then*

$$|\mathcal{F} \vee \mathcal{G}| \geq \min\left\{2^n, \frac{7}{8}(|\mathcal{F}| + |\mathcal{G}|)\right\}. \tag{2.2}$$

Proof. First we consider the case that $\mathcal{F}, \mathcal{G}$ are *not* cross-union. It is easy. If $\mathcal{F}$ and $\mathcal{G}$ are not cross-union then there exist $F \in \mathcal{F}$, $G \in \mathcal{G}$ satisfying $F \cup G = [n]$. Since $\mathcal{F}$ and $\mathcal{G}$ are complexes for all $H \subset [n]$, $F \cap H \in \mathcal{F}$, $G \cap H \in \mathcal{H}$, implying $H \in \mathcal{F} \vee \mathcal{G}$. Thus $\mathcal{F} \vee \mathcal{G} = 2^{[n]}$, proving (2.2). In view of (2.1), while proving (2.2) we may assume that $|\mathcal{F}| + |\mathcal{G}| \leq 2^n$.

Note that a covering complex $\mathcal{H}$ satisfies $|\mathcal{H}| \geq n+1$. Thus $|\mathcal{F}| + |\mathcal{G}| \leq 2^n$ cannot hold for $n < 3$ and even for $n = 3$ the only possibility is $\mathcal{F} = \mathcal{G} = \big\{\emptyset, \{1\}, \{2\}, \{3\}\big\}$. In this case $\mathcal{F} \vee \mathcal{G} = 2^{[3]} \setminus \{[3]\}$, proving (2.2).

Suppose $n > 3$ and apply induction. We distinguish two cases.

(a) $|\mathcal{F}(\bar{i})| + |\mathcal{G}(\bar{i})| > 2^{n-1}$ for all $1 \leq i \leq n$.

Now (2.1) implies $\big([n] \setminus \{i\}\big) \in \mathcal{F}(\bar{i}) \vee \mathcal{G}(\bar{i})$. Since $\mathcal{F}(\bar{i}) \subset \mathcal{F}$, $\mathcal{G}(\bar{i}) \subset \mathcal{G}$, $H \in \mathcal{F} \vee \mathcal{G}$ follows for all $H \subsetneqq [n]$. Thus $|\mathcal{F} \vee \mathcal{G}| \geq 2^n - 1 > \frac{7}{8}2^n$ for $n > 3$.

(b) There exists $j \in [n]$ satisfying $|\mathcal{F}(\bar{j})| + |\mathcal{G}(\bar{j})| \leq 2^{n-1}$.

Since $\mathcal{F}(\bar{j})$ and $\mathcal{G}(\bar{j})$ are covering the induction hypothesis yields

$$|\mathcal{F}(\bar{j}) \vee \mathcal{G}(\bar{j})| \geq \frac{7}{8}\big(|\mathcal{F}(\bar{j})| + \mathcal{G}(\bar{j})|\big). \tag{2.3}$$

Assume by symmetry that $|\mathcal{G}(j)| \geq |\mathcal{F}(j)|$ holds. If $\mathcal{G}(j)$ is not covering, i.e., for some $i \in ([n] \setminus \{j\})$, $\{i\} \notin \mathcal{G}(j)$ then $\{i\} \in \mathcal{F}(\bar{j})$ implies

$$|\mathcal{G}(j) \vee \mathcal{F}(\bar{j})| \geq 2|\mathcal{G}(j)| \geq |\mathcal{F}(j)| + |\mathcal{G}(j)| > \frac{7}{8}\big(|\mathcal{F}(j)| + |\mathcal{G}(j)|\big).$$

In this way we obtain

$$|\mathcal{F} \vee \mathcal{G}| \geq |\mathcal{F}(\bar{j}) \vee \mathcal{G}(\bar{j})| + |\mathcal{F}(\bar{j}) \vee \mathcal{G}(j)| > \frac{7}{8}(|\mathcal{F}| + |\mathcal{G}|).$$

On the other hand, if $\mathcal{G}(j)$ is covering then we first observe that it is a complex. Also, $|\mathcal{F}(\bar{j})| \geq |\mathcal{F}(j)|$ follows from the fact $\mathcal{F}$ is a complex. Using the induction hypothesis these yield

$$|\mathcal{F}(\bar{j}) \vee \mathcal{G}(j)| \geq \frac{7}{8}\big(|\mathcal{F}(\bar{j})| + |\mathcal{G}(j)|\big) \geq \frac{7}{8}\big(|\mathcal{F}(j)| + |\mathcal{G}(j)|\big).$$

Using (2.3) we infer (2.2) again

$$|\mathcal{F} \vee \mathcal{G}| \geq |\mathcal{F}(\bar{j}) \vee \mathcal{G}(\bar{j})| + |\mathcal{F}(\bar{j}) \vee \mathcal{G}(j)| \geq \frac{7}{8}(|\mathcal{F}| + |\mathcal{G}|). \qquad \square$$

Let us now prove Theorem 2. For $n = 1$ the statement is void. For $n = 2$ the only possibilities are $\mathcal{F} = \big\{\emptyset, \{1\}, \{2\}\big\}$ and $\mathcal{G} = \{\emptyset\}$ which satisfy (2).

Let now $n \geq 3$ and let us apply induction. Replacing if necessary $(\mathcal{F}, \mathcal{G})$ by $(\mathcal{F} \cup \mathcal{G}, \mathcal{F} \cap \mathcal{G})$ we may assume that $\mathcal{F} \supset \mathcal{G}$, $\emptyset \in \mathcal{G}$.

Just as above we may assume that for some $j \in [n]$, $\mathcal{F}(\bar{j})$ and $\mathcal{G}(\bar{j})$ are cross-union (on $[n] \setminus \{j\}$). By the induction hypothesis

$$|\mathcal{F}(\bar{j}) \vee \mathcal{G}(\bar{j})| \geq \frac{3}{4}\big(|\mathcal{F}(\bar{j})| + |\mathcal{G}(\bar{j})|\big). \tag{2.4}$$

There are two cases to consider according whether $\mathcal{G}(j)$ is empty or not.

(i) $\mathcal{G}(j) \neq \emptyset$

Since $\mathcal{F}(\bar{j})$ is covering,

$$|\mathcal{F}(\bar{j}) \vee \mathcal{G}(j)| \geq \frac{3}{4}\big(|\mathcal{F}(\bar{j})| + |\mathcal{G}(j)|\big) \geq \frac{3}{4}\big(|\mathcal{F}(j)| + |\mathcal{G}(j)|\big)$$

follows from the induction hypothesis. Now (2.4) yields (2).

(ii) $\mathcal{G}(j) = \emptyset$

Since $\emptyset \in \mathcal{G}(\bar{j})$,

$$|\mathcal{F}(j) \vee \mathcal{G}(\bar{j})| \geq |\mathcal{F}(j)| > \frac{3}{4}|\mathcal{F}(j)|.$$

Adding this to (2.4) yields (2) with strict inequality. $\square$

3 The deduction of Theorem 3

We could not prove Theorem 3 directly. We are going to deduce it from the following recent result of the author

Theorem 3.1 ([F]). *Let $\mathcal{F}, \mathcal{G}, \mathcal{H} \subset 2^{[n]}$ be covering complexes that are cross-union. Then*

$$|\mathcal{F}| + |\mathcal{G}| + |\mathcal{H}| < 2^n. \tag{3.1}$$

The proof of Theorem 3 using (3.1) is easy. First note that since $\mathcal{F}$ and $\mathcal{G}$ are cross 2-union $\mathcal{F} \vee \mathcal{G}$ contains no $(n-1)$-element sets. Consequently $\mathcal{H} \overset{\text{def}}{=} 2^{[n]} \setminus (\mathcal{F} \vee \mathcal{G})^c$ is covering. Since $\mathcal{F}$ and $\mathcal{G}$ are complexes, $\mathcal{F} \vee \mathcal{G}$ and therefore $\mathcal{H}$ also are complexes. Let us show that $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are cross-union.

Since all three are complexes, the contrary means that there are $F \in \mathcal{F}$, $G \in \mathcal{G}$, $H \in \mathcal{H}$ that partition $[n]$. Thus $H = (F \cup G)^c \in (\mathcal{F} \vee \mathcal{G})^c$ contradicting $\mathcal{H} = 2^{[n]} \setminus (\mathcal{F} \vee \mathcal{G})^c$. Applying (3.1) gives

$$|\mathcal{F}| + |\mathcal{G}| + 2^n - |\mathcal{F} \vee \mathcal{G}| < 2^n.$$

Rearranging yields

$$|\mathcal{F}| + |\mathcal{G}| < |\mathcal{F} \vee \mathcal{G}| \quad \text{proving (4).} \qquad \square$$

In [F] the following generalisation of Theorem 3.1 is established in a somewhat lengthy way. Here we provide a much simpler proof.

Theorem 3.2. *Suppose that $r \geq 2$, $\mathcal{F}_1, \ldots, \mathcal{F}_r \subset 2^{[n]}$ are cross-union and covering. Then*

$$\sum_{1 \leq i \leq r} |\mathcal{F}_i| \leq 2^n - (r-2). \tag{3.2}$$

Proof. The case $r = 2$ follows from (2.1). We apply induction on r and use (3.2) to prove it for r replaced by $r+1$. Without loss of generality let $\mathcal{F}_1, \ldots, \mathcal{F}_{r+1}$ be complexes. Note that $\mathcal{F}_r \vee \mathcal{F}_{r+1}$ is a covering complex and that the r families $\mathcal{F}_1, \ldots, \mathcal{F}_{r-1}$, $\mathcal{F}_r \vee \mathcal{F}_{r+1}$ are cross-union.

On the other hand the fact that $\mathcal{F}_1$ is covering implies that $\mathcal{F}_r$ and $\mathcal{F}_{r+1}$ are cross 2-union. Applying the induction hypothesis and (4) yield

$$|\mathcal{F}_1| + \ldots + |\mathcal{F}_{r+1}| \leq |\mathcal{F}_1| + \ldots + |\mathcal{F}_{r-1}| + |\mathcal{F}_r \vee \mathcal{F}_{r+1}| - 1 \leq 2^n - (r-2) - 1 = 2^n - (r-1)$$

as desired. $\square$

Actually in [F] only the slightly weaker statement, $< 2^n$ is proved.

Especially for $n > n_0(r)$ the bound (3.2) seems to be rather far from best possible.

Example 3.3. Let $n > r \geq 3$. Set $\mathcal{G}_1 = \{G \subset [n] : |G| \leq n - r\}$, $\mathcal{G}_2 = \ldots = \mathcal{G}_r = \{G \subset [n] : |G| \leq 1\}$. Then $\mathcal{G}_1, \ldots, \mathcal{G}_r$ are covering and cross-union. Define

$$g(n, r) = |\mathcal{G}_1| + \ldots + |\mathcal{G}_r| = 2^n + (r-1)(n+1) - \sum_{0 \leq j < r} \binom{n}{j}.$$

Note that $g(n, 2) = 2^n$. For $r \geq 3$ fixed and $n \to \infty$ also $g(n, r)/2^n$ tends to 1.

Conjecture 3.1. *Suppose that $\mathcal{F}_1, \ldots, \mathcal{F}_r \subset 2^{[n]}$ are covering and cross-union, $r \geq 3$. Then for $n > n_0(r)$ one has*

$$|\mathcal{F}_1| + \ldots + |\mathcal{F}_r| \leq g(n, r).$$

4 Further applications

Let us use Theorems 1 and 2 to give a new proof for the following recent results from [F].

Theorem 4.1. *Suppose that $\mathcal{A}, \mathcal{B}, \mathcal{C} \subset 2^{[n]}$ are cross-union and $\mathcal{A}, \mathcal{B}$ are covering. Then*

$$|\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| \leq \frac{9}{8} 2^n. \tag{4.1}$$

Theorem 4.2. *Suppose that $\mathcal{A}, \mathcal{B}, \mathcal{C} \subset 2^{[n]}$ are cross-intersecting and $\mathcal{A}$ is covering. Then*

$$|\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| \leq \frac{5}{4} 2^n. \tag{4.2}$$

For a family $\mathcal{H}$ let $\mathcal{H}_*$ be the complex generated by $\mathcal{H}$:

$$\mathcal{H}_* = \{G : \exists H \in \mathcal{H}, G \subset H\}.$$

In both Theorems, replacing $\mathcal{A}, \mathcal{B}, \mathcal{C}$ by $\mathcal{A}_*, \mathcal{B}_*, \mathcal{C}_*$ will not change the union and covering properties and can only increase the size of the families. Therefore in proving (4.1) and (4.2) we may assume that $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are complexes.

Proof of (4.1). Apply (2) for $\mathcal{A} = \mathcal{F}$, $\mathcal{B} = \mathcal{G}$ to obtain

$$\frac{7}{8}|\mathcal{A}| + |\mathcal{B}| \leq |\mathcal{A} \vee \mathcal{B}|. \tag{4.3}$$

Since $\mathcal{A} \vee \mathcal{B}$ and $\mathcal{C}$ are cross-union, we infer from (2.1):

$$|\mathcal{A} \vee \mathcal{B}| + |\mathcal{C}| \leq 2^n.$$

Combining with (4.3) yields

$$\frac{7}{8}|\mathcal{A}| + \frac{7}{8}|\mathcal{B}| + |\mathcal{C}| \leq 2^n.$$

Invoking (3.1) to $\mathcal{A}$ and $\mathcal{B}$ yields

$$\frac{1}{8}|\mathcal{A}| + \frac{1}{8}|\mathcal{B}| \leq \frac{1}{8}2^n.$$

Now adding these two inequalities gives (4.1). □

Proof of (4.3). It is very similar. Using (2.1) for the pairs $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A} \vee \mathcal{B}, \mathcal{C})$ yields

$$\frac{1}{4}|\mathcal{A}| + \frac{1}{4}|\mathcal{B}| \leq \frac{1}{4}2^n,$$
$$|\mathcal{A} \vee \mathcal{B}| + |\mathcal{C}| \leq 2^n.$$

Adding these two inequalities and using

$$|\mathcal{A} \vee \mathcal{B}| \geq \frac{3}{4}(|\mathcal{A}| + |\mathcal{B}|)$$

gives (4.3). □

Let us mention that without covering assumptions (2.1) implies the bound $|\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| \leq \frac{3}{2} \cdot 2^n$ which is best possible as shown by the choice $\mathcal{A} = \mathcal{B} = \mathcal{C} = 2^{[n-1]}$.

One can prove similar statements for r families, $r > 3$ as well, cf. [F].

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