Some exact results for multiply intersecting families

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Abstract

Let $\mathcal{F}$ be a family of subsets of $\{1, 2, \ldots, n\}$, satisfying $|F_1 \cap F_2 \cap F_3| \geq 4$ for all $F_1, F_2, F_3 \in \mathcal{F}$. It is shown that $|\mathcal{F}| \leq 2^{n-4}$ must hold. This best possible bound was conjectured by the author more than 40 years ago. For the proof several related results are proved for one or several families.

1 Introduction

Let $[n] = \{1, \ldots, n\}$ be the standard n-element set. Any subset of the power set $2^{[n]}$ is called a family. For integers $t \geq 1$ and $r \geq 2$ a family $\mathcal{F} \subset 2^{[n]}$ is called *r-wise t-intersecting* if for all choices of $F_i \in \mathcal{F}$, $i = 1, \ldots, r$ one has $|F_1 \cap \ldots \cap F_r| \geq t$. Since for $t > n$ the only possibility is $\mathcal{F} = \emptyset$, in the sequel we tacitly assume $n \geq t$.

Definition 1. Let $m(n, r, t) = \big\{|\mathcal{F}| : F \subset 2^{[n]},\ \mathcal{F} \text{ is } r\text{-wise } t\text{-intersecting}\big\}$.

The easy fact, $m(n, r, 1) = 2^{n-1}$ was proved already by Erdős, Ko and Rado [EKR]. Katona [Kat] determined $m(n, 2, t)$ for all $n \geq t \geq 2$ together with all the families attaining equality. An easy consequence of it is that for every fixed $t \geq 1$,

$$\lim_{n\to\infty} m(n, 2, t)/2^n = 1/2.$$

For $r \geq 3$ the situation is completely different. In [F76] $m(n, 3, 2) = 2^{n-2}$ and

$$m(n, 3, t)/2^n < \left(\frac{\sqrt{5}-1}{2}\right)^t$$

were shown.

For every integer j and $n \geq t + rj$ one defines $\mathcal{A}_j(n,r,t) = \big\{A \subset [n] : \big|[t+rj] \setminus A\big| \leq j\big\}$.

In human language $\mathcal{A}_j(n,r,t)$ consists of those subsets of $[n]$ that miss at most j elements out of the first $t + rj$. Thus r sets miss altogether at most rj, ensuring that their intersection contains at least t elements.

Conjecture 2 ([F76], [F79])**.** *For all $n \geq t \geq 1$, $r \geq 2$*

$$m(n,r,t) = \max_{0 \leq j \leq \frac{n-t}{r}} \big|\mathcal{A}_j(n,r,t)\big|. \tag{1}$$

Easy computation shows that

$$\begin{aligned} \big|\mathcal{A}_0(n,r,t)\big| &= 2^{n-t} && \text{for } n \geq t, \\ \big|\mathcal{A}_1(n,r,t)\big| &= (r+t+1) \cdot 2^{n-t-r} && \text{for } n \geq r+t. \end{aligned}$$

Thus for $n \geq r + t$, $\big|\mathcal{A}_1(n,r,t)\big| \gtreqless \big|\mathcal{A}_0(n,r,t)\big|$ is equivalent to

$$t \gtreqless 2^r - r - 1.$$

In [F91] the validity of $m(n,r,t) = 2^{n-t}$ was established for all (t,r) satisfying $t \geq 2^r - r - 1$ except for the case $t = 3$, $r = 4$. A few years later the author proved this case in a very long and messy way. Since it was not illuminative, he never published it. The present proof is by no means short, however, it proceeds via various results of some independent interest. Hopefully they shed some light on the possible structure of large, r-wise t-intersecting families.

Let us recall that a family is called *trivial* if $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$. One of the very first results on multiply intersecting families is the following beautiful theorem.

Brace–Daykin Theorem ([BD])**.** *Suppose that $\mathcal{F} \subset 2^{[n]}$ is r-wise 1-intersecting and $\mathcal{F}$ is not trivial. Then*

$$|\mathcal{F}| \leq \big|\mathcal{A}_1(n,r,1)\big| = \frac{r+2}{2^{r+1}} \cdot 2^n. \tag{2}$$

Definition 3. Set

$$m^*(n,r,t) = \max\big\{|\mathcal{F}| : \mathcal{F} \subset 2^{[n]},\ \mathcal{F} \text{ is non-trivial and } r\text{-wise } t\text{-intersecting}\big\}.$$

With this terminology the Brace–Daykin Theorem says that $m^*(n, r, 1) = (r + 2) \cdot 2^{n-r-1}$ for $n \geq r + 1$.

In [F91] the more general statement

$$m^*(n, r, t) = (r + t + 1)2^{n-r-t} \tag{3}$$

was proved for all but five pairs (r, t) satisfying $t < 2^r - r - 1$. We proved that (3) holds for these "exceptional" pairs: $(3, 2)$, $(3, 3)$, $(4, 8)$, $(4, 9)$, $(4, 10)$ as well. In the present paper we include the proof for $(3, 2)$ and $(3, 3)$. Basically the same approach applies to the cases with $r = 4$.

Theorem 4.

$$\begin{aligned} \text{(i)} \quad m(n, 3, 4) &= 2^{n-4}, \\ \text{(ii)} \quad m^*(n, 3, 3) &= \frac{7}{8} \cdot 2^{n-3}, \\ \text{(iii)} \quad m^*(n, 3, 2) &= \frac{3}{4} \cdot 2^{n-2}. \end{aligned}$$

Results involving several families, stated in the next section along with (i) are applied to prove the following refinement of (2).

Theorem 5. *Let $\mathcal{F} \subset 2^{[n]}$ be non-trivial, r-wise 1-intersecting, $r \geq 3$. Suppose further that $\mathcal{F}$ is not contained in an isomorphic copy of $\mathcal{A}_1(n, r, 1)$. Then*

$$|\mathcal{F}| \leq \frac{r + 6}{2^{r+2}} \cdot 2^n. \tag{4}$$

2 Tools of proof and further results

Definition 6. For $r \geq 2$, $t \geq 1$ the families $\mathcal{F}_1, \ldots, F_r \subset 2^{[n]}$ are called *cross t-intersecting* if $\left|F_1 \cap \ldots \cap F_r\right| \geq t$ holds for all choices of $F_i \in \mathcal{F}_i$, $1 \leq i \leq r$.

In [EKR] Erdős, Ko and Rado defined a very useful operation, called shifting. It is well known that this operation maintains both r-wise t-intersecting and cross t-intersecting properties (cf. [F87] or [F91] for details). Repeated applications of shifting produce *shifted* families.

Definition 7. A family $\mathcal{F} \subset 2^{[n]}$ is called *shifted* if for all $1 \leq i < j \leq n$ and $F \in \mathcal{F}$, $i \notin F$, $j \in F$ imply that $(F - \{j\}) \cup \{i\}$ is also in $\mathcal{F}$.

Noting that shifting does not change the size, $|\mathcal{F}|$ of the family, in view of the above we are going to assume in the sequel that *all* families are shifted.

For a family $\mathcal{F} \subset 2^{[n]}$ one defines its *upward closure* by $\mathcal{F}^*$:

$$\mathcal{F}^* = \big\{G \subset [n] : \exists F \in \mathcal{F},\ F \subset G\big\}.$$

If $\mathcal{F}$ is r-wise t-intersecting then $\mathcal{F}^*$ is r-wise t-intersecting as well. Obviously, $\mathcal{F}^*$ is a *filter* (*up-set*), i.e., $F \subset G \subset [n]$ and $F \in \mathcal{F}$ imply $G \in \mathcal{F}$. Since we are dealing with upper bounds, we are going to assume that the families in question are filters.

We are going to use the following, very convenient notations.

For $\ell \le n$ and $B \subset [\ell]$ set $\mathcal{F}(B,\ell) = \big\{F \setminus [\ell] : F \in \mathcal{F},\ F \cap [\ell] = B\big\}$.

Note that

$$\sum_{B \subset [\ell]} |\mathcal{F}(B,\ell)| = |\mathcal{F}|. \tag{2.1}$$

If ℓ is fixed and it causes no misunderstanding then we often omit ℓ and write $\mathcal{F}(B)$ instead.

The following important lemma was proved in [F91]. To illustrate the strength and use of shiftedness, we present the easy proof.

Lemma 8. *Let $\mathcal{F}_i \subset 2^{[n]}$, $1 \le i \le r$ be cross t-intersecting, $\ell \ge t$, $B_i \subset [\ell]$, $1 \le i \le r$. Suppose that $\big|B_1 \cap \ldots \cap B_r\big| < t$. Then $\mathcal{F}_1(B_1), \mathcal{F}_2(B_2), \ldots, \mathcal{F}_r(B_r) \subset 2^{[\ell+1,n]}$ are cross s-intersecting for $s = t + (r-1)\ell - \big(|B_1| + \ldots + |B_r|\big)$.*

Proof. Let us prove the statement using induction on s. Let $A_i \in \mathcal{F}_i(B_i)$ and set $F_i = A_i \cup B_i$.

In the case $s = 1$, $\big|F_1 \cap \ldots \cap F_r\big| \ge t$ and $\big|B_1 \cap \ldots \cap B_r\big| < t$ imply $\big|A_1 \cap \ldots \cap A_r\big| \ge 1$, as desired.

Let us do the induction step. Choose an element x, $\ell < x \le n$ contained in $A_1 \cap \ldots \cap A_r$. We distinguish two cases.

(a) $\big|B_1 \cap \ldots \cap B_r\big| \le t - 2$.

Choose an arbitrary element $y \in [\ell]$ that is not contained in *all* of the B_i, $1 \le i \le r$. Since $\ell \ge t$, this is possible. Suppose by symmetry, $y \notin B_r$. Define $\overline{B}_r = B_r \cup \{y\}$, $\overline{A}_r = A_r - \{x\}$ and $\overline{F}_r = \overline{B}_r \cup \overline{A}_r$. Since $\mathcal{F}_r$ is shifted, $\overline{F}_r \in \mathcal{F}_r$. Now $\big|B_1 \cap \ldots \cap B_{r-1} \cap \overline{B}_r\big| \le t - 2 + 1 < t$ and we can apply the induction hypothesis and infer $\big|A_1 \cap \ldots \cap A_{r-1} \cap \overline{A}_r\big| \ge s - 1$. Since $A_1 \cap \ldots \cap A_r$ contains the element x in excess, $\big|A_1 \cap \ldots \cap A_r\big| \ge s$ follows.

(b) $B_1 \cap \ldots \cap B_r \overset{\text{def}}{=} D$ satisfies $|D| = t-1$.

We claim that there is some $y \in [\ell]$ which is contained in at most $r-2$ of the B_i. Indeed, the opposite would mean $|B_1| + \ldots + |B_r| = (t-1) + (r-1)\ell$ implying $s = 1$, a contradiction. Suppose again by symmetry that $y \notin B_r$ and define $\overline{A}_r, \overline{B}_r, \overline{F}_r$ as above.

The careful choice of y implies $B_1 \cap \ldots \cap B_{r-1} \cap \overline{B}_r = B_1 \cap \ldots \cap B_r = D$. Since $|D| = t-1$, we may apply the induction hypothesis and infer $\left|A_1 \cap \ldots \cap A_r\right| = \left|A_1 \cap \ldots \cap A_{r-1} \cap \overline{A}_r\right| + 1 \geq (s-1) + 1 = s$, as desired. □

Note that even if we start with a single family, i.e., $\mathcal{F}_1 = \mathcal{F}_2 = \ldots = \mathcal{F}_r$ the families $\mathcal{F}_1(B_1), \ldots, \mathcal{F}_r(B_r)$ are usually distinct. This is the reason that when proving results for one family, one often needs bounds for several families.

In [F91] various best possible bounds concerning $|\mathcal{F}_1| \cdot |\mathcal{F}_2| \cdot \ldots \cdot |\mathcal{F}_r|$ were proved for cross s-intersecting families.

The following three inequalities will be useful for our proofs. Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \subset 2^{[n]}$ be cross t-intersecting

(2.2) $|\mathcal{F}_1||\mathcal{F}_2||\mathcal{F}_3| \leq 2^{3(n-t)}$ for $t = 1, 2, 3$,

(2.3) $|\mathcal{F}_1||\mathcal{F}_2||\mathcal{F}_3| \leq 2^{3(n-t)} \cdot (\sqrt{5}-1)^{3(t-3)}$ for $t \geq 4$,

(2.4) $|\mathcal{F}_1||\mathcal{F}_2||\mathcal{F}_3| \leq 2^{3n} \cdot \left(\frac{5}{16}\right)^3$ for $t = 1$ *assuming* that $\mathcal{F}_i$ is non-trivial for $i = 1, 2, 3$.

Looking at formula (2.1) one understands that in many cases bounds concerning the sum of $|\mathcal{F}_i|$ are needed. Below we present several such results.

Example 9. Suppose that $n \geq \ell > r \geq 2$ and define

$$\mathcal{H}_1^{(r)} = \left\{H \subset [n] : |H \cap [\ell]| \geq r\right\},$$
$$\mathcal{H}_p^{(r)} = \left\{H \subset [n] : |H \cap [\ell]| \geq \ell - 1\right\}, \quad 2 \leq p \leq r.$$

Claim 10. *$\mathcal{H}_1^{(r)}, \ldots, \mathcal{H}_r^{(r)}$ are cross-intersecting and non-trivial.*

Proof. Non-triviality is obvious. Choose $H_p \in \mathcal{H}_p^{(r)}$, $1 \leq p \leq r$ and set $G_p = [\ell] \setminus H_p$. The definition implies $|G_1| + |G_2| + \ldots + |G_r| \leq (\ell - r) + r - 1 = \ell - 1$. Thus $H_1 \cap \ldots \cap H_r \cap [\ell] \neq \emptyset$ follows. □

For $r = 2$ and ℓ arbitrary, $\left|\mathcal{H}_1^{(2)}\right| + \left|\mathcal{H}_2^{(2)}\right| = 2^n$ is easy to check. On the other hand, setting $\ell = n$ one has

$$\left|\mathcal{H}_1^{(r)}\right| \,/\, 2^n \longrightarrow 1 \text{ as } n \longrightarrow \infty.$$

Theorem 11. *Let $n > r \geq 2$. Suppose that $\mathcal{F}_i \subset 2^{[n]}$, $1 \leq i \leq r$ are non-trivial, cross-intersecting families. Then*

$$|\mathcal{F}_1| + \ldots + |\mathcal{F}_r| \leq 2^n \tag{2.5}$$

with strict inequality unless $r = 2$.

Example 12. Let $n \geq 2$, $r \geq 2$ and define

$$\begin{aligned} \mathcal{B}_1 &= \{B \subset [n] : B \cap [2] \neq \emptyset\}, \\ \mathcal{B}_p &= \{B \subset [n] : [2] \subset B\}, \ 2 \leq p \leq r. \end{aligned}$$

Note that $\mathcal{B}_1, \ldots, \mathcal{B}_r$ are cross-intersecting, $\mathcal{B}_1$ is non-trivial, moreover

$$|\mathcal{B}_1| + \ldots + |\mathcal{B}_r| = 2^n \left(1 + \frac{r-2}{4}\right). \tag{2.6}$$

Example 13. Let $n \geq 3$, $r \geq 2$ and define

$$\begin{aligned} \mathcal{C}_i &= \{C \subset [n] : |C \cap [3]| \geq 2\}, \quad i = 1, 2; \\ \mathcal{C}_j &= \{C \subset [n] : [3] \subset C\}, \qquad 3 \leq j \leq r. \end{aligned}$$

Note that $\mathcal{C}_1, \ldots, \mathcal{C}_r$ are cross-intersecting with $\mathcal{C}_1$ and $\mathcal{C}_2$ non-trivial. One has,

$$|\mathcal{C}_1| + \ldots + |\mathcal{C}_r| = 2^n \left(1 + \frac{r-2}{8}\right). \tag{2.7}$$

We succeeded in proving that both these examples are best possible.

Theorem 14. *Let $s = 1$ or 2, $n > s$, $r \geq 3$. Suppose that $\mathcal{F}_i \subset 2^{[n]}$, $1 \leq i \leq r$, $\mathcal{F}_1, \ldots, \mathcal{F}_r$ are non-empty and cross-intersecting. Suppose also that $\mathcal{F}_i$ is non-trivial for $1 \leq i \leq s$. Then* (i) *or* (ii) *hold.*

(i) $s = 1$ *and*

$$|\mathcal{F}_1| + \ldots + |\mathcal{F}_r| \leq 2^n \left(1 + \frac{r-2}{4}\right). \tag{2.8}$$

(ii) $s = 2$ *and*

$$|\mathcal{F}_1| + \ldots + |\mathcal{F}_r| \leq 2^n \left(1 + \frac{r-2}{8}\right). \tag{2.9}$$

Let us mention that in the 1-intersecting case $\mathcal{F} \subset 2^{[n]}$ is non-trivial iff $\binom{[n]}{n-1} \subset \mathcal{F}$. Therefore if $\mathcal{F}$, $\mathcal{G}$ are both non-trivial then $\mathcal{F} \cap \mathcal{G}$ is non-trivial also. Noting $|\mathcal{F} \cap \mathcal{G}| + |\mathcal{F} \cup \mathcal{G}| = |\mathcal{F}| + |\mathcal{G}|$, this justifies the following general assumption.

Lemma 15. *In the proof of Theorems 11 and 14 we may assume that the families are nested, i.e.,* $\mathcal{F}_1 \supset \ldots \supset \mathcal{F}_r$ *holds.*

Proof. If for some $1 \le a < b \le r$, $\mathcal{F}_a \not\supset \mathcal{F}_b$ then replace them by $\mathcal{F}_a \cup \mathcal{F}_b$ and $\mathcal{F}_a \cap \mathcal{F}_b$. Non-triviality and cross-intersecting properties are maintained and eventually we get nested families. □

Let us close this section with two inequalities.

Lemma 16. *Let* $a \ge b \ge c \ge d \ge e \ge f \ge 0$ *be real numbers satisfying* $a + b \le 16$ *and* $abc \le 125$*. Then* (i) *and* (ii) *hold.*

(i) *If* $a + b + c + d + e + f \ge 30$ *then* $a = b = c = d = e = f = 5$*.*

(ii) *If* $a + b + c + d \ge 20$ *and* $a \le 8$ *then* $a = b = c = d = 5$*.*

Proof. We prove both statements arguing indirectly. Let us choose a counterexample with $a - c$ as large as possible.

(i) Replacing c, d, e, f by their average $(c + d + e + f)/4$ will not alter $a + b + c + d + e + f$ and can only increase $a - c$. Thus we may assume that $c = d = e = f$.

Note that for a positive ε, $(a + \varepsilon)(b - \varepsilon) < ab$. Consequently, the maximality of $a - c$ implies $b = c$.

Define $x \ge 0$ by $b = 5 - x$. Note that $a + b \le 16$ implies $c \ge \frac{30-16}{5} = 3.5$ and thereby $x \le 1.5$. On the other hand $a + b + c + d + e + f \ge 30$ entails $a \ge 5 + 5x = 5(1 + x)$.

Let us conclude the proof by showing that

$$abc \ge 5(1 + x)(5 - x)^2 > 125 \quad \text{holds for} \quad 0 < x < 1.5.$$

Dividing by 5 and rearranging yields

$$x(15 - 9x + x^2) > 0$$

and both terms are positive for $0 < x < \frac{15}{9}$ where $\frac{15}{9} = \frac{5}{3} > 1.5$.

(ii) Again we may assume that $c = d$. Arguing as above, either $b = c = d$ or $a = 8$ follow. In the first case set $x = 5 - b$. Then $a \ge 5 + 3x$, in particular, $0 < x < 1$. Showing that $(5+3x)(5-x)^2 > 125$ we get the desired contradiction. Rearranging yields $25x(1-x)+3x^3 > 0$ which is obviously true for $0 < x < 1$.

The last case is $a = 8$. Now $a+b+c+d \geq 20$ implies $b+c+d = b+2c \geq 12$.

Define x by $c = 4-x$. Then $b \geq 4+2x$, $0 \leq x \leq 2$. In order to get the final contradiction we prove $8(4+2x)(4-x) > 125$ for this range. Rearranging yields

$$3 + 32x - 16x^2 = 3 + 16x(2-x) > 3 \quad \text{for} \quad 0 \leq x \leq 2. \qquad \square$$

3 The proof of Theorems 11 and 14

For $n \leq r$ and an arbitrary choice of r non-trivial families it is clear that they are *not* cross-intersecting. Thus Theorem 11 holds in this range. Theorem 14 is also easy to check for $n \leq 2$.

We prove both theorems simultaneously using induction on n. The proof will consist of two logical steps.

Step 1. We prove Theorem 14 for n, assuming that Theorem 11 holds for the *same* n.

Step 2. We prove Theorem 11 for n, assuming that Theorem 14 holds for all n' with $n' < n$.

Recall that by Lemma 15 we may assume that $\mathcal{F}_1 \supset \ldots \supset \mathcal{F}_r$.

Step 1. In the case $r = 2$ both (2.8) and (2.9) reduce $|\mathcal{F}_1| + |\mathcal{F}_2| \leq 2^n$ which is true by $\mathcal{F}_1 \cap \big\{[n] \setminus F_2 : F_2 \in \mathcal{F}_2\big\} = \emptyset$. We apply induction on r. Suppose that $r \geq 3$ and the statement holds for r replaced by $r-1$.

Let us first deal with (i), i.e., $s = 1$. Since the right-hand side of (2.9) is smaller than that of (2.8), we may assume that $\mathcal{F}_2, \ldots, \mathcal{F}_r$ are trivial. By shiftedness $1 \in F$ for all $F \in \mathcal{F}_i$, $2 \leq i \leq r$.

On the other hand, $\mathcal{F}_1$ is non-trivial, forcing $[2, n] \in \mathcal{F}_1$.

Recall the standard notations:

$$\begin{aligned} \mathcal{F}(i) &= \big\{F \setminus \{i\} : i \in F \in \mathcal{F}\big\}, \\ \mathcal{F}(\bar{i}) &= \big\{F \qquad : i \notin F \in \mathcal{F}\big\}. \end{aligned}$$

Now

$$\big|\mathcal{F}_j(1)\big| = \big|\mathcal{F}_j\big| \quad \text{for} \quad 2 \leq j \leq r$$

since $[2,n] \in \mathcal{F}_1$, $\mathcal{F}_2(1), \ldots, \mathcal{F}_r(1)$ are cross-intersecting. We infer $\big|\mathcal{F}_2(1)\big| + \big|\mathcal{F}_r(1)\big| \leq 2^{n-1}$ implying $\big|\mathcal{F}_r\big| = \big|\mathcal{F}_r(1)\big| \leq 2^n \,/\, 4$. Thus (2.8) follows from the induction hypothesis applied to $\mathcal{F}_1, \ldots, \mathcal{F}_{r-1}$.

Let now $s = 2$. By Theorem 11, $\mathcal{F}_r$ is trivial. Should $|\mathcal{F}_r| = |\mathcal{F}_r(1)| \leq 2^{n-3}$ hold, we would obtain (2.9) by induction on r. From now on we assume that

$$|\mathcal{F}_r(1)| > 2^{n-3}. \tag{3.1}$$

Let us choose q so that $\mathcal{F}_q$ is non-trivial but $\mathcal{F}_{q+1}$ is trivial. By the assumptions $2 \leq q < r$. Apply the induction hypothesis (n replaced by $n-1$) to the two set of families

$\mathcal{F}_1(1), \mathcal{F}_2(\bar{1}), \ldots, \mathcal{F}_q(\bar{1}), \mathcal{F}_{q+1}(1), \ldots, \mathcal{F}_r(1)$ and

$\mathcal{F}_1(\bar{1}), \mathcal{F}_2(1), \ldots, \mathcal{F}_r(1)$.

These are all families on $[2,n]$ and $\mathcal{F}_1(1)$ and $\mathcal{F}_2(1)$ are necessarily non-trivial. Thus we may use (2.8) and infer

(3.2)
$$\big|\mathcal{F}_1(1)\big| + \big|\mathcal{F}_2(\bar{1})\big| + \ldots + \big|\mathcal{F}_q(\bar{1})\big| + \big|\mathcal{F}_{q+1}(1)\big| + \ldots + \big|\mathcal{F}_r(1)\big| \leq 2^{n-1}\left(1 + \frac{r-2}{4}\right),$$

(3.3)
$$\big|\mathcal{F}_1(\bar{1})\big| + \big|\mathcal{F}_2(1)\big| + \ldots + \qquad\qquad + \big|\mathcal{F}_r(1)\big| \leq 2^{n-1}\left(1 + \frac{r-2}{4}\right).$$

In the case $q = 2$, adding (3.2) and (3.3) yields

$$\begin{aligned}\sum_{1 \leq i \leq r} \big|\mathcal{F}_i\big| &\leq 2^n\left(1 + \frac{r-2}{4}\right) - \big|\mathcal{F}_3(1)\big| - \ldots - \big|\mathcal{F}_r(1)\big| \\ &= 2^n\left(1 + \frac{r-2}{8}\right) + \sum_{3 \leq p \leq r}\left(2^{n-3} - \big|\mathcal{F}_p(1)\big|\right) < 2^n\left(1 + \frac{r-2}{8}\right),\end{aligned}$$

as desired.

Suppose next $q = 3$. In view of Theorem 11, $r \geq 4$. Since $\mathcal{F}_3(1)$ is non-trivial, we may apply (2.9) instead of (2.8) and replace (3.3) by

$$\big|\mathcal{F}_1(\bar{1})\big| + \big|\mathcal{F}_2(1)\big| + \ldots + \big|\mathcal{F}_r(1)\big| \leq 2^{n-1}\left(1 + \frac{r-2}{8}\right).$$

Adding this to (3.2) yields

$$\sum_{1\le i\le r} |\mathcal{F}_i| \le 2^n \left(1 + \frac{r-2}{8}\right) + 2^n \cdot \frac{r-2}{16} - \sum_{4\le p\le r} |\mathcal{F}_p(1)|.$$

Using (3.1), (2.9) follows from

$$\begin{aligned} &2^n \frac{r-2}{16} \le \frac{r-3}{8} \cdot 2^n, \quad \text{i.e.,} \\ &r-2 \le 2(r-3) \quad \text{which is valid for } r \ge 4. \end{aligned}$$

The last case is $q \ge 4$. Actually, this is an easy case. We apply (2.9) to the two sets of families

$$\begin{aligned} &\mathcal{F}_1(1), \mathcal{F}_2(1), \mathcal{F}_3(\bar{1}), \ldots, \mathcal{F}_q(\bar{1}), \mathcal{F}_{q+1}(1), \ldots, \mathcal{F}_r(1) \quad \text{and} \\ &\mathcal{F}_1(\bar{1}), \mathcal{F}_2(\bar{1}), \mathcal{F}_3(1), \ldots, \mathcal{F}_r(1). \end{aligned}$$

Adding the two inequalities yields

$$\sum_{1\le i\le r} |\mathcal{F}_i| + \sum_{q<p\le r} |\mathcal{F}_p(1)| \le 2^n \left(1 + \frac{r-2}{8}\right),$$

which is sufficient by large.

Step 2, i.e., the proof of Theorem 11 is harder. We need a lemma of some independent interest.

Proposition 17. *Let $k \ge 1$ and suppose that the non-empty families $\mathcal{F}_1, \ldots, \mathcal{F}_{2^k} \subset 2^{[n]}$ are cross k-intersecting. Then*

$$\sum_{1\le i\le 2^k} |\mathcal{F}_i| \le 2^n. \tag{3.4}$$

Proof. Apply induction on k. The case $k = 1$ is trivial. Also, in the case of $n < k$, the statement holds because there is *no* collection of families satisfying the assumption. If $n = k$, the only choice is $\mathcal{F}_i = \{[k]\}$ and equality holds in (3.4).

We apply induction on n. Without loss of generality $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \ldots \supset \mathcal{F}_{2^k}$. Let us distinguish three cases.

(a) $\mathcal{F}_1(\bar{1}) = \emptyset$.

In this case $|\mathcal{F}_i| = |\mathcal{F}_i(1)|$ for all i.

Since both collections

$$\{\mathcal{F}_i(1) : 1 \le i \le 2^k,\ i \text{ odd}\} \quad \text{and}$$
$$\{\mathcal{F}_i(1) : 1 \le i \le 2^k,\ i \text{ even}\}$$

satisfy the induction hypothesis on $[2, n]$ with k replaced by $k-1$,

$$\sum_{1\le i\le 2^k} |\mathcal{F}_i(1)| \le 2 \cdot 2^{n-1} = 2^n \quad \text{follows.}$$

(b) $\mathcal{F}_2(\overline{1}) \ne \emptyset$.

In this case both $\mathcal{F}_1$ and $\mathcal{F}_2$ are non-trivial. Define

$$\mathcal{G}_i = \begin{cases} \{[2,n]\} & \text{if } \mathcal{F}_i(\overline{1}) = \emptyset, \\ \mathcal{F}_i(\overline{1}) & \text{otherwise.} \end{cases}$$

Let us define two collections of 2^k families on $[2, n]$

$$\{\mathcal{G}_1, \mathcal{F}_2(1), \mathcal{G}_3, \dots, \mathcal{G}_{2^k-1}, \mathcal{F}_{2^k}(1)\} \quad \text{and}$$
$$\{\mathcal{F}_1(1), \mathcal{G}_2, \mathcal{F}_3(1), \dots, \mathcal{F}_{2^k-1}(1), \mathcal{G}_{2^k}\}.$$

We claim that both are collections of 2^k non-empty and cross k-intersecting families. If we could find 2^k sets $H_1, \dots, H_{2^k}$ from the first collection such that their intersection has size less than k, then we just note that $G_1 \in \mathcal{F}_1$ and $H_i \cup \{1\} \in \mathcal{F}_i$ for every $i \ge 2$. This provides us with 2^k sets from the original families which intersect in less than k elements, a contradiction.

The same argument applies to the second collection, proving the claim.

Now, the induction hypothesis with n replaced by $n-1$ gives 2^{n-1} as an upper bound for the sum of the size of the families. Since for every i, $|\mathcal{G}_i| + |\mathcal{F}_i(1)| \ge |\mathcal{F}_i|$, adding these two inequalities proves (3.4).

(c) $\mathcal{F}_1$ is non-trivial but $\mathcal{F}_2(\overline{1}) = \emptyset$.

Since $|\mathcal{F}_i| = |\mathcal{F}_i(1)|$ for $2 \le i \le 2^k$, applying the induction hypothesis to $\mathcal{F}_1(\overline{1}), \mathcal{F}_2(1), \dots, \mathcal{F}_{2^k}(1)$ gives

$$|\mathcal{F}_1(\overline{1})| + \sum_{2\le i\le 2^k} |\mathcal{F}_i| \le 2^{n-1}.$$

Now the obvious inequality $|\mathcal{F}_1(1)| < 2^{n-1}$ shows that (3.4) holds with strict inequality. □

Corollary 18. *Suppose that $r < 2^k$ and $\mathcal{F}_1, \ldots, \mathcal{F}_r \subset 2^{[n]}$ are non-empty, cross k-intersecting families. Then*

$$\sum_{1 \le i \le r} \big|\mathcal{F}_i\big| \le 2^n - (2^k - r). \tag{3.5}$$

Proof. Simply define $\mathcal{F}_j = \{[n]\}$ for $r < j \le 2^k$ and apply (3.4) to the 2^k families $\mathcal{F}_j$, $1 \le j \le 2^k$. □

Proof. Now we return to the proof of Theorem 11. The case $r = 2q$, $q \ge 2$ is relatively easy. We do not even need Step 1.

Let us consider two collections of $2q$ families each:

$$\mathcal{F}_1(1), \ldots, \mathcal{F}_q(1), \mathcal{F}_{q+1}(\bar{1}), \ldots, \mathcal{F}_{2q}(\bar{1}) \subset 2^{[2,n]},$$
$$\mathcal{F}_1(\bar{1}), \ldots, \mathcal{F}_q(\bar{1}), \mathcal{F}_{q+1}(1), \ldots, \mathcal{F}_{2q}(1) \subset 2^{[2,n]}.$$

Applying Lemma 8 with $\ell = 1$ and q of the $B_i = [1]$ and the remaining q being $B_i = \emptyset$ shows that both are collections of non-empty, cross q-intersecting families. Since $2^q \ge q$, we can apply Corollary 18 to both collections:

$$\sum_{1 \le i \le q} \big|\mathcal{F}_i(1)\big| + \sum_{q < j \le 2q} \big|\mathcal{F}_j(\bar{1})\big| \le 2^{n-1}$$

and

$$\sum_{1 \le i \le q} \big|\mathcal{F}_i(\bar{1})\big| + \sum_{q < j \le 2q} \big|\mathcal{F}_j(1)\big| \le 2^{n-1}$$

(with strict inequality unless $q = 2$). Adding the above two inequalities proves Theorem 11 for $r = 2q$, $q \ge 2$.

For $r = 2q + 1$, $q \ge 3$ the same argument works because of $2^q > 2q + 1$. Indeed, even for the 'worse' collection $\mathcal{F}_1(\bar{1}), \ldots, \mathcal{F}_q(\bar{1})$, $\mathcal{F}_{q+1}(1), \ldots, \mathcal{F}_{2q+1}(1)$ we get the cross q-intersecting property and may apply (3.5). Thus only two cases, $r = 3$ and $r = 5$ remain. We deal with them separately.

Step 2, $r = 3$.

It is here that we shall also use the induction hypothesis in its strongest form. Namely, we suppose that Theorem 14 holds for $n - 2$.

We distinguish two subcases.

(i) $\mathcal{F}_i(\bar{1})$ is trivial, $i = 1, 2, 3$.

This means that $\mathcal{F}_i(\emptyset, [2]) = \emptyset$ for $i = 1, 2, 3$.

Let us consider the $3 \times 3 = 9$ families:

$$\mathcal{F}_1([2],[2]),\ \mathcal{F}_2(\{1\},[2]),\ \mathcal{F}_3(\{2\},[2]);$$
$$\mathcal{F}_2([2],[2]),\ \mathcal{F}_3(\{1\},[2]),\ \mathcal{F}_1(\{2\},[2]);$$
$$\mathcal{F}_3([2],[2]),\ \mathcal{F}_1(\{1\},[2]),\ \mathcal{F}_2(\{2\},[2]).$$

Since $[2] \cap \{2\} \cap \{1\} = \emptyset$, these are cross-intersecting on $[3,n]$ and $\mathcal{F}_i([2],[2])$ is non-trivial for $i = 1,2,3$. Applying Theorem 14 (i) with $r = 3$ to each of the three triples of families shows that the sum of their sizes is at most $\frac{5}{4} \cdot 2^{n-2}$. Recalling $\mathcal{F}_i(\emptyset,[2]) = \emptyset$ for $i = 1,2,3$, we infer

$$|\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3| \leq 3 \cdot \frac{5}{4} 2^{n-2} < 2^n, \text{ as desired.}$$

(ii) $\mathcal{F}_1(\bar{1})$ is non-trivial.

We may apply the induction hypothesis (Theorem 11 with n replaced by $n-1$) to $\mathcal{F}_1(\bar{1}), \mathcal{F}_2(1), \mathcal{F}_3(1) \subset 2^{[2,n]}$ and infer

$$|\mathcal{F}_1(\bar{1})| + |\mathcal{F}_2(1)| + |\mathcal{F}_3(1)| < 2^{n-1}.$$

On the other hand, by Lemma 8 the families $\mathcal{F}_1(1)$, $\mathcal{F}_2(\bar{1})$, $\mathcal{F}_3(\bar{1})$ are cross 2-intersecting on $[2,n]$. Now Corollary 18 implies

$$|\mathcal{F}_1(1)| + |\mathcal{F}_2(\bar{1})| + |\mathcal{F}_3(\bar{1})| < 2^{n-1}.$$

Adding these two inequalities yields

$$|\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3| < 2 \cdot 2^{n-1} = 2^n, \quad \text{as desired.}$$

Step 2, $r = 5$. We distinguish two cases again

(i) $\mathcal{F}_1(\bar{1})$ is trivial, i.e., $\mathcal{F}_1(\emptyset,[2]) = \emptyset$.

In this case $\mathcal{F}_i(\emptyset,[2]) = \emptyset$ for $1 \leq i \leq 5$. Therefore $\Sigma|\mathcal{F}_i|$ is the sum of the sizes of the following 15 families: $\mathcal{F}_i(B) := \mathcal{F}_i(B,[2])$, $1 \leq i \leq 5$, $B = \{1\},\{2\}$ or $[2]$. Divide them into the following 3 groups.

$$\mathcal{F}_1([2]), \mathcal{F}_2(\{1\}), \mathcal{F}_3(\{1\}), \mathcal{F}_4(\{2\}), \mathcal{F}_5(\{2\});$$
$$\mathcal{F}_1(\{2\}), \mathcal{F}_2([2]), \mathcal{F}_3([2]), \mathcal{F}_4(\{1\}), \mathcal{F}_5(\{1\});$$
$$\mathcal{F}_1(\{1\}), \mathcal{F}_2(\{2\}), \mathcal{F}_3(\{2\}), \mathcal{F}_4([2]), \mathcal{F}_5([2]).$$

In view of Lemma 8 families of the first group are cross 3-intersecting on $[3,n]$. By the induction hypothesis their sizes add up to less than 2^{n-2}.

The remaining two groups are cross-intersecting and each of them contains two non-trivial families, $\mathcal{F}_i([2])$ ($i = 2, 3$ and $i = 4, 5$). By the induction hypothesis for Theorem 14 (ii), the sum of their sizes is at most $\left(1 + \frac{5-2}{8}\right) 2^{n-2}$, each. Thus the sum of the sizes of the 15 families is less than $3\frac{3}{4} \cdot 2^{n-2} < 2^n$ as desired.

(ii) $\mathcal{F}_1(\bar{1})$ is non-trivial

Since $\mathcal{F}_1(\bar{1}),\ \mathcal{F}_2(1), \ldots, \mathcal{F}_5(1)$ are non-trivial cross-intersecting, the induction hypothesis yields

$$\left|\mathcal{F}_1(\bar{1})\right| + \sum_{2 \le i \le 5} \left|\mathcal{F}_2(1)\right| < 2^{n-1}.$$

On the other hand by Lemma 8 the five families $\mathcal{F}_1(1)$ and $\mathcal{F}_j(\bar{1})$, $2 \le j \le 5$ are cross 4-intersecting on $[2, n]$. By Corollary 18:

$$\left|\mathcal{F}_1(1)\right| + \sum_{2 \le j \le 5} \left|\mathcal{F}_j(\bar{1})\right| < 2^{n-1}.$$

Adding the above two inequalities proves

$$\sum_{1 \le i \le 5} \left|\mathcal{F}_i\right| < 2^n \quad \text{as desired.} \qquad \square$$

4 Proof of Theorem 4 (i)

We prove Theorem 4 also by induction. In this section we assume that both (i) and (ii) hold for lesser values of n and prove the validity of (i). Let us note that for $4 \le n \le 6$ the only 3-wise 4-intersecting families are those containing 4 fixed elements of $[n]$. Also, for $n = 7$ it is easy to check that the only exception is $\mathcal{A}_1(7, 4) = \{A \subset [7] : |A| \ge 6\}$. Let $n \ge 8$ and assume that $\mathcal{F} \subset 2^{[n]}$ is a shifted filter. Moreover,

$$|\mathcal{F}| \ge 2^{n-4}. \tag{4.0}$$

Since $\left|\mathcal{A}_0(n, 4)\right| = \left|\mathcal{A}_1(n, 4)\right| = 2^{n-4}$ we may assume that $\mathcal{F} \not\subset \mathcal{A}_0(n, 4)$ and $\mathcal{F} \not\subset \mathcal{A}_1(n, 4)$ hold. By shiftedness the second implies

$$\mathcal{F}([5], [7]) \ne \emptyset. \tag{4.1}$$

From the first one we infer

$$\text{(4.2)} \qquad \mathcal{F} \ \text{ is non-trivial.}$$

Indeed, if $\mathcal{F}(\bar{1}) = \emptyset$ then $|\mathcal{F}| = |\mathcal{F}(1)|$ and $\mathcal{F}(1)$ is a 3-wise 3-intersecting family on $[2,n]$. Thus $|\mathcal{F}| = |\mathcal{F}(1)| \leq 2^{(n-1)-3} = 2^{n-4}$ follows.

One can say a little more:

$$\text{(4.3)} \qquad \mathcal{F}(1) \ \text{ is non-trivial.}$$

Indeed, by (4.2) one has $\big([n] \setminus \{i\}\big) \in \mathcal{F}$ for all $i \in [n]$. Consequently $\big([2,n] \setminus \{i\}\big) \in \mathcal{F}(1)$ for all $i \in [2,n]$. The induction hypothesis implies

$$\text{(4.4)} \qquad |\mathcal{F}(1)| \leq \frac{7}{8} 2^{n-4}.$$

This and (4.0) entail

$$\text{(4.5)} \qquad \big|\mathcal{F}(\bar{1})\big| > 2^{n-7}.$$

Note that Lemma 8 implies that $\mathcal{F}(\bar{1})$ is 3-wise 6-intersecting.

$$\text{(4.6)} \qquad [2,7] \notin \mathcal{F}(\bar{1}).$$

Indeed, if $[2,7] \in \mathcal{F}(\bar{1}) \subset \mathcal{F}$ then $\big([7] \setminus \{i\}\big) \in \mathcal{F}$ for all $i \in [7]$ follows by shiftedness. This easily implies $\mathcal{F} \subset \mathcal{A}_1(n,4)$, a contradiction.

Combining with (4.5) we conclude that there is some $F \in \mathcal{F}(\bar{1})$ with $[2,7] \not\subset F$. Since $\mathcal{F}$ is a shifted filter, $([2,6] \cup [8,n]) \in \mathcal{F}$. Thus:

$$\text{(4.7)} \qquad \mathcal{F}([2,6],[6]) \ \text{ is non-trivial.}$$

To simplify notation we set $\mathcal{F}(B) = \mathcal{F}(B,[6])$. By shiftedness for every $B \in \binom{[6]}{5}$ one has

$$\mathcal{F}(B) \supset \mathcal{F}([2,6]).$$

Note that if $B_1, B_2, B_3 \in \binom{[6]}{5}$ are distinct then $\big|B_1 \cap B_2 \cap B_3\big| = 3$. Together with (4.7) this implies

(4.8) Let $B_1, B_2, B_3 \in \binom{[6]}{5}$ be distinct.

Then $\mathcal{F}(B_1), \mathcal{F}(B_2), \mathcal{F}(B_3)$ are non-trivial and cross-intersecting.

The following lemma is central for our estimations.

Lemma 19. *Suppose that $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_6 \subset 2^{[m]}$ are non-empty non-trivial families with any three of them cross-intersecting. Then*

$$\sum_{1 \le i \le 6} \left|\mathcal{G}_i\right| / 2^m \le 6 \times \frac{5}{16} = \frac{15}{8}. \tag{4.9}$$

Proof. We want to use Lemma 16. In order to do that we suppose by symmetry $\left|\mathcal{G}_1\right| \ge \dots \ge \left|\mathcal{G}_6\right|$ and denote $\left|\mathcal{G}_i\right| \,/\, 2^{m-4}$, $i = 1, 2, \dots, 6$ in this order by a, b, c, d, e, f.

Since $\mathcal{G}_3$ is non-empty and $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ are cross-intersecting $\left|\mathcal{G}_1\right| + \left|\mathcal{G}_2\right| \le 2^m$, i.e., $a + b \le 16$ follows.

Also, (2.4) implies $abc \le 125$. Now (i) of Lemma 16 implies $a + b + c + d + e + f \le 30$ (with equality iff all are equal to 5), proving (4.9). □

Remark. Let us mention that Lemma 16 (ii) implies that if we knew that $|\mathcal{G}_1| \le 2^{n-1}$, then we could arrive at the analogous conclusion for four families (instead of six). We do not use this fact in the present paper but it might be useful in other situations.

Let us first conclude the proof in the case that $\mathcal{F}([6])$ is *not* intersecting.

This assumption implies the existence of $G, H \in \mathcal{F}$ with $G \cap H = [6]$. Thus $\left|F \cap [6]\right| \ge 4$ for all $F \in \mathcal{F}$.

Since $\mathcal{F} \not\subset \mathcal{A}_0(n, 4)$, $\mathcal{F}([6])$ and $\mathcal{F}([4])$ are cross-intersecting. This yields

$$|\mathcal{F}([6])| + |\mathcal{F}([4])| \le 2^{n-6}. \tag{4.10}$$

For $B \subset \binom{[6]}{4}$, $B \ne [4]$, Lemma 8 implies that $\mathcal{F}(B)$ is 3-wise 4-intersecting. Thus by the induction hypothesis

$$|\mathcal{F}(B)| \le 2^{n-10} \text{ for } B \in \binom{[6]}{4}, \quad B \ne [4]. \tag{4.11}$$

Adding (4.9), (4.10) and (4.11) yields

$$|\mathcal{F}|/2^{n-6} \le \frac{15}{8} + 1 + \frac{14}{16} = 3\frac{3}{4} < 4, \quad \text{as desired.} \tag{4.12}$$

Consequently, we may suppose that $\mathcal{F}([6])$ is intersecting, implying

$$|\mathcal{F}([6])| \le \frac{1}{2} \cdot 2^{n-6}. \tag{4.13}$$

Now we distinguish two cases according whether $|\mathcal{F}([4])| \,/\, 2^{n-6}$ is smaller than 1/4 or not. In the first case (4.10) is improved to

$$|\mathcal{F}([6])| + |\mathcal{F}([4])| < \frac{3}{4} 2^{n-6}. \tag{4.14}$$

Instead of (4.12) we obtain

$$\sum_{\substack{B \subset [6] \\ |B| \leq 4}} \mathcal{F}(B) \,/\, 2^{n-6} < 3\frac{1}{2}.$$

To conclude the proof we need

$$\sum_{\substack{B \subset [6] \\ |B| \leq 3}} |\mathcal{F}(B)| \,/\, 2^{n-6} \leq \frac{1}{2}. \tag{4.15}$$

This will be easy but we need to invoke a simple result from [F91].

Definition 20. For integers $n \geq t \geq 1$ define

$$m(n,t) = \max\big\{|\mathcal{H}| : \mathcal{H} \subset 2^{[n]},\ \mathcal{H} \text{ is 3-wise } t\text{-intersecting}\big\}.$$

In [F91] we proved the following inequality:

$$m(n, t+1) < \big(\sqrt{5} - 1\big) m(n,t)/2. \tag{4.16}$$

Using the induction hypothesis in the form $m(n,4) \,/\, 2^n \leq \frac{1}{16}$ we infer

$$m(n, 4+s) \,/\, 2^n < \left(\frac{\sqrt{s} - 1}{2}\right)^s \cdot \frac{1}{16}. \tag{4.17}$$

Using (4.17) (with n replaced by $n-6$) for $B \subset [6]$, $|B| \leq 3$ and invoking Lemma 8 we get the following upper bounds:

$$|\mathcal{F}(B)| \,/\, 2^{n-6} \leq \left(\frac{\sqrt{5}-1}{2}\right)^3 \cdot \frac{1}{16} < 0.015 \text{ for } B \in \binom{[6]}{3}, \tag{4.18}$$

$$|\mathcal{F}(B)| \,/\, 2^{n-6} \leq \left(\frac{\sqrt{5}-1}{2}\right)^6 \cdot \frac{1}{16} < 0.004 \text{ for } B \in \binom{[6]}{2}, \tag{4.19}$$

$$(4.20) \qquad |\mathcal{F}(B)| \,/\, 2^{n-6} \leq \left(\frac{\sqrt{5}-1}{2}\right)^9 \cdot \frac{1}{16} < 0.001 \text{ for } B \in \binom{[6]}{1},$$

$$(4.21) \qquad |\mathcal{F}(\emptyset)| \,/\, 2^{n-6} \leq \left(\frac{\sqrt{5}-1}{2}\right)^{12} \cdot \frac{1}{16} < 0.0003.$$

The above bounds are easy to compute by hand. Just use $\left(\frac{\sqrt{5}-1}{2}\right)^3 = \sqrt{5} - 2 < \frac{1}{4}$.

Since $20 \times 0.015 + (15+6+1) \times 0.004 = 0.388 < 0.5$, $|\mathcal{F}| < 2^{n-4}$ is proved if (4.14) holds.

In the last case we use $|\mathcal{F}([4])| \,/\, 2^{n-6} \geq \frac{1}{4}$ to improve on $|\mathcal{F}(B)|$ for $B \in \binom{[6]}{4}$, $B \neq [4]$.

In view of Lemma 8 the three families $\mathcal{F}([4]), \mathcal{F}([4]), \mathcal{F}(B)$ are cross 4-intersecting.

For this case in [F91] we proved

$$|\mathcal{F}([4])| \cdot |\mathcal{F}([4])| \cdot |\mathcal{F}(B)| \leq \left(2^{n-6} \cdot \frac{\sqrt{5}-1}{16}\right)^3.$$

Using $|\mathcal{F}([4])| \geq 2^{n-6} \,/\, 4$ we infer

$$|\mathcal{F}(B)| \,/\, 2^{n-6} < \frac{\sqrt{5}-2}{32} < \frac{1}{128}.$$

There are 14 subsets $B \in \binom{[6]}{4}$, $B \neq [4]$. Their contribution was $\frac{14}{16} = \frac{7}{8}$ before. Now it is less than $\frac{14}{128} = \frac{7}{64} < 0.11$.

Together with the sets $B \subset [6]$, $|B| \leq 3$ it is still $0.11 + 0.388 < 0.5 < \frac{7}{8}$ completing the proof of $|\mathcal{F}| < 4 \cdot 2^{n-6} = 2^{n-4}$. $\square$ $\square$

Let us mention that the proof actually gives that (among shifted families) the only families achieving equality are $\mathcal{A}_0(n,4)$ and $\mathcal{A}_1(n,4)$. One can even extract the upper bound $|\mathcal{F}| \leq 2^{n-4} \cdot \frac{15}{16}$ assuming that $\mathcal{F} \not\subset \mathcal{A}_0(n,4)$, $\mathcal{F} \not\subset \mathcal{A}_1(n,4)$.

5 The proof of Theorem 4 (ii) and (iii)

Let us start with the easy proof of (iii).

Let $\mathcal{F} \subset 2^{[n]}$, $\binom{[n]}{n-1} \subset \mathcal{F}$ and $\mathcal{F}$ is 3-wise 2-intersecting. Now Lemma 8 implies that $\mathcal{F}(\bar{1})$ is 3-wise 4-intersecting. By the induction hypothesis $\left|\mathcal{F}(\bar{1})\right| \le 2^{n-1} \cdot \frac{1}{16}$.

Since $[n] \setminus \{2\}$ is in $\mathcal{F}$, $\mathcal{F}(1)$ is a non-trivial 3-wise intersecting family. By the Brace–Daykin Theorem $|\mathcal{F}(1)| \le 2^{n-1} \cdot \frac{5}{16}$.

We infer

$$|\mathcal{F}| = |\mathcal{F}(1)| + \left|\mathcal{F}(\bar{1})\right| = 2^{n-1} \cdot \frac{6}{16} = 2^{n-2} \cdot \frac{3}{4} \quad \text{as desired.} \qquad \square$$

Using Theorem 5 it follows that in case of equality $\mathcal{F}(1) = \big\{G \subset [2,n] : |G \cap [2,5]| \ge 3\big\}$. This in turn easily implies $\mathcal{F} = \big\{F \subset [n] : |F \cap [5]| \ge 4\big\}$.

Now we turn to the proof of (ii). For the rest of this section let $\mathcal{F} \subset 2^{[n]}$ be non-trivial 3-wise 3-intersecting. For $B \subset [5]$ define $\mathcal{F}(B) = \big\{A \subset [6,n] : A \cup B \in \mathcal{F}\big\}$. Of course,

$$|\mathcal{F}| = \sum_{B \subset [5]} |\mathcal{F}(B)|. \tag{5.0}$$

Define $D_j = [5] \setminus \{j\}$, $j = 1, \ldots, 5$. Note that $[4] = D_5$. Note also that shiftedness implies $\mathcal{F}(D_1) \subset \mathcal{F}(D_2) \subset \mathcal{F}(D_3) \subset \mathcal{F}(D_4) \subset \mathcal{F}(D_5)$. For $1 \le a < b < c \le 5$, $\left|D_a \cap D_b \cap D_c\right| = 2$, implying that

$$\mathcal{F}(D_a),\ \mathcal{F}(D_b),\ \mathcal{F}(D_c) \ \text{ are cross-intersecting.} \tag{5.1}$$

Let us first prove that (ii) holds if $\mathcal{F}(D_1)$ is trivial.

If $\mathcal{F}(\{2,3,4,5\})$ is trivial then $\big([2,5] \cup [7,n]\big) \notin \mathcal{F}$. Using shiftedness $[2,6] \subset F$ follows for all $F \in \mathcal{F}(\bar{1})$. Indeed, the contrary means the existence of $F \in \mathcal{F}(\bar{1})$ with $i \notin F$ for some $2 \le i \le 6$. Shifting 6 to i we get a subset of $[2,5] \cup [7,n]$ in $\mathcal{F}$. Now $[2,6] \subset F$ for all $F \in \mathcal{F}(\bar{1})$ yields $\left|\mathcal{F}(\bar{1})\right| \le 2^{n-6} \le \frac{1}{8} 2^{n-3}$.

Since $\mathcal{F}(1)$ is non-trivial, 3-wise 2-intersecting, (iii) gives

$$|\mathcal{F}(1)| \le \frac{3}{4} 2^{n-3}.$$

Thus

$$|\mathcal{F}| = \left|\mathcal{F}(\bar{1})\right| + \left|\mathcal{F}(1)\right| \le \frac{7}{8} \cdot 2^{n-3}, \quad \text{as desired.}$$

From now on we may assume that $\mathcal{F}(D_1)$ is non-trivial. Moreover, $\mathcal{F}(D_1) \subset \mathcal{F}(D_i)$ for $2 \le i \le 5$ implies that $\mathcal{F}(D_i)$ is non-trivial for all $1 \le i \le 5$.

Using Theorem 11 for $r = 3$ gives

$$\big|\mathcal{F}(D_3)\big| + \big|\mathcal{F}(D_4)\big| + \big|\mathcal{F}(D_5)\big| \le 2^{n-5}. \tag{5.2}$$

In view of (5.1), $\mathcal{D}(1)$ and $\mathcal{D}(2)$ are both 3-wise intersecting. Since they are non-trivial,

$$\big|\mathcal{F}(D_i)\big| \le \frac{5}{16} 2^{n-5}, \quad i = 1, 2 \tag{5.3}$$

follows from the Brace–Daykin Theorem.

Let us prove next

$$\big|\mathcal{F}([3])\big| + \big|\mathcal{F}([5])\big| \le 2^{n-5}. \tag{5.4}$$

Indeed, the contrary would force the existence of $G \in \mathcal{F}([3])$ and $H \in \mathcal{F}([5])$ satisfying $G \cap H = \emptyset$. This would entail $[3] \subset F$ for all $F \in \mathcal{F}$, contradicting non-triviality.

For $B \subset [5]$, $B \neq [3]$ using Lemma 8 we infer that $\mathcal{F}(B)$ is 3-wise 4-intersecting. By (ii),

$$|\mathcal{F}(B)| \le 2^{n-5} \cdot \frac{1}{16}. \tag{5.5}$$

Let now $B \in \binom{[5]}{2}$. Then Lemma 8 implies that $\mathcal{F}(B)$ is 3-wise 7-intersecting. Thus

$$|\mathcal{F}(B)| < 2^{n-5} \left(\frac{\sqrt{5}-1}{2}\right)^3 \cdot \frac{1}{16} < 2^{n-5} \cdot \frac{1}{64}. \tag{5.6}$$

Since $\mathcal{F}(B') \subset \mathcal{F}(B)$ is obvious for all $B' \subset B$, we may use (5.6) for all $\binom{5}{2} + \binom{5}{1} + \binom{5}{5} = 16$ sets $B \subset [5]$, $|B| \le 2$. Together with (5.0), (5.2), (5.3), (5.4), (5.5) we obtain

$$|\mathcal{F}| \le 2^{n-5} \left(1 + \frac{10}{16} + 1 + \frac{9}{16} + \frac{4}{16}\right) < 2^{n-3}. \quad \square \quad \square$$

Now the proof of Theorem 4 is complete.

6 The proof of Theorem 5 in the case $r = 3$

Suppose that $\mathcal{F} \subset 2^{[n]}$ is 3-wise intersecting, non-trivial, $\mathcal{F} \not\subset \mathcal{A}_1(n,3)$. For $A \subset [2]$ define $\mathcal{F}(A) = \big\{B \subset [3,n] : A \cup B \in \mathcal{F}\big\}$. Obviously,

$$|\mathcal{F}([2])| + |\mathcal{F}(\{1\})| + |\mathcal{F}(\{2\})| + |\mathcal{F}(\emptyset)| = |\mathcal{F}|. \tag{6.0}$$

Let us first consider the case when $\mathcal{F}(\emptyset) \neq \emptyset$. Note that this implies that $\mathcal{F}([2])$ is intersecting, whence

$$|\mathcal{F}([2])| \le \frac{1}{2}2^{n-2} = 2^{n-3}. \tag{6.1}$$

In view of Lemma 8, $\mathcal{F}(\{1\})$, $\mathcal{F}(\{1\})$, $\mathcal{F}(\{2\})$ are cross 2-intersecting. In view of (2.2),

$$\big(|\mathcal{F}(\{1\})| \,/\, 2^{n-2}\big)^2 \cdot |\mathcal{F}(\{2\})| \,/\, 2^{n-2} \le \left(\frac{1}{4}\right)^3. \tag{6.2}$$

We want to use this to prove

$$|\mathcal{F}(\{1\})| + |\mathcal{F}(\{2\})| \le \frac{9}{16}2^{n-2}. \tag{6.3}$$

Set $b = 16 \cdot |\mathcal{F}(\{1\})| \,/\, 2^{n-2}$, $c = 16 \cdot |\mathcal{F}(\{2\})| \,/\, 2^{n-2}$. Then (6.2) transforms to $b^2c \le 4^3 = 64$. From (6.1) and $\mathcal{F}(\{2\}) \subset \mathcal{F}(\{1\}) \subset \mathcal{F}([2])$, $c \le b \le 8$ follows.

We need a simple, analytical inequality.

Lemma 21. *Let $b \ge c$ be positive real numbers satisfying $b \le 8$ and*

$$b^2c \le 64. \tag{6.4}$$

Then

$$b + c \le 9 \quad \textit{with equality iff} \ \ b = 8,\ c = 1. \tag{6.5}$$

Proof. First note that $b \le 4$ would imply $b+c \le 8$. If $b = 8$ then (6.4) implies $c \le 1$, yielding (6.5). Thus we may assume that $4 < b < 8$. Should (6.4) fail for some pair (b,c) then we can decrease c to $9 - b$ while maintaining (6.4). However,

$$b^2(9-b) - 64 = (8-b)(b^2 - b - 8) \quad \text{is strictly positive}$$

in the range $4 < b < 8$, a contradiction. □

Now (6.3) follows from (6.5). In view of Lemma 8 the family $\mathcal{F}(\emptyset)$ is 3-wise 5-intersecting. Using Theorem 4 (i) and (4.12) we infer

$$|\mathcal{F}(\emptyset)| \leq 2^{n-2} \cdot \frac{1}{16} \cdot \frac{\sqrt{5}-1}{2} < 2^{n-2}/16. \tag{6.6}$$

Together with (6.0), (6.1) and (6.3) we infer

$$|\mathcal{F}| < 2^{n-2}\left(\frac{1}{2} + \frac{9}{16} + \frac{1}{16}\right) = 2^n \cdot \left(\frac{1}{4} + \frac{1}{32}\right) \quad \text{as desired.}$$

Let now $\mathcal{F}(\emptyset) = \emptyset$, that is, $F \cap [2] \neq \emptyset$ for all $F \in \mathcal{F}$. Let us mention that both $\mathcal{A}_0(n,1)$ and $\mathcal{A}_1(n,1)$ have this property.

Since $[2] \cap \{1\} \cap \{2\} = \emptyset$, the three families $\mathcal{F}([2])$, $\mathcal{F}(\{1\})$, $\mathcal{F}(\{2\})$ are cross-intersecting. Note that $\mathcal{F} \not\subset \mathcal{A}_0(n,1)$ implies $\mathcal{F}(\{2\}) \neq \emptyset$. This in turn implies that $\mathcal{F}([2])$ is non-trivial.

If $\mathcal{F}(\{1\})$ is non-trivial, then Theorem 14 (ii) yields

$$|\mathcal{F}| = |\mathcal{F}([2])| + |\mathcal{F}(\{1\})| + |\mathcal{F}(\{2\})| \leq \frac{9}{8} \cdot 2^{n-2} = \left(\frac{1}{4} + \frac{1}{32}\right) 2^n, \quad \text{as desired.}$$

Thus we may assume that $\mathcal{F}(\{1\})$ is trivial. Since $\mathcal{F}(\{2\}) \subset \mathcal{F}(\{1\})$, $\mathcal{F}(\{2\})$ is trivial as well. Consequently, $\left|F \cap [3]\right| \geq 2$ for all $F \in \mathcal{F}$.

Now we switch from [2] to [3] and define

$$\mathcal{D}_i = \{D \subset [4,n] : (D \cup [3] \setminus \{i\}) \in \mathcal{F}\}, \quad i = 1,2,3.$$

Set also

$$\mathcal{D}_0 = \{D \subset [4,n] : D \cup [3] \in \mathcal{F}\}.$$

Since $\{1,2\} \cap \{1,3\} \cap \{2,3\} = \emptyset$, $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ are cross-intersecting. $\mathcal{F} \not\subset \mathcal{A}_0(n,1)$ implies that none of them is empty. If all were trivial, shiftedness would imply $4 \in D$ for all $D \in \mathcal{D}_i$, $i = 1,2,3$. Thereby it would follow that $|F \cap [4]| \geq 3$ for all $F \in \mathcal{F}$. That is, $\mathcal{F} \subset \mathcal{A}_1(n,1)$, a contradiction.

Consequently, we may assume that $\mathcal{D}_3$ is non-trivial. Applying Theorem 14 (i) and $\left|\mathcal{D}_0\right| \leq 2^{n-3}$ we deduce

$$|\mathcal{F}| = \left|\mathcal{D}_0\right| + \left|\mathcal{D}_1\right| + \left|\mathcal{D}_2\right| + \left|\mathcal{D}_3\right| \leq 2^{n-3} + \frac{5}{4}2^{n-3} = 2^{n-2} + 2^{n-5},$$

concluding the proof of Theorem 5. □

7 Doubly non-trivial r-wise intersecting families

Let $r \geq 3$ be an integer, $n \geq r+2$ and define

$$\mathcal{G}(n,r) = \{G \subset [n] : G \cap [r+2] = [r+2] \setminus \{i\},\ 1 \leq i < r\},$$
$$\mathcal{H}(n,r) = \{H \subset [n] : [r-1] \subset H,\ H \cap \{r, r+1, r+2\} \neq \emptyset\}.$$

It is easy to check that $\mathcal{G}(n,r) \cap \mathcal{H}(n,r) = \emptyset$, $|\mathcal{G}(n,r)| = \frac{r-1}{2^{r+2}} \cdot 2^n$, $|\mathcal{H}(n,r)| = \frac{7}{2^{r+2}} \cdot 2^n$ and $\mathcal{D}(n,r) := \mathcal{G}(n,r) \cup \mathcal{H}(n,r)$ is r-wise intersecting with $\mathcal{D}(n,r) \not\subset \mathcal{A}_i(n,r,1)$ for $i = 0, 1$. (We call such a family doubly non-trivial.)

With this terminology Theorem 5 can be stated as:

Theorem 22. *Suppose that n, r are integers, $n \geq r+2 \geq 5$. Let $\mathcal{F} \subset 2^{[n]}$ be doubly non-trivial and r-wise intersecting. Then*

$$|\mathcal{F}| \leq |\mathcal{D}(n,r)| = \frac{r+6}{2^{r+2}} \cdot 2^n. \tag{7.1}$$

Proof. We proved the case $r = 3$ in the preceding section. We prove (7.1) by applying induction on r. Let $r \geq 4$ and suppose that (7.1) holds for $r-1$.

Claim 23. $\mathcal{F}(1) \subset 2^{[2,n]}$ *is doubly non-trivial and $(r-1)$-wise intersecting.*

Proof of the claim. If $G_1, \ldots, G_{r-1} \in \mathcal{F}(1)$ satisfy $G_1 \cap \ldots \cap G_{r-1} = \emptyset$ then $G_i \cup \{1\} \in \mathcal{F}$ for $i = 1, \ldots, r-1$ and these $r-1$ sets intersect in $\{1\}$. Thus $1 \in F$ holds for all $F \in \mathcal{F}$. This contradicts the assumption that $\mathcal{F}$ is (doubly) non-trivial. Since $\binom{[n]}{n-1} \subset \mathcal{F}$, $\binom{[2,n]}{n-2} \subset \mathcal{F}(1)$ is immediate.

Similarly, $\mathcal{F} \not\subset \mathcal{A}_1(n,1)$ implies the existence of $\widetilde{F} \in \mathcal{F}$ satisfying $\widetilde{F} \cap [r+1] = [r-1]$. Now $\widetilde{F} \setminus \{1\} \in \mathcal{F}(1)$ and $\left(\widetilde{F} \setminus \{1\}\right) \cap [2, r+1] = [2, r-1]$ concludes the proof of $\mathcal{F}(1)$ being doubly non-trivial. $\square$

Applying the induction hypothesis to $\mathcal{F}(1)$ yields

$$|\mathcal{F}(1)| \leq \frac{(r-1)+6}{2^{r+1}} \cdot 2^{n-1} = \frac{r+5}{2^{r+2}} \cdot 2^n. \tag{7.2}$$

In order to prove (7.1) it is sufficient to show

$$\left|\mathcal{F}(\bar{1})\right| \leq 2^{n-r-2}. \tag{7.3}$$

Proof of (7.3). Let us first prove that for $2 \le s \le r$, $\mathcal{F}(\bar{1})$ is s-wise $(2r-s)$-intersecting.

For $s = r$ it follows from Lemma 8 (and the fact that $\mathcal{F}(\bar{1}) \subset \mathcal{F}$ is r-wise intersecting). Arguing indirectly, let s be the largest integer such that $\mathcal{F}(\bar{1})$ is not $(2r-s)$-intersecting. Using shiftedness there exist $F_1, \dots, F_s \in \mathcal{F}(\bar{1})$ such that $F_1 \cap \dots \cap F_s = [2, 2r-s]$.

By the maximal choice of s, $\mathcal{F}(\bar{1})$ is $(s+1)$-wise $(2r-s-1)$-intersecting. Consequently $[2, 2r-s] \subset F$ for all $F \in \mathcal{F}(\bar{1})$. For $s \le r-2$ this implies directly that $\left|\mathcal{F}(\bar{1})\right| < 2^{n-r-2}$, i.e., (7.3) holds.

Let $s = r-1$. Define $H_1 = F_1$, $H_i = \big(F_i \setminus \{i\}\big) \cup \{1\}$ for $2 \le i \le r-1$ and $H_r = \widetilde{F}$ (from the proof of Claim 23). Since $H_1 \cap \dots \cap H_r = \emptyset$, we have a contradiction.

For $r \ge 5$ the inequality (7.3) follows from $m(n-1, r-1, r+1) = 2^{(n-1)-(r+1)} = 2^{n-r-2}$.

To prove (7.3) in the last case, $r = 4$ we have to work harder. We distinguish two cases according as $\mathcal{F}(\emptyset, [2])$ is empty or not.

If $\mathcal{F}(\emptyset, [2]) = \emptyset$ then $\left|\mathcal{F}(\bar{1})\right| = \left|\mathcal{F}(\{2\}, [2])\right|$. Since $\mathcal{F}(\bar{1})$ is 3-wise 5-intersecting, $\mathcal{F}(\{2\}, [2])$ is 3-wise 4-intersecting, yielding

$$\left|\mathcal{F}(\bar{1})\right| \le 2^{(n-2)-4} = 2^{n-6} \quad \text{as desired.}$$

Let us suppose now that $\mathcal{F}(\emptyset, [2]) \ne \emptyset$ and forget about (7.3).

Claim 24. $\mathcal{F}([2], [2])$ *is doubly non-trivial* 3*-wise intersecting.*

Proof of the claim. Since $\mathcal{F}(\emptyset, [2]) \ne \emptyset$, $[3, n] \in \mathcal{F}$. By shiftedness $\{1, 2\} \cup [5, n]$ is also in $\mathcal{F}$, i.e., $[5, n] \in \mathcal{F}([2], [2])$, proving doubly non-triviality. Also, $[3, n] \in \mathcal{F}$ implies that $\mathcal{F}([2], [2])$ is 3-wise intersecting. □

Using the case $r = 3$ we obtain

$$|\mathcal{F}([2], [2])| \le \left(\frac{1}{4} + \frac{1}{32}\right) 2^{n-2} = \frac{4.5}{64} 2^n. \tag{7.4}$$

Note that $\mathcal{F}(\emptyset, [2]) \subset \mathcal{F}(\{2\}, [2]) \subset \mathcal{F}(\{1\}, [2])$. Let us show that $\mathcal{F}(\{1\}, [2])$ is 3-wise 4-intersecting. Suppose the contrary and take $G_1, G_2, G_3 \in \mathcal{F}(\{1\}, [2])$ so that $G_1 \cap G_2 \cap G_3 = \{3, 4, 5\}$. By shiftedness $F_i = \big(G_i \setminus \{i\}\big) \cup \{1, 2\}$ is in $\mathcal{F}$ for $i = 3, 4, 5$. However, $[3, n] \cap F_3 \cap F_4 \cap F_5 = \emptyset$, a contradiction.

Using Theorem 4 we infer

$$\left|\mathcal{F}(\{1\}, [2])\right| \le 2^{n-6}. \tag{7.5}$$

Consequently

$$|\mathcal{F}| \le \big|\mathcal{F}([2],[2])\big| + 3\big|\mathcal{F}(\{1\},[2])\big| \le \frac{4.5+3}{64}2^n < \frac{10}{64}\cdot 2^n$$

completing the proof. □

8 Shifted versus non-shifted

As a matter of fact, the proof that we have given for Theorem 22 is only for shifted families.

The problem is that shifting does not alter non-triviality but it might very well happen that it pushes a family inside $\mathcal{A}_1(n,r,1)$.

To deal with this case we need to recall the definition of the process of shifting, namely the $j \to i$ shift, S_{ij}.

Definition 25. Given a family $\mathcal{F} \subset 2^n$ and $1 \le i < j \le n$ one defines $S_{ij}(\mathcal{F}) = \{S_{ij}(F) : F \in \mathcal{F}\}$ where

$$S_{ij}(F) = \begin{cases} F' := (F - \{j\}) \cup \{i\} & \text{if } i \notin F,\ j \in F \text{ and } F' \notin \mathcal{F}, \\ F & \text{otherwise.} \end{cases}$$

For a fixed $r \ge 3$ and a set $B \in \binom{[n]}{r+1}$ define the isomorphic copy $\mathcal{A}(B)$ of $\mathcal{A}_1(n,r,1)$ by $\mathcal{A}(B) = \{F \subset [n] : |F \cap B| \ge r\}$. Note that $\mathcal{A}_1(n,r,1) = \mathcal{A}([r+1])$.

In the proof of Theorem 22 we start with a non-trivial r-wise intersecting family $\mathcal{F} \subset 2^{[n]}$ such that $\mathcal{F} \not\subset \mathcal{A}(B)$ for all $B \in \binom{[n]}{r+1}$. A problem arises if for some $1 \le i < j \le n$, $S_{ij}(\mathcal{F}) \subset \mathcal{A}(B)$. Let us prove that (7.1) holds in this case. To make notation simple we assume $A = [r+1]$, $i = r+1$, $j = r+2$.

Proposition 26. *Suppose that $\mathcal{F} \subset 2^{[n]}$ is doubly non-trivial, r-wise intersecting ($r \ge 3$) and $S_{r+1,r+2}(\mathcal{F}) \subset \mathcal{A}_1(n,r,1)$ then*

$$|\mathcal{F}| < \frac{r+6}{2^{r+2}} \cdot 2^n. \tag{8.1}$$

Proof. First of all let us note that $S_{r+1,r+2}$ does not change the intersection $F \cap [r]$. Thus

(i) $$|F \cap [r]| \ge r-1 \text{ for all } F \in \mathcal{F}.$$

Set $\mathcal{F}_0 = \{F \in \mathcal{F},\ [r] \subset F\}$. Clearly,

(ii) $$|\mathcal{F}_0| \le 2^{n-r} = \frac{4}{2^{r+2}} \cdot 2^n.$$

As a matter of fact, assuming that $|\mathcal{F}|$ is maximal, in view of (i) equality follows in (ii). For $1 \le p \le r$ let us define three families

$$\mathcal{B}_p(q) = \big\{F \in \mathcal{F} : F \cap [r+2] = [r+2] \setminus \{p,q\}\big\} \ \text{ for } q = r+1, r+2;$$
$$\mathcal{B}_p(0) = \big\{F \in \mathcal{F} : F \cap [r+2] = [r+2] \setminus \{p\}\big\}.$$

Now $\mathcal{F} \setminus \mathcal{F}_0$ is partitioned in these $3r$ families. We need to prove

(iii) $$\sum_{1 \le p \le r} \Big(\big|\mathcal{B}_p(r+1)\big| + \big|\mathcal{B}_p(r+2)\big| + \big|\mathcal{B}_p(0)\big|\Big) < \frac{r+2}{2^{r+2}} \cdot 2^n.$$

Without loss of generality $\mathcal{F}$ is a filter, implying

(iv) $$\mathcal{B}_p(q) \subset \mathcal{B}_p(0) \ \text{ for } \ q = r+1,\ r+2.$$

Also, $S_{r+1,r+2}(\mathcal{F}) \subset \mathcal{A}_1(n,r,1)$ implies

(v) $$\mathcal{B}_p(r+1) \cap \mathcal{B}_p(r+2) = \emptyset.$$

Since $\mathcal{F} \not\subset \mathcal{A}_1(n,r,1)$, we may assume by symmetry that

(vi) $$\mathcal{B}_1(r+1) \neq \emptyset.$$

Since $\mathcal{F} \not\subset \mathcal{A}\big([r] \cup \{r+2\}\big)$, we may assume by symmetry that either (vii) or (viii) hold.

(vii) $$\mathcal{B}_2(r+2) \neq \emptyset,$$

(viii) $\mathcal{B}_1(r+2) \neq \emptyset$ but $\mathcal{B}_p(q) = \emptyset$ for all $2 \le p \le r,\ q = r+1, r+2$.

Note that (iv) and (v) imply

(ix) $$\big|\mathcal{B}_p(r+1)\big| + \big|\mathcal{B}_p(r+2)\big| + \big|\mathcal{B}_p(0)\big| \le \frac{2}{2^{r+2}} \cdot 2^n.$$

In case of (viii) we immediately obtain

$$\big|\mathcal{F} \setminus \mathcal{F}_0\big| \le (2 + (r-1))\frac{2^n}{2^{r+2}} < \frac{(r+2)}{2^{r+2}} \cdot 2^n, \quad \text{as desired.}$$

Thus we may assume that (vii) holds.

Let us note that $|\mathcal{B}_p(0)| \le 2^{n-r-3}$ would imply that (ix) holds with $\frac{1}{2^{r+2}} \cdot 2^n$ for the corresponding p. Should this be the case for all $1 \le p \le r$, we would get

$$|\mathcal{F} \setminus \mathcal{F}_0| \le \frac{r}{2^{r+2}} 2^n < \frac{r+2}{2^{r+2}} \cdot 2^n.$$

Thus we may assume by symmetry that $\mathcal{B}_3(0) > \frac{1}{2} \cdot 2^{n-r-2}$, whence $\mathcal{B}_3(0)$ is non-trivial on $[r+3, n]$.

Note that $\mathcal{B}_p(0) = \emptyset$ would imply $\mathcal{F} \subset \mathcal{A}\big([r+2] \setminus \{p\}\big)$, contradicting doubly non-triviality.

Thus we infer that the r families $\mathcal{B}_1(r+1)$, $\mathcal{B}_2(r+2)$, $\mathcal{B}_3(0)$, $\ldots$, $\mathcal{B}_r(0)$ are all non-empty with $\mathcal{B}_3(0)$ non-trivial. Applying Theorem 14 (i) we obtain

$$\text{(x)} \qquad |\mathcal{B}_1(r+1)| + |\mathcal{B}_2(r+2)| + \sum_{3 \le p \le r} |\mathcal{B}_p(0)| \le \frac{r+2}{4} \cdot 2^{n-r-2}.$$

If both $\mathcal{B}_1(r+2)$ and $\mathcal{B}_2(r+1)$ are non-empty then the same argument applies to the r-wise cross-intersecting families $\mathcal{B}_1(r+2), \mathcal{B}_2(r+1), \mathcal{B}_3(0), \ldots, \mathcal{B}_r(0)$. Adding the corresponding inequality to (x) and using $|\mathcal{B}_p(r+1)| + |\mathcal{B}_p(r+2)| \le |\mathcal{B}_p(0)|$ we infer

$$|\mathcal{F} \setminus \mathcal{F}_0| \le |\mathcal{B}_0(1)| + |\mathcal{B}_0(2)| + \frac{r+2}{2} \cdot 2^{n-r-2} \le \frac{r+6}{2} \cdot 2^{n-r-2}.$$

Since $\frac{r+6}{2} < r+2$ for $r \ge 3$, we are done.

The case $\mathcal{B}_1(r+2) = \mathcal{B}_2(r+1) = \emptyset$ is even easier. We can take the double of (ix) and arrive at the same conclusion.

Using symmetry there is only one more case left, $\mathcal{B}_1(r+2) \ne \emptyset$ but $\mathcal{B}_2(r+1) = \emptyset$.

For this last case we note that doubling (ix) still yields

$$|\mathcal{F} \setminus \mathcal{F}_0| \le |\mathcal{B}_0(1)| + |\mathcal{B}_0(2)| + |\mathcal{B}_1(r+2)| - |\mathcal{B}_1(r+1)| + \frac{r+2}{2} 2^{n-r-2}.$$

For $r \ge 4$, $\frac{r+2}{2} + 3 \le r+2$ and we still obtain $|\mathcal{F} \setminus \mathcal{F}_0| < (r+2) \cdot 2^{n-r-2}$.

The only remaining subcase is $r = 3$. If $|\mathcal{B}_2(0)| \le \frac{1}{2} \cdot 2^{n-r-2}$, $|\mathcal{B}_2(0)| + |\mathcal{B}_2(\{r+1\})| + |\mathcal{B}_2(\{r+2\})| \le 2^{n-r-2}$ and $|\mathcal{F} \setminus \mathcal{F}_0| \le (2+1+2) \cdot 2^{n-r-2} = \frac{5}{32} 2^{n-5}$ follows.

Since we could exchange 2 and 3, we may assume that $\mathcal{B}_3(r+1) = \emptyset$ and $\mathcal{B}_3(r+2) \ne \emptyset$.

Apply (2.8) to $\mathcal{B}_1(r+2)$, $\mathcal{B}_2(0)$ and $\mathcal{B}_3(r+2)$.

$$\text{(xi)} \qquad \left|\mathcal{B}_1(r+1)\right| + \left|\mathcal{B}_2(0)\right| + \left|\mathcal{B}_3(r+2)\right| \le \frac{5}{4} \cdot 2^{n-r-2}.$$

Using (x):

$$\left|\mathcal{F} \setminus \mathcal{F}_0\right| \le \left|\mathcal{B}_1(r+2)\right| + \left|\mathcal{B}_1(0)\right| - \left|\mathcal{B}_1(r+1)\right| + \frac{5}{2}2^{n-5} < \frac{9}{2}2^{n-5} < 5 \cdot 2^{n-5},$$

completing the proof. □

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