A stability result for families with fixed diameter

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Abstract

Let $\mathcal{F} \subset 2^{[n]}$ be a family of subsets. The *diameter* of $\mathcal{F}$ is the maximum of the size of symmetric differences among pairs of members of $\mathcal{F}$. In 1966 Kleitman determined the maximum of $|\mathcal{F}|$ for fixed diameter. However, this important classical result lacked a characterisation of the families meeting the bound. This is remedied in the present paper where a best possible stability result is established as well.

In Section 4 we introduce a "parity trick" that provides an easy way of deducing the odd case from the even case in both Kleitman's original theorem and in the stability version of it.

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1 Introduction

Let $[n] = \{1, \dots, n\}$ be the standard n-element set. The power set of $[n]$ is denoted by $2^{[n]}$. A subset $\mathcal{F}$ of $2^{[n]}$ is called a *family*.

For $i \in [n]$ we define $\mathcal{F}(i) = \big\{F - \{i\} : \ i \in F \in \mathcal{F}\big\}$ and $\mathcal{F}(\bar{i}) = \{F \in \mathcal{F} : \ i \notin F\}$. Note that both $\mathcal{F}(i)$ and $\mathcal{F}(\bar{i})$ are subsets of $2^{[n]-\{i\}}$ and $|\mathcal{F}| = |\mathcal{F}(i)| + |\mathcal{F}(\bar{i})|$ holds.

Sometimes, in particular in coding theory, $2^{[n]}$ is considered a metric space with the distance of $A, B \subset [n]$ defined as the size of the symmetric difference. That is,

$$d(A, B) = |A \setminus B| + |B \setminus A|.$$

$2^{[n]}$ can be regarded as an elementary Abelian group of order 2^n as well. One defines the addition modulo 2, i.e., $A + B = \{i : \ i$ is contained in exactly one of A and $B\}$. Of course, $A+B = A\setminus B \cup B\setminus A$ is the symmetric difference of A and B.

The *diameter* $\Delta(\mathcal{F})$ of a family $\mathcal{F}\subset 2^{[n]}$ is simply

$$\begin{aligned}&\max\{d(A,B):\ A,B\in\mathcal{F}\}\ \text{ or equivalently,}\\ &\max\{|A+B|:\ A,B\in\mathcal{F}\}.\end{aligned}$$

Let us note the obvious inequality

$$|A+B| \leqq |A\cup B|. \tag{0}$$

One can argue that extremal set theory emerged as an independent field inside combinatorics through the many problems and conjectures posed by Paul Erdős. Let us mention here two important classical results.

First define the function $m(n,s)$ for $n > s \geqq 0$ by

$$m(n,s) = \begin{cases} \sum\limits_{0\leqq i\leqq s/2} \binom{n}{i} & \text{if } s \text{ is even,}\\ 2\cdot \sum\limits_{0\leqq i\leqq s/2} \binom{n-1}{i} & \text{if } s \text{ is odd.}\end{cases}$$

Katona Theorem ([Kat] (1964))**.** *Suppose that $\mathcal{F}\subset 2^{[n]}$ satisfies*

$$|F\cup F'| \leqq s \quad \textit{for all} \quad F,F'\in\mathcal{F}. \tag{1}$$

Then for all $n > s \geqq 0$

$$|\mathcal{F}| \leqq m(n,s) \quad \textit{holds.} \tag{2}$$

Moreover, if $n \geqq s+2$ then equality holds in (2) iff $\mathcal{F}$ is of the following form:

(i) *s is even, $\mathcal{F} = \left\{F\subset[n]:\ |F|\leqq \frac{s}{2}\right\} \overset{\text{def}}{=} \mathcal{K}(n,s)$,*

(ii) *s is odd, $\mathcal{F} = \left\{F\subset[n]:\ \big|F\cap([n]-\{y\})\big| \leqq \frac{s}{2}\right\} \overset{\text{def}}{=} \mathcal{K}_y(n,s)$, for some fixed element $y\in[n]$.*

Kleitman Diameter Theorem ([Kle] (1966))**.** *Suppose that $\mathcal{F}\subset 2^{[n]}$ satisfies*

$$|F+F'| \leqq s \quad \textit{for all} \quad F,F'\in\mathcal{F}. \tag{3}$$

Then for all $n > s \geqq 0$

$$|\mathcal{F}| \leqq m(n,s) \quad \textit{holds.} \tag{4}$$

In view of (0), the bound (4) is stronger than (2). On the other hand, no uniqueness is proved.

One of the aims of the present paper is to remedy this problem and for $n \geqq s+2$ determine all families attaining equality in (4).

Let us mention that Kleitman proves (4) by reducing the problem on $|F+F'|$ to that on $|F \cup F'|$. For this reason he introduces the very useful operation of *down-shift*, S_j.

Definition 1. Let $\mathcal{F} \subset 2^{[n]}$, $j \in [n]$. Define $S_j(\mathcal{F}) = \{S_j(F) : \ F \in \mathcal{F}\}$ where

$$S_j(F) = \begin{cases} F - \{j\} & \text{if } j \in F \in \mathcal{F} \ \ \textit{and } (F - \{j\}) \notin \mathcal{F}, \\ F & \text{otherwise.} \end{cases}$$

Claim 1 (Kleitman [Kle])**.** $|S_j(\mathcal{F})| = |\mathcal{F}|$ *and* $\Delta\big(S_j(\mathcal{F})\big) \leqq \Delta(\mathcal{F})$ *hold.* $\square$

The following are easy to verify.

$$\widetilde{\mathcal{F}}(j) = \mathcal{F}(j) \cap \mathcal{F}(\bar{j}), \tag{5}$$

$$\widetilde{\mathcal{F}}(\bar{j}) = \mathcal{F}(j) \cup \mathcal{F}(\bar{j}) \ \text{ and} \tag{6}$$

$$|\widetilde{\mathcal{F}}(i)| = |\mathcal{F}(i)|, \ \widetilde{\mathcal{F}}(\bar{i})| = |\mathcal{F}(\bar{i})| \ \text{ for all } i \neq j. \tag{7}$$

Definition 2. For a family $\mathcal{G} \subset 2^{[n]}$ and a set $S \subset [n]$ we define $\mathcal{G} + S$, the *translation* of $\mathcal{G}$ by S, in the following way:

$$\mathcal{G} + S = \{G + S : \ G \in \mathcal{G}\}.$$

Note that $\Delta(\mathcal{G} + S) = \Delta(\mathcal{G})$ holds.

Our main result is the following.

Theorem 1. *Let $n \geqq s+2$, $s \geqq 0$ and let $\mathcal{F} \subset 2^{[n]}$ satisfy $\Delta(\mathcal{F}) \leqq s$. Then*

$$\begin{aligned} &|\mathcal{F}| = m(n,s) \ \ \textit{implies that } \mathcal{F} \textit{ is a translate of} \\ &\quad \mathcal{K}(n,s) \ \ \textit{(for } s \textit{ even) or of } \ \mathcal{K}_y(n,s) \ \ \textit{(for } s \textit{ odd).} \end{aligned} \tag{8}$$

Moreover, the following stability results hold.

If $s = 2d$ and $\mathcal{F}$ is not contained in any translate of $\mathcal{K}(n,s)$ then

$$|\mathcal{F}| \leqq \sum_{0 \leqq i \leqq d} \binom{n}{i} - \binom{n-d-1}{d} + 1. \tag{9}$$

If $s = 2d+1$ and $\mathcal{F}$ is not contained in any translate of $\mathcal{K}_y(n,s)$ (for any $y \in [n]$) then

$$(10)\qquad |\mathcal{F}| \leqq 2 \sum_{0 \leqq i \leqq d} \binom{n-1}{i} - \binom{n-d-2}{d} + 1 \quad \textit{holds.}$$

Let us mention that both (9) and (10) are best possible. We shall discuss it in the next section.

2 Tools of proofs

Let us recall that a family $\mathcal{C} \subset 2^{[n]}$ is called a *complex* (or down-set) if for all $D \subset C \in \mathcal{C}$, $D \in \mathcal{C}$ holds.

Claim 2 (Kleitman [Kle]). *If $\mathcal{C}$ is a complex then*

$$|C \cup C'| \leqq \Delta(\mathcal{C}) \quad \textit{holds for all} \quad C, C' \in \mathcal{C}.$$

Proof. Set $C'' = C' \setminus C$. Then $C'' \in \mathcal{C}$ and $C \cup C' = C \cup C'' = C + C''$, implying the statement. □

As a matter of fact Kleitman deduced (4) from (2) by repeatedly applying the down-shift S_j, $1 \leqq j \leqq n$. If we start with a family $\mathcal{F} \subset 2^{[n]}$ satisfying $\Delta(\mathcal{F}) \leqq s$ then we end up with a complex $\mathcal{C}$. In view of Claims 1 and 2 the complex $\mathcal{C}$ satisfies $|\mathcal{C}| = |\mathcal{F}|$, $\Delta(\mathcal{C}) \leqq s$ and even $|C \cup C'| \leqq s$ for all $C, C' \in \mathcal{C}$. Thus applying (2) to $\mathcal{C}$ yields (4).

We are going to imitate Kleitman's approach. However, since $S_j(\mathcal{F})$ changes the structure of $\mathcal{F}$, we have to be careful. On the other hand, if the "end product" $\mathcal{C}$ is a complex which is not contained in $\mathcal{K}(n,s)$ or $\mathcal{K}_y(n,s)$ then we can apply the following, recent stability theorem.

Theorem 2 ([F]). *Suppose that $\mathcal{F} \subset 2^{[n]}$ $(n \geqq s+2 \geqq 2)$ satisfies $|F \cup F'| \leqq s$ for all $F, F' \in \mathcal{F}$. Then*

(i) *$s = 2d$ and $\mathcal{F} \not\subset \mathcal{K}(n,s)$ then*

$$(11)\qquad |\mathcal{F}| \leqq \sum_{0 \leqq i \leqq d} \binom{n}{i} - \binom{n-d-1}{d} + 1.$$

(ii) $s = 2d+1$ *and there is no* $y \in [n]$ *such that* $\mathcal{F} \subset \mathcal{K}_y(n,s)$ *then*

$$|\mathcal{F}| \leqq \sum_{0 \leqq i \leqq d} \binom{n}{i} + \binom{n-1}{d} - \binom{n-d-2}{d} + 1 \quad \textit{holds.} \tag{12}$$

For the unexperienced reader it might be not clear that the RHS of (12) is less than $m(n, 2d+1)$. However, it follows from

$$2 \sum_{0 \leqq i \leqq d} \binom{n-1}{i} = \sum_{0 \leqq i \leqq d} \binom{n}{i} + \binom{n-1}{d}.$$

Let us show the constructions giving equality in (11) and (12).

First we suppose that $s = 2d$ and let $D \in \binom{[n]}{d+1}$ be a fixed $d+1$-element set. Define

$$\mathcal{H}(n,s) = \big\{H \subset [n] : |H| \leqq d\big\} \cup \{D\} \Big\backslash \left\{H \in \binom{[n]}{d} : \; H \cap D = \emptyset\right\}.$$

Next consider the case $s = 2d+1$ and fix a $D \in \binom{[n-1]}{d+1}$. Define first the intersecting family $\mathcal{H}_0(n,s)$:

$$\mathcal{H}_0(n,s) = \left\{H \in \binom{[n]}{d+1} : \; n \in H, \; H \cap D \neq \emptyset\right\} \cup \{D\}.$$

Then define

$$\mathcal{H}(n,s) = \{H \subset [n] : |H| \leqq d\} \cup \mathcal{H}_0(n,s).$$

It is easy to check that in both cases $|H \cup H'| \leqq s$ holds for all $H, H' \in \mathcal{H}(n,s)$. Consequently, $\Delta(\mathcal{H}(n,s)) \leqq s$. Therefore $\Delta(\mathcal{H}(n,s) + S) \leqq s$ for all $S \subset [n]$ as well.

In [F] it is also proven that unless $s = 5$, $\mathcal{H}(n,s)$ are the only families for which equality holds in (11) and (12).

A family $\mathcal{H} \subset \binom{[n]}{k}$ is called *intersecting* if $H \cap H' \neq \emptyset$ for all $H, H' \in \mathcal{H}$. It is called *non-trivial* if $\bigcap\limits_{H \in \mathcal{H}} H = \emptyset$. We shall use the following classical result.

Hilton–Milner Theorem ([HM])**.** *If* $\mathcal{H} \subset \binom{[n]}{k}$ *is a non-trivial intersecting family,* $n > 2k$*, then*

$$|\mathcal{H}| \leqq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1 \quad \textit{holds.} \qquad \square$$

During the proof we shall need also the following inequality. Let us recall that two families $\mathcal{A}, \mathcal{B} \subset 2^{[n]}$ are called *cross-intersecting* if $A \cap B \neq \emptyset$ holds for all $A \in \mathcal{A}$, $B \in \mathcal{B}$.

Proposition 1 (Frankl–Tokushige [FT], cf. also Wang–Zhang [WZ])**.** *Suppose that $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$, $n > 2k$, are cross-intersecting, non-empty and $\mathcal{A} \cap \mathcal{B} = \emptyset$. Then*

$$|\mathcal{A}| + |\mathcal{B}| \leqq \binom{n}{k} - \binom{n-k}{k} \quad \textit{holds.} \tag{13}$$ □

Note that (13) is a slight improvement over an inequality used by Hilton and Milner [HM].

3 Proof of Theorem 2

We start with a family $\mathcal{F} \subset 2^{[n]}$ satisfying $\Delta(\mathcal{F}) \leqq s$. Define

$$S = \left\{ i : \ |\mathcal{F}(i)| > |\mathcal{F}(\bar{i})| \right\} \subset [n].$$

Then the translated family $\widetilde{\mathcal{F}} = \mathcal{F} + S$ satisfies $\Delta(\widetilde{\mathcal{F}}) = \Delta(\mathcal{F})$ and $\left|\widetilde{\mathcal{F}}(i)\right| \leqq \left|\widetilde{\mathcal{F}}(\bar{i})\right|$ for all $i \in [n]$. Without loss of generality we can consider $\mathcal{F} + S$ instead of $\mathcal{F}$. Thus we assume that $|\mathcal{F}(i)| \leqq |\mathcal{F}(\bar{i})|$ holds for all $i \in [n]$ at the start.

If $\mathcal{F}$ is a complex then by Claim 2 one has $|F \cup F'| \leqq s$ for all $F, F' \in \mathcal{F}$ and the statements follow directly from Theorem 1.

Suppose that $\mathcal{F}$ is not a complex. Since repeated applications of the down-shift S_j, $1 \leqq j \leqq n$, eventually turn $\mathcal{F}$ into a complex, there must be an intermediary family $\mathcal{G}$ satisfying the following:

$$|\mathcal{G}| = |\mathcal{F}|, \quad \Delta(\mathcal{G}) \leqq \Delta(\mathcal{F}) \leqq s$$

and $\mathcal{G}$ is not a complex but $S_j(\mathcal{G})$ is a complex. To have part of the argument unified for s even and s odd, we use the symbol $\mathcal{K}(n, s)$ for s odd as well, having $\mathcal{K}_y(n, s)$ in mind for some unspecified $y \in [n]$. If $S_j(\mathcal{G}) \not\subset \mathcal{K}(n, s)$, applying Theorem 1 concludes the proof. Thus we assume that $S_j(\mathcal{G}) \subset \mathcal{K}(n, s)$ holds. It might happen that $\mathcal{G} \subset \mathcal{K}(n, s)$. In that case we backtrack and consider the last $\mathcal{G}$ with $\mathcal{G} \not\subset \mathcal{K}(n, s)$ but $S_j(\mathcal{G}) \subset \mathcal{K}(n, s)$ (by abuse of notation we use the same letter j).

Let us first consider the case $s = 2d$. There must exist some $D \in \mathcal{G} \cap \binom{n}{d+1}$ such that $j \in D$ and $D - \{j\} \notin \mathcal{G}$. Define the two families $\mathcal{A}$ and $\mathcal{B}$ as follows.

$$\mathcal{A} = \left\{ A \in \binom{[n]}{d} \ : \ j \notin A, \ (A \cup \{j\}) \in \mathcal{G} \right\}.$$
$$\mathcal{B} = \left\{ B \in \binom{[n]}{d} \ : \ B \in \mathcal{G} \right\}.$$

Claim 3. (i) $\mathcal{A} \neq \emptyset$;

(ii) *for* $A \in \mathcal{A}$ *one has* $A \notin \mathcal{G}$;

(iii) $\mathcal{A} \cap \mathcal{B} = \emptyset$;

(iv) $\mathcal{A}$ *and* $\mathcal{B}(\bar{j})$ *are cross-intersecting.*

Proof. (i) follows from $D \in \mathcal{A}$.

(ii) Since $S_j(\mathcal{G})$ contains no members of size exceeding d, $S_j(A \cup \{j\}) = A$ must hold for $A \in \mathcal{A}$. This implies $A \notin \mathcal{G}$.

(iii) $A \in \mathcal{A} \cap \mathcal{B}$ then $j \notin A$ and both A and $A \cup \{j\}$ are in $\mathcal{G}$. Thus $S_j(A \cup \{j\}) = A \cup \{j\}$, contradicting $S_j(\mathcal{G}) \subset \mathcal{K}(n, s)$.

(iv) This follows from $\left|(A \cup \{j\}) + B\right| \leqq 2d$. □

Let us note that $\mathcal{B}(\bar{j}) = \emptyset$ would imply

$$|\mathcal{G}| \leqq 2|\mathcal{G}(\bar{j})| \leqq 2 \sum_{0 \leqq i < d} \binom{n-1}{i} = \sum_{0 \leqq i < d} \binom{n}{i} + \binom{n-1}{d-1}$$

which is smaller than the RHS of (9). Thus we may assume that $\mathcal{B}(\bar{j}) \neq \emptyset$ holds.

Applying Proposition 1 to $\mathcal{A}$ and $\mathcal{B}(\bar{j}) \subset \binom{[n]-\{j\}}{d}$ gives $|\mathcal{A}| + |\mathcal{B}(\bar{j})| \leqq \binom{n-1}{d} - \binom{n-1-d}{d}$.

Since $|\mathcal{B}(j)| \leqq \binom{n-1}{d-1}$, $|\mathcal{A}| + |\mathcal{B}| \leqq \binom{n}{d} - \binom{n-d-1}{d}$ follows.

Note that there are at most $|\mathcal{A}| + |\mathcal{B}|$ sets of size d in $S_j(\mathcal{G})$. Using $S_j(\mathcal{G}) \subset \mathcal{K}(n, s)$,

$$|\mathcal{F}| = |S_j(\mathcal{G})| \leqq \sum_{0 \leqq i \leqq d} \binom{n}{i} - \binom{n-d-1}{d} \quad \text{follows.}$$

Now we consider the case $s = 2d + 1$. We set $\mathcal{K}(n, s) = \mathcal{K}_y(n, s)$ for an unspecified $y \in [n]$, i.e., $\mathcal{K}(n, s) = \big\{K \subset [n] \ : \ |K \cap ([n] - \{y\})| \leqq d\big\}$,

$\mathcal{G} \not\subset \mathcal{K}(n,s)$ together with $S_j(\mathcal{G}) \subset \mathcal{K}(n,s)$ imply the existence of some $D \subset [n]$ satisfying $\left|D \cap ([n] - \{y\})\right| \geqq d+1$, $D \in \mathcal{G}$, $D \notin S_j(\mathcal{G})$ and more importantly $\left|S_j(D) \cap ([n] - \{y\})\right| \leqq d$. This is possible only for $j \neq y$ and $\left|D \cap ([n] - \{y\})\right| = d+1$ and only in the case $(D - \{j\}) \notin \mathcal{G}$.

First we take care of the case when

$$|G| \leqq d+1 \text{ holds for all } G \in \mathcal{G}. \tag{14}$$

Let us consider the subfamily $\mathcal{H} \overset{\text{def}}{=} \mathcal{G} \cap \binom{[n]}{d+1}$ of all $(d+1)$-sets in $\mathcal{G}$.

Since $\Delta(\mathcal{H}) \leqq \Delta(\mathcal{G}) \leqq 2d+1$, $\mathcal{H}$ is an intersecting family. Should $\mathcal{H}$ be a *star*, i.e., should there exist an element $z \in [n]$ with $z \in H$ for all $H \in \mathcal{H}$, then using (14), $\mathcal{G} \subset \mathcal{K}_z(n,s)$ follows. This is a contradiction.

Consequently we may apply the Hilton–Milner Theorem to $\mathcal{H}$ and obtain $|\mathcal{H}| \leqq \binom{n-1}{d} - \binom{n-d-2}{d} + 1$. This yields

$$|\mathcal{G}| \leqq \sum_{0 \leqq i \leqq d} \binom{n}{i} + \binom{n-1}{d} - \binom{n-d-2}{d} + 1,$$

as desired. Now suppose that there is some $G \in \mathcal{G}$ with $|G| \geqq d+2$.

Claim 4. *$\{j,y\} \subset G$ and $|G| = d+2$.*

Proof. $S_j(\mathcal{G}) \subset \mathcal{K}_y(n,s)$ implies $\left|S_j(G) \cap ([n] - \{y\})\right| \leqq d$. Thus $S_j(G) \neq G$, i.e., $j \in G$ and $S_j(G) = G - \{j\}$. Since $|S_j(G)| = |G| - 1$, $|G| = d+2$ and $y \in G$ follow as well. □

Let us define again two families of sets

$$\mathcal{A} = \{A \subset ([n] - \{j\}) : |A \cap ([n] - \{y\})| = d,\ A \cup \{j\} \in \mathcal{G}\},$$
$$\mathcal{B} = \{B \subset ([n] - \{j\}) : |B \cap ([n] - \{y\})| = d,\ B \in \mathcal{G}\},$$

In view of $S_j(\mathcal{G}) \subset \mathcal{K}_y(n,s)$, $\mathcal{A} \cap \mathcal{B} = \emptyset$ holds.

We are going to consider the four families $\mathcal{A}(y)$, $\mathcal{A}(\overline{y})$, $\mathcal{B}(y)$, $\mathcal{B}(\overline{y}) \subset \binom{[n]-\{y,j\}}{d}$. $\Delta(\mathcal{G}) \leqq 2d+1$ implies that $\mathcal{A}(y)$ and $\mathcal{B}(\overline{y})$ and also $\mathcal{A}(\overline{y})$ and $\mathcal{B}(y)$ are cross-intersecting.

By definition, G from Claim 4 provides us with a set, namely $G - \{j,y\}$, belonging to $\mathcal{A}(y)$. If $\mathcal{B}(\overline{y}) \neq \emptyset$, then (13) yields

$$|\mathcal{A}(y)| + |\mathcal{B}(\overline{y})| \leqq \binom{n-2}{d} - \binom{n-2-d}{d}.$$

Since $\mathcal{A} \cap \mathcal{B} = \emptyset$, $|\mathcal{A}(\overline{y})| + |\mathcal{B}(y)| \leqq \binom{n-2}{d}$ holds as well.

$$\left|\left\{G \in \mathcal{G} :\ j \in G,\ \left|G \cap ([n] - \{y\})\right| = d\right\}\right| \leqq 2\binom{n-2}{d-1} \quad \text{is obvious.}$$

We infer,

$$\begin{aligned} |\mathcal{G}| &\leqq 2 \sum_{0 \leqq i < d} \binom{n-1}{i} + 2\binom{n-2}{d-1} + 2\binom{n-2}{d} - \binom{n-2-d}{d} \\ &= 2 \sum_{0 \leqq i \leqq d} \binom{n-1}{i} - \binom{n-2-d}{d}, \quad \text{as desired.} \end{aligned}$$

Thus we may assume that $\mathcal{B}(\overline{y}) = \emptyset$. Absolutely the same argument works if both $\mathcal{B}(y)$ and $\mathcal{A}(\overline{y})$ are nonempty. To conclude the proof we distinguish two cases according to $\mathcal{A}(\overline{y}) = \emptyset$ or $\mathcal{B}(y) = \emptyset$.

(a) $\mathcal{A}(\overline{y}) = \emptyset$

Recall that $|\mathcal{F}(\overline{y})| \geqq |\mathcal{F}(y)|$ held at the start and this is not altered by the down-shift (cf. (5)–(7)). Therefore,

$$|\mathcal{G}| = |\mathcal{G}(y)| + |\mathcal{G}(\overline{y})| \leqq 2|\mathcal{G}(\overline{y})| \quad \text{follows.}$$

Note that the only sets of size at least d in $\mathcal{G}(\overline{y}) = \mathcal{G}(\overline{y}) - (\mathcal{A}(\overline{y}) \cup \mathcal{B}(\overline{y}))$ are sets $H \subset \binom{[n]-\{y\}}{d}$ with $j \in H$. These are at most $\binom{n-2}{d-1}$ sets, implying

$$|\mathcal{G}(\overline{y})| \leqq \sum_{0 \leqq i < d} \binom{n-1}{i} + \binom{n-2}{d-1}.$$

Therefore,

$$|\mathcal{G}| \leqq 2 \sum_{0 \leqq i \leqq d-1} \binom{n-1}{i} + 2\binom{n-2}{d-1}.$$

To conclude the proof in this case we need

$$2\binom{n-2}{d-1} \leqq 2\binom{n-1}{d} - \binom{n-2-d}{d}.$$

Rearranging we get

$$\binom{n-d-2}{d} \leqq 2\binom{n-2}{d} \quad \text{which holds trivially.}$$

(b) $\mathcal{B}(y) = \emptyset$

Since $\mathcal{B}(\overline{y}) = \emptyset$ also, $\mathcal{B} = \emptyset$ follows. This means that $\mathcal{G} + \{j\} \subset \mathcal{K}_y(n,s)$. What can be the members of $\mathcal{G}(\overline{j})$? If $H \in \mathcal{G}(\overline{j})$, then $j \notin H$ implies $H \in S_j(\mathcal{G})$. Thus $\big|H \cap ([n] - \{y\})\big| \leqq d$. Using $\mathcal{B} = \emptyset$, even $\big|H \cap ([n] - \{y\})\big| \leqq d - 1$ follows. Since $j \notin H$ implies $H \subset ([n] - \{j\})$, we infer

$$|\mathcal{G}(\overline{j})| \leqq 2 \sum_{0 \leqq i \leqq d-1} \binom{n-2}{i} = \sum_{0 \leqq i \leqq d} \binom{n-1}{i} - \binom{n-2}{d}.$$

Consequently,

$$|\mathcal{G}| \leqq 2|\mathcal{G}(\overline{j})| \leqq 2 \sum_{0 \leqq i \leqq d} \binom{n-1}{i} - 2\binom{n-2}{d}.$$

Since $2\binom{n-2}{d} > \binom{n-2}{d} > \binom{n-2-d}{d}$, the proof is complete. □

4 The parity trick

Let $p(n,s)$ denote the maximum-value of $\mathcal{F} \subset 2^{[n]}$ satisfying $\Delta(\mathcal{F}) \leqq s$. By Kleitman's theorem we know that $p(n,s)$ equals $m(n,s)$ from Katona's Theorem.

In this section we are going to give a simple proof of the following:

Proposition 4.1.

(15) $\quad p(n, 2d+1) = 2p(n-1, 2d)$ *holds for* $n \geqq 2d + 2$.

Looking at the formulae for $m(n,s)$ one can easily verify that (15) holds. However, we are going to prove it without assuming *any* knowledge of the actual formula for $m(n,s)$ or $p(n,s)$.

Proof. Let $\mathcal{A} \subset 2^{[n-1]}$ satisfy $|\mathcal{A}| = p(n-1, 2d)$ and $\Delta(\mathcal{A}) \leqq 2d$. Define $\mathcal{B} = \mathcal{A} \cup \{A \cup \{n\} : A \in \mathcal{A}\}$. Then $|\mathcal{B}| = 2|\mathcal{A}|$ and $\Delta(\mathcal{B}) = \Delta(\mathcal{A}) + 1 \leqq 2s + 1$ hold. This proves $p(n, 2d+1) \geqq 2p(n-1, 2d)$.

Let us prove the opposite inequality. For $\mathcal{F} \subset 2^{[n]}$ satisfying $\Delta(\mathcal{F}) \leqq 2d + 1$ and $|\mathcal{F}| = p(n, 2d+1)$ define the partition $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$ by

$$\mathcal{F}_i = \big\{F \in \mathcal{F} : |F| \equiv i \pmod 2\big\}.$$

Note that for $F, F' \in \mathcal{F}_i$ one has $|F + F'| \equiv 0 \pmod 2$ implying

$$\Delta(\mathcal{F}_i) \leqq 2d \quad \text{for} \quad i = 0, 1.$$

At the first sight this gives only the bound $|\mathcal{F}_i| \leqq p(n, 2d)$ which is insufficient to prove (15).

Fortunately, one can do one more trick. Define $\mathcal{G}_i = \{F \cap [n-1] : F \in \mathcal{F}_i\} \subset 2^{[n-1]}$. The point is that if $F, F' \in \mathcal{F}_i$ are distinct then $|F| \equiv |F'| \pmod 2$ implies $|F + F'| \geqq 2$. Consequently, $F \cap [n-1] \neq F' \cap [n-1]$ holds. Therefore $|\mathcal{G}_i| = |\mathcal{F}_i|$ and $\Delta(\mathcal{G}_i) \leqq 2d$ hold. By definition,

$$p(n, 2d+1) = |\mathcal{F}_0| + |\mathcal{F}_1| = |\mathcal{G}_0| + |\mathcal{G}_1| \leqq 2p(n-1, 2d)$$

follows concluding the proof of (15). □

Proposition 4.1 shows that the odd case is a consequence of the even case in Kleitman's Diameter Theorem.

Let us elaborate this approach and sketch how the above parity trick can be used to derive the odd case of Theorem 1 from the even case.

Let $\mathcal{F} \subset 2^{[n]}$ satisfy $\Delta(\mathcal{F}) \leqq 2d+1$. Define $\mathcal{F}_i$ and $\mathcal{G}_i$, $i = 0, 1$, as above.

If $|\mathcal{F}| = p(n, 2d+1) = 2p(n-1, 2d)$, then $|\mathcal{F}_i| = |\mathcal{G}_i| = p(n-1, 2d)$ holds for $i = 0, 1$. Applying the even case of Theorem 1, we infer that there exist subsets S_0 and S_1 of $[n-1]$ such that

$$\mathcal{G}_i = \{G \subset [n-1] : |G + S_i| \leqq d\}, \quad i = 0, 1.$$

There are three cases: $S_0 = S_1$, $|S_0 + S_1| = 1$ and $|S_0 + S_1| \geqq 2$.

In the first case, $\mathcal{G}_0 = \mathcal{G}_1$ and

$$\mathcal{F}_0 \cup \mathcal{F}_1 = \{F \subset [n] : |F \cap [n-1]| \leqq d\} \quad \text{follow.}$$

In the second case defining j by $S_0 + S_1 = \{j\}$,

$|\mathcal{F}_0 \cup \mathcal{F}_1| + S' = \{F \subset [n] : |F \cap ([n] - \{j\})| \leqq d\}$ follows for some appropriate S'.

In the third case one can easily find $E_i \subset [n-1]$, $|E_i| \leqq d$, $i = 0, 1$ such that

$$|(S_0 + E_0) + (S_1 + E_1)| \geqq \min\{n-1, 2d + |S_0 + S_1|\} \geqq 2d + 2.$$

Since $S_0 + E_0 \in \mathcal{G}_0$, $S_1 + E_1 \in \mathcal{G}_1$ we infer the existence of $F_i \in \mathcal{F}_i$, $i = 0, 1$ with $|F_0 + F_1| \geqq 2d + 2$, a contradiction.

If either $\mathcal{G}_0$ or $\mathcal{G}_1$ satisfy $\mathcal{G}_i \not\subset \mathcal{K}(n-1, 2d) + S$ for all choices of $S \subset [n]$, then applying (9) with $n - 1$ gives

$$|\mathcal{G}_i| \leqq \sum_{0 \leqq i \leqq d} \binom{n-1}{i} - \binom{n-d-2}{d} + 1 \tag{16}$$

for the corresponding i. For $i' = 1 - i$ we still have

$$|\mathcal{G}_{i'}| \leqq \sum_{0 \leqq i \leqq d} \binom{n-1}{i}.$$

Adding these two inequalities gives (10). The hardest case is when $\mathcal{G}_i \subset \mathcal{K}(n, s) + S_i$ holds for an appropriate choice of S_i, $i = 0, 1$. In the case $|S_0 + S_1| \leqq 1$, $\mathcal{F}_0 \cup \mathcal{F}_1 = \mathcal{F} \subset \mathcal{K}(n, s) + S$ follows for a suitable choice of S.

Finally, if $|S_0 + S_1| \geqq 2$ and neither $\mathcal{G}_0$ nor $\mathcal{G}_1$ verifies (16), then $\Delta(\mathcal{G}_0 \cup \mathcal{G}_1) \geqq 2d + 2$ can be shown easily, the final contradiction. □

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