Resilient hypergraphs with fixed matching number

by Peter Frankl, Rényi Institute, Budapest, Hungary

Abstract

Let $\mathcal{H}$ be a hypergraph of rank k, that is, $|H| \leqq k$ for all $H \in \mathcal{H}$. Let $\nu(\mathcal{H})$ denote the matching number, the maximum number of pairwise disjoint edges in $\mathcal{H}$. For a vertex x let $\mathcal{H}(\overline{x})$ be the hypergraph consisting of the edges $H \in \mathcal{H}$ with $x \notin H$. If $\nu(\mathcal{H}(\overline{x})) = \nu(\mathcal{H})$ for all vertices, $\mathcal{H}$ is called resilient. The main result is the complete determination of the maximum number of 2-element sets in a resilient hypergraph with matching number s. For $k = 3$ it is $\binom{2s+1}{2}$ while for $k \geqq 4$ the formula is $k \cdot \binom{s+1}{2}$. The results are used to obtain a stability theorem for k-uniform hypergraphs with given matching number.

1 Introduction

Let $[n] = \{1, \ldots, n\}$ be the standard n-element set and let $\mathcal{F} \subset 2^{[n]}$ be a family of sets. The *matching number* $\nu(\mathcal{F})$ is the maximum number of pairwise disjoint members of $\mathcal{F}$. If $\emptyset \notin \mathcal{F}$ then $\nu(\mathcal{F}) \leqq n$ holds.

Set $\mathcal{F}(\bar{i}) = \{F \in \mathcal{F} : i \notin F\}$ and $\mathcal{F}(i) = \big\{F - \{i\} : i \in F \in \mathcal{F}\big\}$.

Definition. A family $\mathcal{F}$ is called *resilient* if $\nu\big(\mathcal{F}(\bar{i})\big) = \nu(\mathcal{F})$ for all $i \in [n]$.

In the present paper we shall investigate mainly resilient k-graphs $\mathcal{H} \subset \binom{[n]}{k}$ with prescribed matching number s, i.e., $\nu(\mathcal{H}) = s$.

Nevertheless let us recall an important conjecture of Erdős [E].

First we define two families $\mathcal{A}_1(n, k, s)$ and $\mathcal{A}_k(n, k, s)$ of matching number s.

$$\mathcal{A}_1(n, k, s) = \left\{ A \in \binom{[n]}{k} : A \cap [s] \neq \emptyset \right\},$$

$$\mathcal{A}_k(n,k,s) = \binom{[k(s+1)-1]}{k}.$$

Erdős Matching Conjecture ([E]). *Suppose that $n \geqq k(s+1)$ and $\mathcal{F} \subset \binom{[n]}{k}$ satisfies $\nu(\mathcal{F}) \leqq s$. Then*

$$|\mathcal{F}| \leqq \max\bigl\{|\mathcal{A}_1(n,k,s)|, |\mathcal{A}_k(n,k,s)|\bigr\}. \tag{1}$$

Let us note that $\mathcal{A}_k(n,k,s)$ is resilient but $\mathcal{A}_1(n,k,s)$ is not. For convenience let us introduce the function $e(n,k,s) = \max\Bigl\{|\mathcal{F}| : \mathcal{F} \subset \binom{[n]}{k}, \nu(\mathcal{F}) \leqq s\Bigr\}$. Using the obvious inequality $|\mathcal{F}(i)| \leqq \binom{n-1}{k-1}$, one has

$$|\mathcal{F}| \leqq e(n-1,k,s-1) + \binom{n-1}{k-1} \quad \text{if } \mathcal{F} \text{ is not resilient.} \tag{2}$$

In case $e(n-1,k,s-1) = \bigl|\mathcal{A}_1(n-1,k,s-1)\bigr| = \binom{n-1}{k} - \binom{n-s}{k}$, (2) gives $|\mathcal{F}| \leqq \binom{n}{k} - \binom{n-s}{k} = \bigl|\mathcal{A}_1(n,k,s)\bigr|$, establishing (1). In [F1] it is shown that even if $e(n-1,k,s-1) = \bigl|\mathcal{A}_k(n-1,k,s-1)\bigr|$ holds, (2) is sufficient to prove (1). That is, induction works if $\mathcal{F} \subset \binom{[n]}{k}$ is not resilient.

This fact shows that in order to solve the Erdős Matching Conjecture it is sufficient to consider resilient families.

One can approach resilience from a different side.

Let us define the rank, $r(\mathcal{F})$ of a family as $\max\{|F| : F \in \mathcal{F}\}$. Having a family $\mathcal{F} \subset 2^{[n]}$, $r(\mathcal{F}) = k$ with $\nu(\mathcal{F}) = s$ one can add to it successively subsets $G \subset [n]$, $|G| \leqq k$ such that $\nu(\mathcal{F} \cup \{G\}) = s$ holds as well. Eventually, we obtain a family $\mathcal{F}_{\max}$ of rank k, $\nu(\mathcal{F}_{\max}) = s$ to which no more sets of size at most k can be added without increasing the matching number. Such a family is called *saturated*. Let us note that $\mathcal{F}_{\max}$ may be not unique.

Note that $\nu\bigl(\mathcal{F}(\bar{i})\bigr) < s$ is equivalent to $\nu(\mathcal{F} \cup \{i\}) = s$. That is, we have

$$\begin{aligned} &\mathcal{F} \ \text{ is resilient iff there is no saturated family} \\ &\mathcal{F}_{\max} \ \text{ for } \ \mathcal{F} \ \text{ containing a 1-element set.} \end{aligned} \tag{3}$$

Let $\mathcal{F} \subset \binom{[n]}{k}$ be resilient and let us *fix* a saturated family $\mathcal{F}_{\max} \subset 2^{[n]}$ of rank k, containing $\mathcal{F}$. Partition $\mathcal{F}_{\max}$ according to the size of its members:

$$\mathcal{F}_{\max} = \mathcal{F}_2 \cup \ldots \cup \mathcal{F}_k \ \text{ where } \ \mathcal{F}_\ell = \bigl\{F \in \mathcal{F}_{\max} : |F| = \ell\bigr\}.$$

Let us remark that if $\mathcal{F}$ is resilient then $\mathcal{F}_{\max}$ is resilient as well.

Consider the following family:

$$\mathcal{A}_2(n,k,s) = \left\{ A \in \binom{[n]}{k} : \bigl|A \cap [2s+1]\bigr| \geqq 2 \right\}.$$

Note that $\nu\bigl(\mathcal{A}_2(n,k,s)\bigr) = s$ for $n \geqq ks$ and that in every saturated family $\mathcal{F}_{\max}$ containing $\mathcal{A}_2(n,k,s)$ one has $\mathcal{F}_2 = \binom{[2s+1]}{2}$ as long as $n \geqq k(s+1)$.

Let us define a quite different resilient family. First consider the $(k-1)s$-element set $[s+2, ks+1]$ and let $H_1 \cup H_2 \cup \ldots \cup H_s$ be a partition of it where H_1 consists of the largest $k-1$ elements, H_2 of the next largest $k-1$ elements, etc., H_s consists of the smallest $k-1$ elements of $[s+2, ks+1]$. Next set $F_i = \{i+1\} \cup H_i$, $i = 1, \ldots, s$. Note that $F_1, F_2, \ldots, F_s$ partition $[2, ks+1]$.

Let us define the graph $\mathcal{G} = \mathcal{G}(k,s) \subset \binom{[ks+1]}{2}$ by defining all edges (a,b) with $a < b$. These are

$$\begin{aligned}
&(1,b) \text{ for } \ b \leqq ks+1\\
&(2,b) \text{ for } \ b \leqq k \times (s-1) + 2\\
&(3,b) \text{ for } \ b \leqq k \times (s-2) + 3 \ \text{ etc.}\\
&(s,b) \text{ for } \ b \leqq k \times 1 + s.
\end{aligned}$$

Then $|\mathcal{G}| = sk + (s-1)k + \ldots + k = k\binom{s+1}{2}$ holds. For $k \geqq 4$ this is more than $\binom{2s+1}{2}$.

Now we can define the family $\mathcal{B}(n,k,s)$.

$$\mathcal{B}(n,k,s) = \{F_1, \ldots, F_s\} \cup \left\{ F \in \binom{[n]}{k} : \exists E \in \mathcal{G},\ E \subset F \right\}.$$

Definition. We say that $\mathcal{F} \subset \binom{[n]}{k}$ is *saturated* if $\nu(\mathcal{F} \cup \{H\}) > \nu(\mathcal{F})$ holds for all $H \in \left(\binom{[n]}{k} - \mathcal{F}\right)$.

In the same way a family $\mathcal{F}$ of rank k is called saturated if $\nu(\mathcal{F} \cup \{H\}) > \nu(\mathcal{F})$ holds for all $|H| \leqq k$, $H \notin \mathcal{F}$.

Proposition 1.1. *$\nu(\mathcal{B}(n,k,s)) = s$ and $\mathcal{B}(n,k,s)$ is saturated.*

Proof. First let us show that $\nu(\mathcal{B}(n,k,s)) = s$ holds. The pairwise disjoint sets $F_1, \ldots, F_s$ show that $\nu(\mathcal{B}(n,k,s)) \geqq s$. On the other hand it should be clear that $B \cap [s+1] \neq \emptyset$ for all $B \in \mathcal{B}(n,k,s)$. Should $B_0, \ldots B_s$ be $s+1$

pairwise disjoint members of $\mathcal{B}(n,k,s)$ then after suitably renumbering them $B_i \cap [s+1] = \{i+1\}$ must hold. Since the only member $B \in \mathcal{B}(n,k,s)$ satisfying $B \cap [s] = \emptyset$ is $B = F_s$, $B_s = F_s$.

Next let us look at B_{s-1} satisfying $B_{s-1} \cap [s+1] = \{s\}$. Since all the edges $E \in \mathcal{G}$ satisfying $E \cap [s+1] = \{s\}$ are joining $\{s\}$ to a vertex of F_s, $E \not\subset B_{s-1}$ is impossible. Thus $B_{s-1} = F_{s-1}$ follows.

Repeating this argument we can prove $B_{s-2} = F_{s-2}, \dots, B_1 = F_1$. Then looking at B_0 satisfying $B_0 \cap [s+1] = \{1\}$ find again that $E \subset B_0$ is impossible because the neighbourhood of 1 in $\mathcal{G}$ is exactly $F_1 \cup \ldots \cup F_s$. Therefore there is no admissible choice for B_0. This proves $\nu(\mathcal{B}(n,k,s)) \leqq s$.

Let $H \in \binom{[n]}{k}$, $H \notin \mathcal{B}(n,k,s)$. By definition, H cannot contain any edge of $\mathcal{G} = \mathcal{F}_2$. In particular, $|H \cap [s]| \leqq 1$ holds.

Suppose first that $H \cap [s] = \emptyset$. Note that in $\mathcal{B}(n,k,s)$ there is a unique set, F_s which is disjoint to $[s]$. Since $H \notin \mathcal{F}$, $H \neq F_s$ holds. Let y_s be an arbitrary element of $F_s \setminus H$. Then $(s, y_s) \in \mathcal{G}$ and it is disjoint to H. Look at the set of neighbours of j, $1 \leqq j < s$, exceeding s. There are $(s+1-j)(k-1)+1$ of them. This guarantees that we can choose successively the neighbours y_j of j for $j = s-1, \dots, 1$ such that $y_j > s$ and they are distinct and disjoint to H. In this way we obtain $s+1$ pairwise disjoint sets $H, (s, y_s), \dots, (1, y_1)$. Using $n \geqq k(s+1)$, we can lift them to $s+1$ pairwise disjoint sets in $\mathcal{B}(n,k,s)$.

Let now H satisfy $\bigl|H \cap [s]\bigr| = \{i\}$ for some $1 \leqq i \leqq s$. Define $H_0 = H - \{i\}$. Since in $\mathcal{G}$ the vertex i is connected to all j, $s+1 \leqq j \leqq (s+1) + ((k-1)s+1-i) \overset{\text{def}}{=} i(k,s)$, $H_0 \subset [i(k,s)+1, n]$ must hold. The formula for $i(k,s)$ might look horrible but $i(k,s)$ is simply the last element of F_{i+1}. I.e., the requirement is that $H_0 \cap \bigl([s+1] \cup F_s \cup \ldots \cup F_{i+1}\bigr) = \emptyset$. This shows already that $H, F_s, \dots, F_{i+1}$ are pairwise disjoint. We are going to adjoin to them i more sets, actually edges of $\mathcal{G}$ that are disjoint to them and also pairwise disjoint among themselves. In this way we show $\nu\bigl(\mathcal{F}_{\max} \cup \{H\}\bigr) = s+1$ and conclude the proof.

To accomplish our task first note that $H \notin \mathcal{F}$ implies $H \neq F_i$. Choose $y_i \in F_i \setminus H$ and take the edge $(i, y_i) \in \mathcal{G}$.

Then just as in the above case, we can continue by successively choosing $(\ell, y_\ell) \in \mathcal{G}$ for $\ell = i-1, i-2, \dots, 1$ because ℓ is having more and more neighbours in $\mathcal{G}$ as ℓ decreases. $\square$

Our main result is the following.

Theorem 1. *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be resilient with $\nu(\mathcal{F}) = s$. Then for $k = 3$*

$$(4) \quad |\mathcal{F}_2| \leqq \binom{2s+1}{2} \quad \text{holds for all saturated families } \mathcal{F}_{\max} \text{ containing } \mathcal{F}.$$

For $k \geqq 4$ one has

$$(5) \qquad |\mathcal{F}_2| \leqq k\binom{s+1}{2} \quad \text{for all saturated families } \mathcal{F}_{\max}, \mathcal{F} \subset \mathcal{F}_{\max}.$$

The paper is organised as follows. In the next section we discuss shifting, an important operation on families that does not increase the matching number. Then we give the easy proof that (4) and (5) hold for shifted families.

The reason for considering the shifted case separately is two-fold. First, if one is working toward a proof of the Erdős Matching Conjecture, it is sufficient to consider shifted families (cf. e.g. [F2]). Second, the proof for this special case is much simpler but still conveys the flavour of the proof of the general case.

In Section 3 we prove (4) and (5) in full generality.

In Section 4 Theorem 1 is used to prove a Hilton–Milner-type result.

The final section is devoted to further problems.

2 Shifted families

For a family $\mathcal{F} \subset 2^{[n]}$ and fixed integers i and j, $1 \leqq i < j \leqq n$ one defines the (i,j)-shift $S_{i,j}(\mathcal{F}) = \{S_{i,j}(F) : F \in \mathcal{F}\}$ where

$$S_{i,j}(F) = \begin{cases} F' = (F - \{j\}) \cup \{i\} & \text{if } i \notin F,\ j \in F,\ F' \notin \mathcal{F}; \\ F & \text{otherwise.} \end{cases}$$

Note that $|F'| = |F|$ and the definition imply that the (i,j)-shift does not change the size of the sets in $\mathcal{F}$, neither $|\mathcal{F}|$. The fact that $\nu\big(S_{i,j}(\mathcal{F})\big) \leqq \nu(\mathcal{F})$ is easy to verify.

Let us consider the pentagon with edge-set

$$\mathcal{E} = \big\{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,1\}\big\}.$$

Applying the $(1,5)$-shift produces

$$S_{1,5}(\mathcal{E}) = \big\{\{1,2\},\{2,3\},\{3,4\},\{4,1\},\{5,1\}\big\}$$

which is no longer resilient. I.e., resilience is not invariant under shifting.

Definition. The family $\mathcal{F} \subset 2^{[n]}$ is called *shifted* if $\big((F - \{j\}) \cup \{i\}\big) \in \mathcal{F}$ holds for all $1 \leqq i < j \leqq n$ provided $i \notin F$, $j \in F$.

In view of the above, trying to prove the Erdős Matching Conjecture, one can always assume that $\mathcal{F} \subset \binom{[n]}{k}$ is shifted. For two distinct k-element sets $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$, where $a_1 < \cdots < a_k$, $b_1 < \cdots < b_k$ we say that A is smaller than B if $a_i \leqq b_i$ for all $1 \leqq i \leqq k$. This defines the so-called shifting partial order. By giving priority to smaller sets (in the shifting partial order) throughout the adding process one can assume that the saturated family $\mathcal{F}_{\max}$ is shifted as well.

Theorem 2.1. *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is shifted, resilient with $\nu(\mathcal{F}) = s$. Then for any saturated family $\mathcal{F}_{\max}$ containing $\mathcal{F}$ one has*

$$|\mathcal{F}_2| \leqq \binom{2s+1}{2} \quad \text{for} \quad k = 3 \quad \text{and} \tag{2.1}$$

$$|\mathcal{F}_2| \leqq k\binom{s+1}{2} \quad \text{for} \quad k \geqq 4. \tag{2.2}$$

Proof. Since $\mathcal{F}$ is resilient, $\nu\big(\mathcal{F}(\bar{1})\big) = s$ holds. Let us choose a matching $F_1, \ldots, F_s$ in $\mathcal{F}_{\max}(\bar{1})$ such that the size of $Z \overset{\text{def}}{=} F_1 \cup \ldots \cup F_s$ is as small as possible. Let us reorder F_i so that for some q, $0 \leqq q \leqq s$, $|F_i| = 2$ for $i \leqq q$ and $|F_i| \geqq 3$ for $i > q$ hold.

For notational convenience we set $\mathcal{G} = \mathcal{F}_2$, the collection of 2-element sets in $\mathcal{F}_{\max}$. Then $\mathcal{G} \subset \mathcal{F}_{\max}$ is a shifted graph satisfying $\nu(\mathcal{G}) \leqq s$.

Claim 2.1. *We can suppose that $Z = [2, |Z| + 1]$.*

Proof. Suppose that $j \in Z$ for some $j > |Z| + 1$. Then there is some $1 < i \leqq |Z| + 1$ with $i \notin Z$. Choose the unique set F_r from the matching with $j \in F_r$. Since $\mathcal{F}_{\max}$ is shifted, we can replace F_r by $\big(F_r - \{j\}\big) \cup \{i\}$. Repeating this operation leads eventually to $Z = [2, |Z| + 1]$. □

Since $\nu(\mathcal{F}_{\max}) = s$, $F \cap [2, |Z| + 1] \neq \emptyset$ holds for all $F \in \mathcal{F}_{\max}$. In particular, $(1, j) \notin \mathcal{G}$ for $j > |Z| + 1$. Using the shiftedness of $\mathcal{G}$, $(i, j) \notin \mathcal{G}$ follows for all i, that is $\mathcal{G} \subset \binom{[|Z|+1]}{2}$ holds.

For two disjoint sets P and Q let $e(P, Q)$ denote the number of edges in $\mathcal{G}$ between P and Q.

Claim 2.2. *For $1 \leqq i < j \leqq s$, one of the following holds.*

$$j \leqq q \quad \textit{and} \quad e(F_i, F_j) \leqq 4, \tag{2.3}$$

$$j > q \quad \textit{and} \quad e(F_i, F_j) \leqq \max\{|F_i|, |F_j|\}. \tag{2.4}$$

Proof. If $j \leqq q$ then $|F_i| = |F_j| = 2$ imply (2.3). Let $j > q$. We claim that there are no two disjoint edges $E, E' \in \mathcal{G}$ connecting F_i and F_j. Indeed, otherwise we could replace F_i and F_j by E and E' to obtain a matching of total size smaller than $|Z|$. This would contradict the minimal choice of Z.

Now (2.4) follows from the fact that the only bipartite graphs with matching number one are the stars. □

Observing that for $j > q$, by the minimal choice of Z, F_j contains no edge of $\mathcal{G}$, we can list and estimate the types of possible edges in $\mathcal{G}$.

(a) $(1, r)$: at most $|Z|$, one for each $r \in Z$;

(b) inside F_i: one for each $1 \leqq i \leqq q$;

(c) between F_i, F_j: at most 4 for $j \leqq q$, at most k otherwise.

Using $|Z| = |F_1| + \ldots + |F_s| \leqq 2s + (k-2)(s-q)$ let us deduce the desired bounds.

(i) $k = 3$

$$|\mathcal{G}| \leqq 2s+(s-q)+q+\left(\binom{s}{2} - \binom{q}{2}\right)\times 3+\binom{q}{2}\times 4 \leqq 3s+4\binom{s}{2} = \binom{2s+1}{2}.$$

(ii) $k \geqq 4$

$$|\mathcal{G}| \leqq 2s + q + (k-2)(s-q) + \binom{s}{2} \times k \leqq k\left(s + \binom{s}{2}\right) = k\binom{s+1}{2}.$$

Let us now analyse the case of equality. If $k = 3$ then $k < 4$ implies that the inequality is strict unless $q = s$. That is, $|Z| = 2s$ and $\mathcal{G} \subset \binom{[2s+1]}{2}$ hold. Consequently, $|\mathcal{G}| = \binom{2s+1}{2}$ holds iff $\mathcal{G}$ is the complete graph on the vertex set $[2s+1]$.

If $k \geqq 4$ then in order to have equality in (ii), $|F_1| = \ldots = |F_s| = k$ must hold.

To determine the exact structure of $\mathcal{G}$ let a_i be the smallest element of F_i for $1 \leqq i \leqq s$ and let us number the F_i in increasing order: $a_1 < a_2 < \ldots < a_s$. Since we have equality in (ii), there are exactly k edges between F_i and F_j for every $1 \leqq i < j \leqq s$.

Claim 2.3. *The k edges between F_i and F_j are (a_i, b) such that $b \in F_j$ (for all $1 \leqq i < j \leqq s$).*

Proof. Since there are no two independent edges between F_i and F_j, the induced bipartite graph must be a star. By shiftedness the center of the star is a_i or a_j. However, if it is a_j then we can take a vertex $c \in F_i - \{a_i\}$ and look at $(a_j, c) \in \mathcal{G}$. Recalling that $\mathcal{G}$ is shifted and $a_i < a_j$, $(a_i, c) \in \mathcal{G}$ follows. Replacing F_i by (a_i, c) we get a contradiction with the minimal choice of Z. $\square$

In the same way we obtain that every element of $F_i - \{a_i\}$ is larger than every element of $F_j - \{a_j\}$. It follows that $a_i = i + 1$ for $i = 2, \ldots, s$. Also $F_i - \{a_i\} = \big\{(k-1)(s-i)+s+2, (k-1)(s-i)+s+3, \ldots, (k-1)(s-i)+s+k\big\}$. $\square$

3 General families

Let us consider $\mathcal{F} \subset \binom{[n]}{k}$, $\nu(\mathcal{F}) = s$, $\mathcal{F}$ is both saturated and resilient. Let $\mathcal{F}_{\max}$ be a non-extendable family of rank k containing $\mathcal{F}$. Then $\mathcal{F}_{\max}$ itself is saturated as well. Our goal is to estimate $|\mathcal{G}|$ where $\mathcal{G} = \mathcal{F}_{\max} \cap \binom{[n]}{2}$ is the graph of 2-element sets in $\mathcal{F}_{\max}$.

Let x be a vertex of maximal degree in $\mathcal{G}$. Let us fix a matching $\{F_1, ..., F_s\} \subset \mathcal{F}_{\max}(\overline{x})$ for which $Z \overset{\text{def}}{=} F_1 \cup \ldots \cup F_s$ has *minimal* size. Many of the simple statements that we had for shifted families remain valid.

To make the proofs smooth and short, let us describe the situations that cannot occur.

For disjoint sets P, Q let $\mathcal{B}(P, Q)$ denote the bipartite graph with partite sets P and Q, edge-set $(P \times Q) \cap \mathcal{G}$. Set $Y = [n] - (\{x\} \cup Z)$. In what follows *edge* always means an edge in $\mathcal{G}$.

- (α) Any edge from x to a vertex in Y would increase $\nu(\mathcal{F}_{\max})$.
- (β) If $\mathcal{B}(F_i, Y) \neq \emptyset$ then $|F_i| = 2$ as otherwise replacing F_i by that edge decreases $|Z|$.
- (γ) Even for $|F_i| = 2$, $\nu\big(\mathcal{B}(F_i, Y)\big) \leqq 1$ (otherwise $\nu(\mathcal{F}_{\max}) > s$).
- (δ) If $|F_j| > 2$ then $\nu\big(\mathcal{B}(F_i, F_j)\big) \leqq 1$ for all $i \neq j$ (otherwise replacing F_i, F_j by two independent edges makes $|Z|$ smaller.

Let us note that a bipartite graph of matching number 1 is a star. We use this to divide the 2-sets among the F_i into 4 groups. Let $|F_1| = \ldots = |F_t| = 2$, $|F_j| \geqq 3$ for $t < j \leqq s$. We partition $[t]$ into $T_0 \cup T_1 \cup T_2 \cup T_3$ as follows.

$i \in [t]$ is in T_0 (T_1) if $|\mathcal{B}(F_i, Y)| = 0$ $(= 1)$, respectively.

$i \in T_2$ if there is $x_i \in F_i$ which is joined to at least two vertices in Y. (Both for $i \in T_1$ and T_2 we set $F_i = \{x_i, z_i\}$, where $\mathcal{B}(z_i, Y) = \emptyset$.)

$i \in T_3$ if there is a vertex $y = y(F_i) \in Y$ which is joined to both vertices of F_i.

The following statements follow easily from $\nu(\mathcal{F}_{\max}) = s$ or the minimality of $|Z|$.

(ε) $\mathcal{B}(x, F_i) = \emptyset$ if $i \in T_3$.

(ζ) If $i \in T_3$ and $j > t$ then $\mathcal{B}(F_i, F_j) = \emptyset$.

For $i \in T_1 \cup T_2$ let $F_i = \{x_i, z_i\}$ where z_i is the vertex with no neighbours in Y.

(η) $(x, z_i) \notin \mathcal{G}$, for $i \in T_1 \cup T_2$.

(ϑ) $(z, z_i) \notin \mathcal{G}$ for $z \in F_j$, $j \in T_3$ and $i \in T_2$.

(ι) If $i \in T_1 \cup T_2$ then $\mathcal{B}(z_i, F_j) = \emptyset$ for $t < j \leqq s$.

We cannot say much about T_0 unconditionally. However, knowing that both or at least one of its vertices are connected to x some restrictions arise. Define $F_i = \{x_i, z_i\}$ for $i \in T_0$.

(κ) If $(x, x_i), (x, z_i) \in \mathcal{G}$ then $\mathcal{B}(F_i, F_j) = \emptyset$ for $i \in T_0$, $j \in T_3$.

(λ) If $(x, x_i) \in \mathcal{G}$ then $(z_i, z_{i'}) \notin \mathcal{G}$ for $i \in T_0$, $i' \in T_1 \cup T_2$.

Now the preparation is finished, we are ready to prove Theorem 1. Let us restate it with $\mathcal{G}$ instead of $\mathcal{F}_2$.

Theorem 3.1. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be saturated for $\nu(\mathcal{F}) = s$. Let $\mathcal{F} \subset \mathcal{F}_{\max}$, $r(\mathcal{F}_{\max}) = s$ and $\mathcal{F}_{\max}$ is saturated, $\mathcal{G} = \mathcal{F}_{\max} \cap \binom{[n]}{2}$. If $\mathcal{F}$ is resilient then* (i) *and* (ii) *hold.*

(i) *If $k = 3$ then $|\mathcal{G}| \leqq \binom{2s+1}{2}$.*

(ii) *If $k \geqq 4$ then $|\mathcal{G}| \leqq k\binom{s+1}{2}$.*

Proof of the theorem. We continue using the notation of this section. To estimate $|\mathcal{G}|$ let x be a vertex of maximal degree in $\mathcal{G}$. We freely use the statements $(\alpha) \sim (\lambda)$. First let us observe

$$|\mathcal{B}(F_i, F_j)| \leqq k \text{ holds (by } (\delta)) \tag{3.1}$$

whenever $t < j \leqq s$ and

$$\bigl|\mathcal{B}(F_i, F_j)\bigr| \leqq 4 \text{ for } 1 \leqq i < j \leqq t. \tag{3.2}$$

Let us also note that we showed above that $(z_i, z_j) \notin \mathcal{G}$ in many cases. If x is connected to some vertex of $F_i \cup F_j$ then (3.2) can be replaced by

$$\bigl|\mathcal{B}(F_i, F_j)\bigr| \leqq 3 \text{ if } (i, j) \subset T_1 \cup T_2 \cup T_3. \tag{3.3}$$

(Indeed, $\mathcal{B}(F_i, F_j) = 4$ implies that $|F_i \cup F_j| = 4$ and these four vertices span a complete graph.) If $i \in T_0$ and $j \in T_1 \cup T_2 \cup T_3$ but $\mathcal{B}(x, F_i) \neq \emptyset$ then (3.3) holds as well.

Let $d = |\mathcal{G}(x)|$ be the degree of x. Then

$$d \leqq 2 \times |T_0| + |T_1| + |T_2| + |F_{t+1}| + \ldots + |F_s| \text{ holds.} \tag{3.4}$$

We are going to prove the theorem by altering the graph $\mathcal{G}$ gradually. For notational convenience we keep denoting the newer and newer graphs by the same letter $\mathcal{G}$. We are going to be careful in keeping x as a vertex of maximum degree and safeguarding the validity of (3.4). We achieve the latter by never connecting x to any vertex outside $F_1 \cup \ldots \cup F_s$.

First consider F_j with $j \in T_3$. By (ζ) there are no edges in $\mathcal{B}(F_i, F_j)$ for $i \in [t+1, s]$. However, in (3.1) we allow for k edges in it.

Let us remove the vertices $y(F_j)$ for all $j \in T_3$ and put in edges (x, z) for all $z \in F_j$, $j \in T_3$.

We maintained the number of edges in $\mathcal{G}$ and to our disadvantage we increased $d = |\mathcal{G}(x)|$ by $2 \times |T_3|$.

In the new situation $T_3 = \emptyset$.

Next we consider the vertex x_i in F_i for $i \in T_1 \cup T_2$. If $(x, x_i) \notin \mathcal{G}$ then we put this edge into $\mathcal{G}$ and remove one edge (x_i, y) with $y \in Y$. This increases $|\mathcal{G}(x)|$ but leaves $|\mathcal{G}|$ unchanged. If in the new graph $\mathcal{B}(x_i, Y) = \emptyset$, we move i into T_0. For the remaining i we rename $T_1 \cup T_2$ simply to T_1.

Finally, we join x to all vertices in F_i for $i \in T_0$ (if not previously joined). Note that the degree of x still verifies (3.4) with the new T_0 and T_1 (and $T_2 = \emptyset$). The RHS corresponds to the size of

$$\left(\bigcup_{i \in T_0} F_i\right) \cup \{x_i : i \in T_1\} \cup (F_{t+1} \cup \ldots \cup F_s).$$

Our next plan is to gradually replace the edges (x_i, y) for some $y \in Y$ for $x_i \in F_i$, $i \in T_1$ by edges of the form (x_i, z) for $z \in Z$.

If $(x_i, y) \in \mathcal{G}$ and $(x_i, z) \notin \mathcal{G}$ for some $z \in F_j$, $t < j \leqq s$, then we can do it in view of (ι) without destroying the validity of $\left|\mathcal{B}(F_i, F_j)\right| \leqq k$. Once $(x_i, z) \in \mathcal{G}$ for all $z \in F_{t+1} \cup \ldots \cup F_s$, we continue with (x_i, z) where $z \in \bigcup_{i \in T_0} F_i$.

If even after saturating these vertices some edges of the form (x_i, y) remain in $\mathcal{G}$ with $y \in Y$ then we turn to our final resource $\{x_{i'} : i' \in T_1\}$. Here we must be careful, because if say $(x_i, y) \in \mathcal{G}$, $(x_{i'}, y') \in \mathcal{G}$ are two edges with $y, y' \in Y$ we should not replace them by a single edge $(x_i, x_{i'})$ thereby diminishing $|\mathcal{G}|$.

Here we use that $i, i' \in T_1$ at present means that both x_i and $x_{i'}$ were joined to at least two vertices in $Y \cup \{x\}$ in $\mathcal{G}$ forcing $(z_i, z_{i'}) \notin \mathcal{G}$. Thus we can replace $(x_i, y), (x_{i'}, y')$ by $(x_i, x_{i'})$ and $(z_i, z_{i'})$. At this point, i.e., if for some $i \in T_1$ all $(x_i, x_{i'})$ are also in $\mathcal{G}$ for $i \not\equiv i' \in T_1$ then the degree of x_i (not forgetting the edge (x_i, z_i)) is at least $2|T_0| + |T_1| + |F_{t+1}| + \ldots + |F_k|$, i.e., it is equal to $|\mathcal{G}(x)|$. This shows that no more edges join x_i to Y. Therefore, the moment the above process terminates $\mathcal{G} \subset \binom{Z \cup \{x\}}{2}$ holds.

Let us do the final computation. The edges in $\mathcal{G}$ fall into three groups. First, those in $\{x\} \cup F_1 \cup \ldots \cup F_t$. Second, those in $F_{t+1} \cup \ldots \cup F_s$. Third, those connecting these two groups. We bound the respective sizes by $\binom{2t+1}{2}$, $\binom{s-t}{2} \times k$ and $(t+1)(s-t)k$. For $k = 3$ we need to show the following

$$\binom{2t+1}{2} + 3\binom{s-t}{2} + 3(t+1)(s-t) \leqq \binom{2s+1}{2}. \tag{3.5}$$

Rearranging we get

$$\frac{3(s-t)}{2}(s+t+1) \leqq (s-t)(2s+2t+1).$$

We have equality for $s = t$. If $s > t$ then dividing by $\frac{s-t}{2}$ gives $3s + 3t + 3 \leqq 4s + 4t + 2$. This is a strict inequality unless $s = 1$, $t = 0$, in which case equality holds. Thus (3.5) is proven.

For $k \geqq 4$ we need

$$\binom{2t+1}{2} + k\binom{s-t}{2} + k(t+1)(s-t) \leqq k\binom{s+1}{2}. \tag{3.6}$$

Rearranging gives

$$\binom{2t+1}{2} \leqq k\binom{t+1}{2}.$$

The RHS is an increasing function of k and already for $k = 4$ we have

$$\binom{2t+1}{2} = 2t^2 + t \leqq 2t^2 + 2t = 4\binom{t+1}{2} \quad \text{with equality only for} \quad t = 0.$$

Thus (3.6) and the theorem are proved. □

Remark. Analyzing the case of equality in (3.5) shows that equality holds either if $s = t$ and $\mathcal{G}$ is the complete graph on vertex set $\{x\} \cup F_1 \cup \ldots \cup F_t$ or $s = 1$, $t = 0$, i.e., $\mathcal{G}$ is a star with three edges (joining x to the vertices of F_1). In case of (3.6) equality implies $t = 0$ and $|F_1| = \ldots = |F_s| = k$. Moreover, x is joined to all vertices in $F_1 \cup \ldots \cup F_k$ and $\mathcal{B}(F_i, F_j)$ is a star of k edges for every choice of $1 \leqq i < j \leqq s$.

Let us make a tournament on the vertex set $[s]$ by drawing an edge from i to j if the center of the star $\mathcal{B}(F_i, F_j)$ is in F_i.

Claim. *This defines a transitive tournament.*

Indeed, otherwise we can find $i_1, i_2, i_3 \in [s]$ such that the three edges are $i_1 \to i_2 \to i_3 \to i_1$. Let x_{i_j} be the center of the star and z_{i_j} a different vertex for $j = 1, 2, 3$. Then using the three independent edges $(x_{i_1}, z_{i_2}), (x_{i_2}, z_{i_3}), (x_{i_3}, z_{i_1})$ to replace F_{i_j}, $j = 1, 2, 3$ shows that Z was not minimal, a contradiction.

In view of the claim after renumbering F_i we can choose $x_i \in F_i$, $i = 1, \ldots, s$ such that, setting $x_0 = x$ the edges in $\mathcal{G}$ are $\big\{(x_i, z_j) : 0 \leqq i < j \leqq s,\ z_j \in F_j\big\}$. That is, $\mathcal{G}$ has exactly the same structure as in $\mathcal{B}(n, k, s)$.

For a family $\mathcal{F} \subset \binom{[n]}{k}$, with $\nu(\mathcal{F}) = s$ let $\mathcal{F}_{\max}$ be a non-extendable family of rank k containing $\mathcal{F}$. Let $\mathcal{F}_i$, $2 \leqq i \leqq k$ be the collection of minimal (for containment) i-element sets in $\mathcal{F}_{\max}$. In particular, $\mathcal{F}_2 = \mathcal{G}$. It is well known that $|\mathcal{F}_i|$ is bounded. E.g., $\mathcal{F}_i$ contains no sunflower of size $ks + 1$ (cf. the proof of Claim 4.1 below). By a classical result of Erdős and Rado [ER],

$$|\mathcal{F}_0| \leqq (kis)^i i!$$

This implies

$$|\mathcal{F}| = |\mathcal{G}|\binom{n-2}{k-2} + \sum_{3 \leqq i \leqq k} i!(kis)^i \binom{n-i}{k-i}. \tag{3.7}$$

Thus we can deduce:

Theorem 3.2. *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is resilient with matching number s. Then*

(i) $|\mathcal{F}| \leqq \binom{2s+1}{2}n + O(1)$ *for* $k = 3$ *and*

(ii) $|\mathcal{F}| \leqq k\binom{s+1}{2}\binom{n-2}{k-2} + O(n^{k-3})$ *for* $k \geqq 4$.

Moreover, these bounds are best possible.

4 Hilton–Milner-type results

Let us recall that a family $\mathcal{F}$ is called *intersecting* if $F \cap F' \neq \emptyset$ holds for all $F, F' \in \mathcal{F}$. That is, a non-empty family $\mathcal{F}$ is intersecting iff $\nu(\mathcal{F}) = 1$ holds. The Erdős Matching Conjecture is known to be true for $s = 1$. In fact, no doubt it was this case that served as a motivation for Erdős to consider the problem for $s \geqq 2$. Namely, the $s = 1$ case follows from the Erdős–Ko–Rado Theorem.

Theorem ([EKR])**.** *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$, $n \geqq 2k$ and $\nu(\mathcal{F}) = 1$. Then*

$$|\mathcal{F}| \leqq \binom{n-1}{k-1} \quad \text{holds.} \tag{4.1}$$

Looking at all k-sets through a fixed vertex shows that (4.1) is best possible. An intersecting family $\mathcal{F}$ is called *non-trivial* if $\bigcap_{F \in \mathcal{F}} F = \emptyset$ holds. For $s = 1$ non-trivial coincides with our notion of resilience.

Hilton–Milner Theorem ([HM])**.** *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$, $n > 2k$ and $\mathcal{F}$ is a non-trivial intersecting family. Then*

$$|\mathcal{F}| \leqq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1 \quad \text{holds.} \tag{4.2}$$

Moreover, in case of equality either $\mathcal{F} = \mathcal{B}(n, k, 1)$ or $k = 3$ and $\mathcal{F} = \mathcal{A}_2(n, 3, 1)$ hold.

For $n > n_0(s)$ one can sharpen Theorem 3.2:

Theorem 4.1. *Suppose that $\mathcal{F} \subset \binom{[n]}{3}$ is resilient with $\nu(\mathcal{F}) = s$. Then for $n \geqq n_0(s)$*

$$|\mathcal{F}| \leqq \left|\mathcal{A}_2(n,3,s)\right| \quad \textit{holds.} \tag{4.3}$$

Moreover, for $s \geqq 2$ the inequality is strict unless $\mathcal{F}$ is isomorphic to $\mathcal{A}_2(n,3,s)$.

Proof. Defining $\mathcal{F}_{\max}$ and $\mathcal{F}_2$ as before, recall that we proved: $|\mathcal{F}_2| < \binom{2s+1}{2}$ unless $\mathcal{F}_2$ is a complete graph on $2s+1$ vertices.

Now, if $|\mathcal{F}_2| \leqq \binom{2s+1}{2} - 1$ then define $\mathcal{H} = \{H \in \mathcal{F} : \nexists E \in \mathcal{F}_2,\ E \subset H\}$.

Claim 4.1. $|\mathcal{H}| = O(1)$, *i.e., it is bounded independent of n.*

Proof. Let us suppose that $\mathcal{H}$ contains a *sunflower* of size $3s+1$, that is $3s+1$ members $H_0, \ldots, H_{3s} \in \mathcal{H}$ such that *all* pairwise intersections $H_i \cap H_j$ are the same, $0 \leqq i < j \leqq 3s$. Set $A = H_1 \cap H_2$.

We claim that $\nu(\mathcal{H} \cup \{A\}) = s$. Indeed, if there are $F_1, \ldots, F_s \in \mathcal{F}$ forming together with A a collection of $s+1$ pairwise disjoint sets, then using $|F_1 \cup \ldots \cup F_s| = 3s$, we can find an i, $0 \leqq i \leqq 3s$ for which $(H_i - A) \cap (F_1 \cup \ldots \cup F_s) = \emptyset$. Thus $H_i, F_1, \ldots, F_s$ are $s+1$ pairwise disjoint sets, a contradiction.

There are two possibilities for $|A|$. If $|A| = 1$, then $\nu(\mathcal{F} \cup \{A\}) = s$ contradicts the resilience of $\mathcal{F}$. If $|A| = 2$, then A can be added to $\mathcal{F}_2$, i.e., it contradicts the fact that $\mathcal{F}_{\max}$, the family used in defining $\mathcal{F}_2$ was saturated.

Consequently, $\mathcal{H}$ contains no sunflower of size $3s+1$. In view of an old theorem of Erdős and Rado [ER], this concludes the proof of the claim.

As a matter of fact the bounds from [ER] show $|\mathcal{H}| \leqq 6 \cdot (3s)^3$. □

Now using $\mathcal{F}_1 = \emptyset$ (by resilience),

$$|\mathcal{F}| \leqq \left(\binom{2s+1}{2} - 1\right)(n-2) + 6 \cdot (3s)^3 \ \text{ follows.}$$

Noting

$$\left|\mathcal{A}_2(n,3,s)\right| = \binom{2s+1}{3} + \binom{2s+1}{2}(n-2s-1),$$

for sufficiently large n we infer that (4.3) holds with strict inequality unless $\mathcal{F}_2 = \binom{R}{2}$ for some $R \in \binom{[n]}{2s+1}$.

If H is any set with $|H \cap R| \leqq 1$ then one can find s pairwise disjoint pairs in $\binom{R}{2}$ which are disjoint to H as well. That is, $\mathcal{F}_2 = \binom{R}{2}$ and $\nu(\mathcal{F}) = s$ imply $\mathcal{F} \subset \left\{B \in \binom{[n]}{3} : |B \cap R| \geqq 2\right\}$, completing the proof of the theorem. $\square$

For $k \geqq 4$ as well one can classify the optimal families.

Theorem 4.2. *Let $k \geqq 4$ and $n > n_0(k,s)$. Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is resilient with $\nu(\mathcal{F}) = s$. Then*

$$|\mathcal{F}| \leqq |\mathcal{B}(n,k,s)| \tag{4.4}$$

with strict inequality unless $\mathcal{F}$ is isomorphic to $\mathcal{B}(n,k,s)$.

Proof. We shall rely on the uniqueness part of Theorem 3.1 (ii). Let $\mathcal{G}$ be the collection of two element sets in $\mathcal{F}_{\max}$ and suppose that $|\mathcal{G}| = k\binom{s+1}{2}$ holds. Then there exist vertices $x_0, x_1, \ldots, x_s$ and pairwise disjoint edges $F_1, \ldots, F_s \in \mathcal{F}$ such that $x_i \in F_i$ and the edges of $\mathcal{G}$ are the pairs (x_i, y_j) for $0 \leqq i < j \leqq s$, $y_j \in F_j$. That is, x_i is having $i + (s-i)k$ neighbours (the i coming from the edges $(x_{i'}, x_i)$, $i' < i$).

Since we want to prove an upper bound for $|\mathcal{F}|$, we may assume that $\mathcal{F}$ is saturated. In particular, all $F \in \binom{[n]}{k}$ satisfying $E \subset F$ for some $E \in \mathcal{G}$ are in $\mathcal{F}$. However, together with the matching $F_1, \ldots, F_s$ they form an isomorphic copy of $\mathcal{B}(n,k,s)$.

On the other hand Proposition 1.1 says that $\mathcal{B}(n,k,s)$ is saturated. This proves that $\mathcal{F}$ is isomorphic to $\mathcal{B}(n,k,s)$.

Finally we have to prove that if $|\mathcal{G}| < k\binom{s+1}{2}$ then $|\mathcal{F}| < |\mathcal{B}(n,k,s)|$. We shall do it in a more general way. A member G of $\mathcal{F}_{\max}$ is called minimal if there is no $H \in \mathcal{F}_{\max}$, $H \subsetneqq G$. Let us define

$$\mathcal{G}_\ell = \left\{G \in \binom{[n]}{\ell} : G \in \mathcal{F}_{\max},\ G \text{ is minimal}\right\}.$$

Since every $F \in \mathcal{F}_{\max}$ contains at least one minimal element,

$$\left|\mathcal{F}_{\max}\right| \leqq |\mathcal{G}|\binom{n-2}{k-2} + \sum_{3 \leqq \ell \leqq k} |\mathcal{G}_\ell|\binom{n-\ell}{k-\ell} \quad \text{holds.} \tag{4.5}$$

To conclude the proof it is sufficient to show that $|\mathcal{G}_\ell|$ is bounded by some function of k and s.

Claim 4.2. *$\mathcal{G}_\ell$ contains no sunflower of size $ks + 1$.*

Proof. Suppose that for some A, $|A| < \ell$ there exist $G_0, \ldots, G_{ks} \in \mathcal{G}_\ell$ with $G_i \cap G_j = A$, $0 \leqq i < j \leqq ks$. We claim that $\nu\big(\mathcal{F}_{\max} \cup \{A\}\big) = s$. The contrary would mean the existence of pairwise disjoint sets $H_i \in \mathcal{F}_{\max}$, $1 \leqq i \leqq s$ that are disjoint to A as well.

Since $\big|H_1 \cup \ldots \cup H_s\big| \leqq ks$, at least one of the $ks + 1$ pairwise disjoint sets $G_j - A$, $0 \leqq j \leqq sk$ is disjoint to $H_1 \cup \ldots \cup H_s$. Consequently, $G_j, H_1, \ldots, H_s$ are $s + 1$ pairwise disjoint sets in $\mathcal{F}_{\max}$, a contradiction to $\nu(\mathcal{F}_{\max}) = \nu(\mathcal{F}) = s$. □

Now the above-cited result of Erdős and Rado [ER] implies that $\mathcal{G}_\ell$ is bounded. For example,

$$|\mathcal{G}_\ell| \leqq \ell!(ks)^\ell.$$

This concludes the proof of (4.5) and also that of the theorem. □

Remark. The Hilton–Milner Theorem holds for all $n > 2k$. However, we could prove Theorems 4.1 and 4.2 only for substantially larger values of n with respect to k and s. It would be desirable to prove these results for $n > cks$ for some absolute constant c.

5 Some related problems

One can define higher resilience too.

Definition. For $1 \leqq t < k$ we say that $\mathcal{F} \subset \binom{[n]}{k}$ is t-resilient if $\nu\big(\mathcal{F}(\overline{T})\big) = \nu(\mathcal{F})$ holds for all $T \in \binom{[n]}{t}$. Let us define

$$m_t(n, k, s) = \max\left\{|\mathcal{F}| : \mathcal{F} \subset \binom{[n]}{k},\ \nu(\mathcal{F}) = s,\ \mathcal{F} \text{ is } t\text{-resilient}\right\}.$$

Note that for $s = 1$ the notion of t-resilience is equivalent to saying $\tau(\mathcal{F}) > t$, i.e., there is no t-element set meeting all members of $\mathcal{F}$.

Probably the most investigated question in this direction is estimating $m_{k-1}(n, k, 1)$. In their seminal paper Erdős and Lovász [EL] proved that

(5.1) $$m_{k-1}(n, k, 1) \leqq k^k, \text{ independent of } n.$$

This was the motivation for many of the related research. In [FOT2] a lower bound slightly better than $\left(\frac{k}{2}\right)^{k-1}$ is proven. However, it is still a large gap which seems to be difficult to diminish.

$m_2(n, k, 1)$ was determined by the author in [F3] for $k \geqq 3$, $n > n_0(k)$.

$m_3(n, k, 1)$ was determined in [FOT1] for $k \geqq 9$, $n > n_0(k)$ and by Furuya–Takatou ([FT1], [FT2]) for $5 \leqq k \leqq 8$.

Conlon and Rödl [CR] observed that $m_{k-1}(n, k, s) \leqq (sk)^k$ can be proved in the same way as (5.1). However, there are no constructions coming anywhere close to it.

For the case $k = 2$, which we did not mention so far,

$$m_1(n, 2, s) = \binom{2s+1}{2}$$

can be easily derived e.g. using the Berge–Tutte formula for the matching number. It does actually follow from our proof for the case $k = 3$ as well.

Let us close this paper by the following conjecture.

Conjecture ([FOT2])**.** *There exists a function $p(k, t)$ of k and t, which is a polynomial of degree at most $t - 3$ in k such that*

$$m_t(n, k, 1) = \left(k^{t-1} - \binom{t-1}{2} k^{t-2} + p(k, t) + o(1)\right) \binom{n-t}{k-t} \tag{5.2}$$

holds for $k > k_0(t)$.

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