Proof of the Erdős Matching Conjecture in a New Range

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Abstract

Let $s > k \geqq 2$ be integers. It is shown that there is a positive real $\varepsilon = \varepsilon(k)$ such that for all integers n satisfying $(s+1)k \leqq n < (s+1)(k+\varepsilon)$ every k-graph on n vertices with no more than s pairwise disjoint edges has at most $\binom{(s+1)k-1}{k}$ edges in total. This proves a part of an old conjecture of Erdős.

1 Introduction

Let $[n] = \{1, \ldots, n\}$ be the standard n-element set. For an integer $k \geqq 2$ a family $\mathcal{F} \subset \binom{[n]}{k}$ is called a k-graph. The matching number, $\nu(\mathcal{F})$ of $\mathcal{F}$ is the *maximum* number of pairwise disjoint edges in $\mathcal{F}$. Obviously, $\nu(\mathcal{F}) \leqq n/k$ holds.

Consequently for every k and $s \geqq 1$ the family $\mathcal{A} = \binom{[(s+1)k-1]}{k}$ has matching number equal to s. Note that $\mathcal{A}$ is independent of n for $n \geqq (s+1)k - 1$. Define

$$\mathcal{B} = \mathcal{B}(n,k,s) = \left\{ B \in \binom{[n]}{k} : B \cap [s] \neq \emptyset \right\}. \tag{1.1}$$

Fixing k and s we always assume $n \geqq (s+1)k$. From definition (1.1) it should be clear that $\nu(\mathcal{B}) = s$ holds.

Erdős Matching Conjecture (Erdős [E])**.** *Let $k, s \geqq 1$ be fixed integers, $\mathcal{F} \subset \binom{[n]}{k}$ a k-graph satisfying $\nu(\mathcal{F}) \leqq s$. Then*

$$|\mathcal{F}| \leqq \max\left\{|\mathcal{A}|, |\mathcal{B}|\right\} \quad \textit{holds.} \tag{1.2}$$

Let us note that the case $k = 1$ is trivial. For $k = 2$ Erdős and Gallai [EG] proved (1.2). The case $k = 3$ is completely settled now. Łuczak and Mieczkowska [LM] proved (1.2) for s very large and then the present author [F2] for all s.

The case $s = 1$ is the classical Erdős–Ko–Rado Theorem [EKR], in that case for $n \geqq 2k$ the maximum is always attained by the second term, its value is $\binom{n-1}{k-1}$.

Erdős [E] proved (1.2) for $n > n_0(k, s)$ for a certain value of $n_0(k, s)$. Over the years the bound on $n_0(k, s)$ was successively improved by Bollobás, Daykin and Erdős [BDE], Füredi and the present author (unpublished), Huang, Loh and Sudakov [HLS], Frankl, Łuczak and Mieczkowska [FLM]. The current best bound is due to the author [F1] and it shows that (1.2) is true for $n \geqq 2(s+1)k$. However, in *all* these cases the maximum is given by $|\mathcal{B}|$.

The biggest difference in the behaviour of $|\mathcal{B}|$ and $|\mathcal{A}|$ is that the first one is a strictly increasing function of n while the second one is a constant. That is, (1.2) asserts that in the range where $|\mathcal{A}| > |\mathcal{B}|$, adding an extra vertex does not help, the maximum of $|\mathcal{F}|$ remains unchanged.

Easy computation shows that for $n \geqq \left(k + \frac{1}{2}\right)(s+1)$ already $|\mathcal{B}|$ is greater. Thus in (1.2), $|\mathcal{A}|$ is greater only in an interval of length at most $(s+1)/2$. The aim of the present paper is to confirm (1.2) for a small but positive proportion of this interval.

Theorem. *For every $k \geqq 2$ there is a positive $\varepsilon = \varepsilon(k)$ such that* (1.2) *holds for $k(s+1) \leqq n \leqq (k+\varepsilon)(s+1)$. Moreover, the only family $\mathcal{F}$ attaining equality is $\binom{Q}{k}$ for some $Q \subset [n]$, $|Q| = (s+1)k - 1$.*

Let us stress that this is the first result proving (1.2) for a range where $|\mathcal{A}| > |\mathcal{B}|$.

2 Preliminaries

Let $(a_1, a_2, \ldots, a_r)$ denote the set $\{a_1, \ldots, a_r\}$ if we know and want to stress that the elements are listed in increasing order: $a_1 < a_2 < \ldots < a_r$.

Definition 2.1 (Shifting partial order)**.** Let us define the shifting partial order $\prec$ where $(a_1, \ldots, a_r) \prec (b_1, \ldots, b_r)$ iff $a_i \leqq b_i$ for all i. A family $\mathcal{F} \subset \binom{[n]}{k}$ is called *shifted* if whenever $F, G \in \binom{[n]}{k}$ with $F \prec G$, $G \in \mathcal{F}$ then $F \in \mathcal{F}$ holds as well.

It is well known (cf. [F3] for a proof) that in proving the theorem (or (1.2) in general) one can assume that $\mathcal{F}$ is shifted. Throughout the paper we consider s and k fixed and let $\mathcal{F} \subset \binom{[n]}{k}$ be a family of maximal size with respect to $\nu(\mathcal{F}) = s$. Moreover, we suppose that $\mathcal{F}$ is shifted. Define $\mathcal{T} = \mathcal{T}(\mathcal{F})$ the *trace* of $\mathcal{F}$ on $[k(s+1)-1]$ by

$$\mathcal{T} = \bigl\{F \cap [k(s+1)-1] : F \in \mathcal{F}\bigr\}.$$

Of course $\mathcal{T}$ needs not to be k-uniform but $|T| \leqq k$ for all $T \in \mathcal{T}$.

Proposition 2.1 ([F3])**.**

$$\nu(\mathcal{T}) = s. \tag{2.1}$$

Since $|\mathcal{F}|$ is maximal, $\mathcal{F}$ is determined by $\mathcal{T}$:

$$\mathcal{F} = \left\{F \in \binom{[n]}{k} : \exists\, T \in \mathcal{T},\, T \subset F\right\}. \tag{2.2}$$

Let $\mathcal{T} = \mathcal{T}^{(1)} \cup \ldots \cup \mathcal{T}^{(k)}$ where $\mathcal{T}^{(d)} = \bigl\{T \in \mathcal{T} : |T| = d\bigr\}$. Set also $t(d) = |\mathcal{T}^{(d)}|$. For notational convenience we set $\overline{n} = n - (k(s+1)-1)$. In view of (2.2) we have

$$|\mathcal{F}| = \sum_{1 \leqq d \leqq k} t(d) \binom{\overline{n}}{k-d}. \tag{2.3}$$

Note that in case of $\mathcal{A}$ one has

$$\mathcal{T}(\mathcal{A}) = \mathcal{A} = \binom{[k(s+1)-1]}{k},$$

that is there are no sets of size less than k in the trace on $[k(s+1)-1]$. Eventually we want to prove that the same is true for our family $\mathcal{F}$.

Because of the Erdős–Ko–Rado Theorem we can and we will suppose that $s \geqq 2$. Let us mention that for the initial case: $n = k(s+1)$ the validity of (1.2), that is $|\mathcal{F}| \leqq \binom{k(s+1)-1}{k}$ was proved by Kleitman [K] and the uniqueness of the optimal families is shown in [F3]. Thus we may assume in the sequel that $n > (s+1)k$. If $\varepsilon = \varepsilon(k) < \frac{1}{k}$, then for $s < k$ one has $\varepsilon(k)(s+1) < 1$ and the Theorem is true (the interval $\bigl[k(s+1), (k+\varepsilon(k))(s+1)\bigr]$ contains no integer except for $k(s+1)$).

As a matter of fact our $\varepsilon(k)$ is going to be much smaller. Therefore we will assume that $s \geqq k$ holds.

Since $\mathcal{F}$ is shifted and $\nu(\mathcal{F}) = s$ there is a matching $F_1, \dots, F_s \in \mathcal{F}$ such that $F_j \subset [k(s+1)-1]$ for $j = 1, \dots, s$. (A *matching* is a collection of pairwise disjoint sets.) Define $F_0 = F_0(F_1, \dots, F_s) = [k(s+1)-1] - (F_1 \cup \ldots \cup F_s)$. Then $|F_0| = k(s+1) - 1 - ks = k - 1$ holds.

In view of $\nu(\mathcal{T}) = s$, $F_0 \notin \mathcal{T}$. Let us recall the definition of the lexicographic order $\lessdot$. For two distinct sets $A, B \subset [n]$ one has $A \lessdot B$ iff either $A \subset B$ or $\min\{x : x \in A \setminus B\} < \min\{y : y \in B \setminus A\}$. Let us define G_0 as the smallest $(k-1)$-subset of $[k(s+1)-1]$ in the lexicographic order that is *not* a member of $\mathcal{T} = \mathcal{T}(\mathcal{F})$. The key point is that, by the maximality of $|\mathcal{F}|$, $\nu\big(\mathcal{F} \cup \{G_0\}\big) \geqq s+1$ holds. That is, we can find pairwise disjoint sets $F_1, \dots, F_s \in \mathcal{F}$ that are disjoint to G_0 as well.

Now we fix G_0 for the rest of the proof and consider a matching $G_1, ..., G_s \in \mathcal{F}$ satisfying $G_i \cap G_0 = \emptyset$ and $G_i \in \binom{[k(s+1)-1]}{k}$. (Since $\mathcal{F}$ is shifted, to every matching $F_1, \dots, F_s \in \mathcal{F}$ satisfying $F_1 \cap G_0 = \ldots = F_s \cap G_0 = \emptyset$ we can easily find $G_i \prec F_i$ such that $G_1, \dots, G_s$ is a matching as required above.)

Note that $G_0 \cup \ldots \cup G_s = [k(s+1)-1]$ holds. We fix the matching $G_1, \dots, G_s$ as well. To justify the definition of G_0 let us prove a simple statement that we are going to use in the proof of the Theorem.

Proposition 2.2. (i) *If $R \subset [(s+1)k-1]$ satisfies $|R| \leqq k$, $R \not\subset G_0$ and $R \lessdot G_0$ then $R \in \mathcal{T}(\mathcal{F})$.*

(ii) *If $(b_1, \dots, b_k) \in \mathcal{F}$ is disjoint to G_0 then $G_0 \cup \{b_1\} \in \mathcal{F}$ holds.*

Proof. Suppose first $|R| \leqq k-1$. Assume $R \notin \mathcal{T}(\mathcal{F})$. Then the maximality of $|\mathcal{F}|$ implies the existence of a matching $F_1, \dots, F_s \in \mathcal{F}$ such that $F_i \cap R = \emptyset$ for all i. Using shiftedness we can assume that $F_i \subset [(s+1)k-1]$, $1 \leqq i \leqq s$. Define $\widetilde{R} = [(s+1)k-1] - (F_1 \cup \ldots \cup F_s)$. Then $R \subset \widetilde{R}$ and $|\widetilde{R}| = k-1$. Since $R \lessdot G_0$ implies $\widetilde{R} \lessdot G_0$, we get a contradiction with the choice of G_0.

If $|R| = k$ let R_1 be the $(k-1)$-set that we obtain from R by removing the largest element. Then $R_1 \lessdot G_0$ and the above argument imply $R_1 \in \mathcal{T}(\mathcal{F})$. Now $R \in \mathcal{T}(\mathcal{F})$ follows from $R_1 \subset R$, concluding the proof of (i).

To prove (ii) we distinguish two cases. Let $g_{k-1}^{(0)}$ be the largest element of $G_0 = \big(g_1^{(0)}, \dots, g_{k-1}^{(0)}\big)$.

- If $g_{k-1}^{(0)} < b_1$ then $g_1^{(0)} < ... < g_{k-1}^{(0)} < b_1 < ... < b_k$ imply $\big(g_1^{(0)}, ..., g_{k-1}^{(0)}, b_1\big) \prec (b_1, \dots, b_k)$ and the statement follows by shiftedness.

- If $b_1 < g^{(0)}_{k-1}$ then $G' \overset{\text{def}}{=} \big(g^{(0)}_1, \dots, g^{(0)}_{k-2}, b_1\big) \prec G_0$ implies $G' \in \mathcal{T}(\mathcal{F})$. Now the statement follows from $G' \subset G_0 \cup \{b_1\}$. $\square$

3 The counting formula

The main use of G_0 and the carefully picked matching is a formula that tells us the size $|\mathcal{F}|$ of $\mathcal{F}$ from *local* information.

Definition. For a set $T \in \mathcal{T}(\mathcal{F})$ let us define its width, $v(T)$ by

$$v(T) = \big|\big\{i : 1 \leqq i \leqq s,\ T \cap G_i \neq \emptyset\big\}\big|.$$

Note that $i = 0$ is not permitted in the above definition. Thus $v(T)$ is the number of edges in the matching that have non-empty intersection with T. This implies $v(T) \leqq |T|$ with equality iff $|T \cap G_i| = 1$ holds for exactly $|T|$ values of $1 \leqq i \leqq s$. We call such a T a *transversal*. If further $|T| = k$ then we say that T is a *full* transversal.

It is very important to notice that $G_0 \notin \mathcal{T}(\mathcal{F})$ implies that $v(T) \geqq 1$ for every $T \in \mathcal{T}(\mathcal{F})$.

Let $M = (m_1, m_2, \dots, m_k)$ be a k-subset of $[s]$. To avoid double indices we set $B_i = G_{m_i}$ and consider the $k^2 + k - 1$-element set

$$G_0 \cup B_1 \cup \ldots \cup B_k \overset{\text{def}}{=} G(M).$$

Our local information is related to $T \in \mathcal{T}(\mathcal{F})$ satisfying $T \subset G(M)$.

For every pair c, d, $1 \leqq c \leqq d \leqq k$ define

$$\mathcal{T}_M(c,d) = \big\{T \in \mathcal{T}(\mathcal{F}) : T \subset G(M), v(T) = c, |T| = d\big\} \quad \text{and}$$
$$t_M(c,d) = |\mathcal{T}_M(c,d)|.$$

Claim 3.1. *Every set $T \in \mathcal{T}(\mathcal{F})$ with $v(T) = c$, $|T| = d$ satisfies $T \in \mathcal{T}_M(c,d)$ for $\binom{s-c}{k-c}$ choices of $M \in \binom{[s]}{k}$.*

Proof. Set $C = \{i, 1 \leqq i \leqq s,\ T \cap G_i \neq \emptyset\}$. Then $T \in \mathcal{T}_M(c,d)$ iff $C \subset M$. $\square$

Lemma 3.1 (Counting Formula)**.**

$$|\mathcal{F}| = \sum_{M \in \binom{[s]}{k}} \sum_{1 \leqq c < d \leqq k} t_M(c,d) \cdot \frac{\binom{\overline{n}}{k-d}}{\binom{s-c}{k-c}}. \tag{3.1}$$

Proof. The above formula is almost evident. In formula (2.3) every $T \in \mathcal{T}(\mathcal{F})$ is added with coefficient $\binom{\overline{n}}{k-d}$ to produce $|\mathcal{F}|$. However, in (3.1) each $T \in \mathcal{T}$ with $v(T) < k$ is counted several times. That multiplicity is exactly $\binom{s-c}{k-c}$ which proves the veracity of (3.1). $\square$

Let us define the *weight*, $w(T)$ for $T \in \mathcal{T}$ by $w(T) = \binom{\overline{n}}{k-d} \Big/ \binom{s-c}{k-c}$. Then (3.1) is equivalent to

$$\sum_{M \in \binom{[s]}{k}} \Bigg(\sum_{T \in \mathcal{T}, T \subset G(M)} w(T) \Bigg) = |\mathcal{F}|. \tag{3.2}$$

Let us define the weight, $w(M)$ of a matching $M \in \binom{[s]}{k}$ as the sum in the bracket, that is

$$w(M) = \sum_{T \in \mathcal{T}, T \subset G(M)} w(T).$$

Note that for the family $\mathcal{A}_k = \mathcal{A}$, all $\binom{k^2+k-1}{k}$ k-subsets of $G(M)$ are in $\mathcal{T}$. (However, no sets of size $k-1$ or less.) Define

$$w(\mathcal{A}_k) = \sum_{T \in \binom{G(M)}{k}} w(T).$$

Then (3.2) implies

$$w(\mathcal{A}_k) = |\mathcal{A}_k| \Big/ \binom{s}{k} = \binom{(s+1)k-1}{k} \Big/ \binom{s}{k}.$$

Our plan is to prove that

$$w(M) \leqq w(\mathcal{A}_k) \quad \text{holds for all} \quad M \in \binom{[s]}{k}. \tag{3.3}$$

In view of (3.2) this will imply $|\mathcal{F}| \leqq |\mathcal{A}_k|$. In order to achieve that, let us compare the weights of $T \in \mathcal{T}_M(c,d)$ for some values of c and d. Let us put them into a table for $\overline{n} = \varepsilon s$, $s > k$.

$c = d = k$	$w(T) = 1$
$c = d = k-1$	$w(T) = \varepsilon s / (s-k+1)$
$c = k-1,\ d = k$	$w(T) = 1/(s-k+1)$
$c < d = k-1$	$w(T) \leqq \varepsilon s \Big/ \binom{s-k+2}{2}$
$c = d \leqq k-2$	$w(T) \leqq 2\varepsilon^2$

Table 1

We are going to choose $\varepsilon = \varepsilon(k) = k^{-2k-1}/2$. Therefore it is sufficient to consider the case $s \geqq 2k^{2k+1}$.

Since the total number of subsets $T \subset G(M)$ with $|T| \leqq k-1$ is

$$\sum_{0 \leqq d < k} \binom{k^2+k-1}{k-1} < k\binom{k^2+k-1}{k-1} < k(k+1)^{2(k-1)} < k^{2k+1},$$

the total weight of such sets is less than 1.

On the other hand, every full transversal $T \in \mathcal{T}$, that is a set T with $v(T) = |T| = k$ has weight 1. This shows that (3.3) holds unless $\mathcal{T}$ contains all full transversals in $G(M)$.

Therefore in the sequel we assume that all $|B_1| \cdot |B_2| \cdot \ldots \cdot |B_k| = k^k$ full transversals are in $\mathcal{T} = \mathcal{T}(\mathcal{F})$. Let us use this assumption to prove:

Proposition 3.1. *There is no $T \in \mathcal{T}(\mathcal{F})$ satisfying $v(T) = |T| < k$.*

Proof. Let $B_i = \left(b_1^{(i)}, \ldots, b_k^{(i)}\right)$, $i = 1, \ldots, k$. Suppose for contradiction that $T \in \mathcal{T}(\mathcal{F})$ and $v(T) = |T| < k$. Let us define $V = \{i : T \cap B_i \neq \emptyset\}$. Since $v(T) = |T|$, $|V| = |T|$ holds, proving that $|T \cap B_i| = 1$ for each $i \in V$.

Define $\widetilde{T} = \{b_1^{(i)} : i \in V\}$. Since $b_1^{(i)}$ is the smallest element of B_i for each i, $\widetilde{T} \prec T$ follows. Thus $\widetilde{T} \in \mathcal{T}$.

Let j be an arbitrary element of $[k] \setminus V$. By Proposition 2.2 (ii) the set $G_0 \cup \{b_1^{(j)}\}$ is in $\mathcal{T}$ as well. Define the $k-1$ transversal sets $T_2, \ldots, T_k$ by $T_\ell = \left\{b_\ell^{(1)}, b_\ell^{(2)}, \ldots, b_\ell^{(k)}\right\}$.

Now $\widetilde{T}$, $G_0 \cup \{b_1^{(j)}\}$ along with $T_2, \ldots, T_k$ are $k+1$ pairwise disjoint sets. Together with the $s-k$ edges of the fixed matching, $G_u : u \notin M$ we found $s+1$ pairwise disjoint edges contradicting $\nu(\mathcal{T}) = s$. □

4 The last part of the proof

In view of the results of the preceding section we may assume that $\mathcal{T} = \mathcal{T}(\mathcal{F})$ contains all full transversals in $G(M)$ and every $T \in \mathcal{T}$ with $|T| < k$ satisfies $v(T) < |T|$.

Looking at Table 1 we see that these sets T have weight at most $\varepsilon s \Big/ \binom{s-k+2}{2} < 2\varepsilon/s$ if $|T| = k-1$ and much smaller weight for $|T| < k-1$.

On the other hand a k-set $P \subset G(M)$ satisfying $v(P) = k-1$ has weight $1/(s-k+1) > 1/s$. Using $2\varepsilon < k^{-2k-1}$, we infer that a single set P

of the above type has more weight than all possible sets $T \subset G(M)$ with $c(T) < |T| < k$.

Since $\mathcal{A}_k$ contains all such P, (3.3) holds automatically unless $\mathcal{T}$ contains all such P as well. Therefore from now on we suppose that every $P \in \binom{G(M)}{k}$ with $v(P) \geqq k-1$ is in $\mathcal{T}$.

If $\mathcal{T}$ contains no set of size less than k then (3.3) follows. Thus we may assume that there is some $T \subset G(M)$ with $|T| < k$ and $T \in \mathcal{T}$.

We need a simple lemma.

Lemma 4.1. *Suppose that $T \subset G(M)$, $|T| < k$. Then there exist pairwise disjoint sets $Q_1, \ldots, Q_k \in \binom{G(M)}{k}$ satisfying $v(Q_i) = k$ and $|Q_i \cap T| \leqq 1$ for $1 \leqq i \leqq k$.*

Note that the conditions imply that each Q_i is a full transversal and that they partition $B_1 \cup \ldots \cup B_k$.

Proof of the lemma. Let $Q_1 \dot\cup \ldots \dot\cup Q_k$ be a partition of $B_1 \cup \ldots \cup B_k$ in full transversals such that $\big|\{i : |Q_i \cap T| \geqq 2\}\big|$ is as small as possible If this number is 0, we have nothing to prove.

Suppose by symmetry that $|Q_k \cap T| = r \geqq 2$. Note the inequality

$$\sum_{1 \leqq i \leqq k} |Q_i \cap T| \leqq |T| \leqq k-1$$

which is a direct consequence of the pairwise disjointness of the Q_i. Using $|Q_k \cap T| = r$,

$$\sum_{1 \leqq i < k} |Q_i \cap T| \leqq k-1-r \quad \text{follows.}$$

Consequently we can choose $1 \leqq i_1 < \ldots < i_r < k$ such that $Q_i \cap T = \emptyset$.

Set $Q_k \cap T = \{a_1, \ldots, a_r\}$. Renumbering $B_1, \ldots, B_k$ if necessary, we may suppose that $a_j \in B_j$ for $1 \leqq j \leqq r$.

Let c_j be the unique element in the intersection of the full transversal Q_{i_j} with B_j.

Define

$$\begin{aligned} \widetilde{Q}_{i_j} &= \big(Q_{i_j} \cup \{a_j\}\big) - \{c_j\}, \quad j = 1, \ldots, r, \\ \widetilde{Q}_k &= \big(Q_k \cup \{c_1, \ldots, c_r\}\big) - \{a_1, \ldots, a_r\} \quad \text{and} \\ \widetilde{Q}_i &= Q_i \quad \text{for} \quad i \notin \{i_1, \ldots, i_r, k\}. \end{aligned}$$

Then the $\widetilde{Q}_i$ are full transversals partitioning $B_1 \cup \ldots \cup B_k$ but the number $\left|\{i : |\widetilde{Q}_i \cap T| \geqq 2\}\right|$ is one smaller. This contradiction concludes the proof. $\square$

Now the final contradiction is immediate. Since $|G_0| = k - 1 \geqq |T|$, $|G_0 \setminus T| \geqq |T \setminus G_0|$ holds. Let $T \setminus G_0 = \{x_1, \ldots, x_p\}$ and let $y_1, \ldots, y_p$ be p distinct elements of $G_0 \setminus T$. Renumbering the Q_i if necessary we may assume that $x_i \in Q_i$ for $i = 1, \ldots, p$. Then define $Q_i^* = (Q_i - \{x_i\}) \cup \{y_i\}$, $i = 1, \ldots, p$ and note that $v(Q_i^*) = k - 1$, implying $Q_i^* \in \mathcal{T}(\mathcal{F})$. Set $Q_j^* = Q_j$ for $p < j \leqq k$. Then T and $Q_1^*, \ldots, Q_p^*, Q_{p+1}, \ldots, Q_k$ are $k+1$ pairwise disjoint members of $\mathcal{T}$. Together with the remaining $s - k$ edges G_ℓ of the fixed matching ($\ell \in [s] - M$) we get $s + 1$ pairwise disjoint members of $\mathcal{T} = \mathcal{T}(\mathcal{F})$, the final contradiction.

Thus we proved that (3.3) holds for all $M \in \binom{[s]}{k}$ and the inequality is strict unless $\mathcal{T}(\mathcal{F})$ is k-uniform. Therefore $|\mathcal{F}| \leqq \binom{k(s+1)-1}{k}$ follows with equality holding only for $\mathcal{F} = \binom{[k(s+1)-1]}{k} = \mathcal{A}_k$. $\square$ $\square$

Remark. We proved the theorem with $\varepsilon = k^{-2k-1}/2$. With some effort we can increase ε to something around k^{-k} but the real challenge is to prove the statement for a positive constant, i.e., an ε independent of k.

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