Linear independence, a unifying approach to shadow theorems

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Abstract

The intersection shadow theorem of Katona is an important tool in extremal set theory. The original proof is purely combinatorial. The aim of the present paper is to show how it is using linear independence latently.

1 Introduction

Let $[n] = \{1, 2, \ldots, n\}$ be the standard n-element set and let $\binom{[n]}{k}$ denote the collection of all its k-element subsets.

A family $\mathcal{F} \subset \binom{[n]}{k}$ is called *t-intersecting* if $|F \cap F'| \geqq t$ holds for all $F, F' \in \mathcal{F}$, $n \geqq k \geqq t > 0$.

For an integer s, $0 \geqq s \leqq k$, the *s-shadow*, $\Delta_s(\mathcal{F})$ is defined by

$$\Delta_s(\mathcal{F}) = \{G : \ |G| = s, \ \exists F \in \mathcal{F}, \ G \subset F\}.$$

Katona Intersecting Shadow Theorem ([K])**.** *If $\mathcal{F} \subset \binom{[n]}{k}$ is t-intersecting then*

$$\big|\Delta_s(\mathcal{F})\big| \geqq |\mathcal{F}| \times \binom{2k-t}{s} \Big/ \binom{2k-t}{k} \tag{1}$$

holds for all s, $k - t \leqq s \leqq k$.

Let us note that in the above range the factor of $|\mathcal{F}|$ on the right-hand side is at least one.

Choosing $\mathcal{F} = \binom{[2k-t]}{k}$ shows that (1) is best possible.

Katona's proof was purely combinatorial. It relied on *shifting*, an operation on subsets and families of sets, invented by Erdős, Ko and Rado [EKR]. We are not giving here the rather technical definition but mention that it maintains the size of the subsets, the size of the family, the t-intersecting property and it does not increase the size of the shadow.

Repeated applications of shifting produce a family having the following property.

(2)

For $1 \leqq i < j \leqq n,$ if $j \in F \in \mathcal{F}$ and $i \notin F$ then $(F - \{j\}) \cup \{i\} \in \mathcal{F}$ also.

A family $\mathcal{F}$ satisfying (2) is called *shifted*.

The author proved the following

Claim 1 ([F1]). *If $\mathcal{F} \subset \binom{[n]}{k}$ is t-intersecting and shifted then for every $F \in \mathcal{F}$ there exists an $\ell = \ell(F)$, $0 \leqq \ell \leqq k - t$ such that*

$$\big|F \cap [t + 2\ell]\big| \geqq t + \ell \quad \text{holds.} \tag{3}$$

Note that for a fixed ℓ one can define

$$\mathcal{A}_\ell(n,k,t) = \left\{ A \in \binom{[n]}{k} : \ A \cap [t + 2\ell] \geqq t + \ell \right\}.$$

These are usually called the Frankl-families and they are t-intersecting. It was conjectured in [F1] and proved by Ahlswede and Khachatrian [AK] that

$$|\mathcal{F}| \leqq \max_\ell \big|\mathcal{A}_\ell(n,k,t)\big| \tag{4}$$

holds for every t-intersecting family $\mathcal{F} \subset \binom{[n]}{k}$, $n \geqq 2k - t$. Let us define $\mathcal{F}(n,k,t)$ as the family of all $F \in \binom{[n]}{k}$ satisfying (3) for *some* ℓ. Obviously, $\mathcal{F}(n,k,t)$ is the union of $\mathcal{A}_\ell(n,k,t)$ for $0 \leqq \ell \leqq k - t$. In view of Claim 1 we have

Claim 2. *If $\mathcal{F} \subset \binom{[n]}{k}$ is t-intersecting and shifted then $\mathcal{F} \subset \mathcal{F}(n,k,t)$ holds.*

Note that $\mathcal{F}(n,k,t)$ is no longer t-intersecting for $n > 2k - t$. However, the author proved that it still verifies (1).

Proposition 1 ([F2]). *If $\mathcal{F} \subset \mathcal{F}(n,k,t)$ then (1) holds.*

2 Inclusion matrices and statement of the results

Let $0 \leqq r < k \leqq n$ be integers and $\mathcal{F} \subset \binom{[n]}{k}$ a family of subsets. One defines the inclusion matrix $M(r, \mathcal{F})$ as a $(0-1)$-matrix whose rows are indexed by the subsets $G \in \binom{[n]}{r}$, and columns are indexed by the members F of $\mathcal{F}$. The general entry is 1 if $G \subset F$ and 0 otherwise.

The ordering of the rows is not essential but for convenience we use the *colex* order i.e., G precedes G' iff the maximal element of $G \setminus G'$ is smaller than that of $G' \setminus G$.

Note that the (column) vector $\overrightarrow{w}(F)$ is a vector of length $\binom{n}{r}$ having $\binom{k}{r}$ entries equal 1, corresponding to the sets in $\binom{F}{r}$.

Füredi and the author proved the following.

Theorem ([FF]). *If the rank of $M(k-t, \mathcal{F})$ is $|\mathcal{F}|$ then $\mathcal{F}$ verifies* (1).

Let us note that if $\mathcal{F} \subset \binom{[n]}{k}$ is t-intersecting then $M(k-t, \mathcal{F})$ has rank $|\mathcal{F}|$. In [FW] it was proved in a much more general setting. Therefore the above theorem generalizes the Katona Intersecting Shadow Theorem. This way we have two seemingly different generalisations of the Katona Intersecting Shadow Theorem: Proposition 1 from the introduction with a purely combinatorial proof and the above theorem using linear independence.

The aim of the present paper is to show that $M\big(k-t,\ \mathcal{F}(n,k,t)\big)$ has rank $|\mathcal{F}(n,k,t)|$, i.e., Proposition 1 can be proved via linear independence.

As a matter of fact $|\mathcal{F}(n,k,t)| = \binom{n}{k-t}$ holds (cf. [F1]). That is, $M\big(k-t,\ \mathcal{F}(n,k,t)\big)$ is a square matrix.

Let us call a column vector a *standard basis* vector if it has exactly one position equal to 1 (all the others are 0). Now our statements about linear independence follow once we prove the following.

Theorem 2.1. *All the standard basis vectors can be expressed by linear combinations over the rationals of the column vectors of $M\big(k-t,\ \mathcal{F}(n,k,t)\big)$.*

3 Proof of the Theorem

We fix $t > 0$ and apply double induction on n and k. Note that the statement is trivially true for $k = t$. Also in the case $n = 2k - t$ one has $\mathcal{F}(n,k,t) =$

$\binom{[2k-t]}{k}$ and the statement is folklore (cf. e.g. [W]). Thus the base cases are cleared.

Suppose that the statement was proved for n and k' where $n \geqq 2k-t$, $t \leqq k' \leqq k$. Let us prove it for $n+1$ and k, $k > t$.

Let us partition $\mathcal{F}(n+1,k,t)$ into two parts according containment of the element $n+1$. $\mathcal{F}_0 = \big\{F \in \mathcal{F},(n+1,k,t) :\ n+1 \notin F\big\}$. Note that $\mathcal{F}_0$ is simply $\mathcal{F}(n,k,t)$.

We prefer to look at the remaining sets without the element $n+1$ and define

$$\mathcal{F}_1 = \big\{F - \{n+1\} :\ n+1 \in F \in \mathcal{F}(n+1,k,t)\big\}.$$

Note that $\mathcal{F}_1 = \mathcal{F}(n,k-1,t)$.

$M(k-t,\mathcal{F}(n,k,t))$	$M(k-t,\mathcal{F}(n,k-1,t))$
0	$M(k-t-1,\mathcal{F}(n,k-1,t))$

Figure The structure of $M(k-t,\mathcal{F}(n+1,k,t))$

Let $\overrightarrow{b}(G)$ denote the *standard* basis vector of length $\binom{n+1}{k-t}$ where the coordinates are indexed in colex order by the $k-t$-subsets of $[n+1]$, $\overrightarrow{b}(G)$ has 1 in position G and 0 everywhere else.

If $n+1 \notin G$, i.e., $G \subset [n]$, then $\overrightarrow{b}(G)$ is the extension by $\binom{n}{k-t-1}$ extra zeros of the corresponding standard basis vector of length $\binom{n}{k-t}$, considered in $\binom{[n]}{k-t}$. Thus by the induction hypothesis $b(\overrightarrow{G})$ can be expressed as a linear combination of the vectors $\overrightarrow{w}(F)$, $F \in \mathcal{F}_0$. This takes care of the standard basis vectors in the left-hand side of the figure.

Let now $n+1 \in H \in \binom{[n+1]}{k-t}$. Consider the standard basis vector $\overrightarrow{b}(H-\{n+1\})$ over $\binom{[n]}{k-t-1}$. By the induction hypothesis there exist rational numbers $c(F)$, $F \in \mathcal{F}_1$ such that

$$\sum_{F \in \mathcal{F}_1} c(F)\overrightarrow{w}(F) = \overrightarrow{b}(H - \{n+1\}) \ \text{ holds.} \tag{3.1}$$

Next we lift (3.1) up to $\binom{[n+1]}{k-t}$. That is we consider the corresponding linear combination

$$\sum_{F \in \mathcal{F}_1} c(F)\overrightarrow{w}(F \cup \{n+1\}) \stackrel{\text{def}}{=} \overrightarrow{b}. \tag{3.2}$$

In (3.2) all vectors are of length $\binom{n+1}{k-t}$ but the coordinates indexed by sets $K \in \binom{[n+1]}{k-t}$, $n+1 \in K$ are in one-to-one correspondence with those indexed by $H - \{n+1\}$ in (3.1). Therefore the vector $\overrightarrow{b}$ might not be equal to the standard basis vector $\overrightarrow{b}(H)$ but they coincide on all coordinates indexed by sets K, $n+1 \in K \in \binom{[n+1]}{k-t}$. That is we have the corresponding standard basis vector in the lower right part of the figure, having exactly one coordinate equal to 1, zeroes in the rest. Let $\overrightarrow{v}$ be the part of this vector in the upper right part of the figure. If $\overrightarrow{v}$ is the all-zero vector then we have nothing to prove.

Now let us use the standard basis vectors which we obtained using the upper left part of the figure to express $-\overrightarrow{v}$ as a rational linear combination. Since all these vectors have only zeroes in the bottom half, adding this linear combination to $\overrightarrow{b}$ will cancel out $\overrightarrow{v}$ in the top part and provide us with the desired standard basis vector. $\square$

As a combinatorialist, the author feels that it would be desirable to find some new combinatorial conditions implying (1) that go beyond linear independence.

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