On the maximum number of edges in a hypergraph with given matching number

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Abstract

The aim of the present paper is to prove that the maximum number of edges in a 3-uniform hypergraph on n vertices and matching number s is

$$\max\left\{\binom{3s+2}{3}, \binom{n}{3} - \binom{n-s}{3}\right\}$$

for all n, s, $n \geq 3s+2$.

1 Introduction

Let $[n] = \{1, 2, \ldots, n\}$ be a finite set and $\mathcal{F} \subset \binom{[n]}{k}$ a k-uniform hypergraph. The matching number $\nu(\mathcal{F})$ is the maximum number of pairwise disjoint edges in $\mathcal{F}$. One of the classical problems in extremal set theory is to determine the maximum number of edges in a k-uniform hypergraph with matching number 1. This was solved by Erdős, Ko and Rado [4], who proved that for $n \geq 2k$ this maximum is $\binom{n-1}{k-1}$. There are two natural ways to generalize this problem for k-uniform hypergraphs. One is to consider the the maximum number of edges for matching number 2, 3, etc. This is the problem that we shall solve in this paper for $k = 3$. The other one is to make the restriction "matching number is one" stronger by requiring that any two edges intersect in at least t elements (t is a fixed integer, $k > t > 1$). Such a family is called *t-intersecting*. Let us consider the following construction.

$$\mathcal{F}(n, k, t, i) = \{F \subset [n] \mid |F| = k, |F \cap [t + 2i]| \geq t + i\}.$$

In 1976 the author [7] made the following conjecture. For all n, k and t, such that $n > 2k - t$, and for every k-uniform t-intersecting hypergraph $\mathcal{F}$ on n vertices one has

$$|\mathcal{F}| \leq \max_i |\mathcal{F}(n, k, t, i)|.$$

In 1987 Füredi and the author [6] showed that for every i, in the range, that $|\mathcal{F}(n, k, t, i)|$ is the maximal term the conjecture is true for all but a FINITE number of values of t. However, it was not until ten years later that Ahlswede and Khachatrian [1] succeeded in proving the conjecture completely.

Fixing the matching number, say s, there are two very natural constructions for k-graphs with that matching number:

$$\mathcal{A}_k = \binom{[ks + k - 1]}{k}, \quad \text{and}$$
$$\mathcal{A}_1(n) = \left\{ F \in \binom{[n]}{k} : F \cap [s] \neq \emptyset \right\}.$$

In 1965 Paul Erdős made the following.

Conjecture 1.1 (Matching Conjecture) ([5])**.** *If* $\mathcal{F} \subset \binom{[n]}{k}$ *satisfies* $\nu(\mathcal{F}) = s$ *then*

$$|\mathcal{F}| \leq \max\{|\mathcal{A}_1(n)|, |\mathcal{A}_k|\}.$$

In the same paper Erdős proved the conjecture for $n > n_1(k, s)$. Let us mention that the conjecture is trivial for $k = 1$, and it was proved for graphs ($k = 2$) by Erdős and Gallai [3].

There were several improvements on the bound $n_1(k, s)$. Bollobás, Daykin and Erdős [2] proved $n_1(k, s) \leq 2k^3 s$ and recently Huang, Loh and Sudakov [13] improved it to $n_1(k, s) \leq 3k^2 s$. The current record is due to the present author [11], it is $n_1(k, s) \leqq (2s + 1)k - s$.

The aim of the present paper is to prove

Theorem 1.1. *The conjecture is true for* $k = 3$.

We should mention that our proof relies partly on ideas from Frankl–Rödl–Ruciński [10], who proved $n_1(3, s) \leq 4(s + 1)$ and the recent result of Łuczak and Mieczkowska [15] who proved the conjecture for $k = 3$, $s > s_0$.

Let us mention that the best general bound, true for all k, s and $n \geq k(s + 1)$ is due to the author (cf. [8] or [9]) and it says

$$|\mathcal{F}| \leq s\binom{n-1}{k-1}. \tag{1.1}$$

Note that for $n = k(s+1)$, (1.1) reduces to $|\mathcal{F}| \le |\mathcal{A}_k|$. This special case, the first non-trivial instance of the conjecture, was proved implicitly by Kleitman [14]. The case $s = 1$ of (1.1) is the classical Erdős–Ko–Rado Theorem [4].

2 Notation, tools

For a family $\mathcal{H} \subset 2^{[n]}$ and an element $i \in [n]$ we define $\mathcal{H}(i)$ and $\mathcal{H}(\bar{i})$ by

$$\mathcal{H}(i) = \big\{H - \{i\} : i \in H \in \mathcal{H}\big\},$$
$$\mathcal{H}(\bar{i}) = \big\{H \in \mathcal{H} : i \notin H\big\}.$$

For a subset $H = \{h_1, \dots, h_q\}$ we denote it <u>also</u> by $(h_1, \dots, h_q)$ whenever we know for <u>certain</u> that $h_1 < h_2 < \dots < h_q$.

For subsets $H = (h_1, \dots, h_q)$, $G = (g_1, \dots, g_q)$ we define the partial order, $\ll$ by

$$H \ll G \quad \text{iff} \quad h_i \le g_i \quad \text{for} \quad 1 \le i \le q.$$

Definition 2.1. *The family $\mathcal{F} \subset \binom{[n]}{k}$ is called <u>stable</u> if $G \ll F \in \mathcal{F}$ implies $G \in \mathcal{F}$.*

In Frankl [8] (cf. also [9]) it was proved that it is sufficient to prove the Matching conjecture for stable families. Therefore throughout the paper we assume that $\mathcal{F}$ is stable and use stability without restraint.

An easy consequence of stability is the following. Let $\mathcal{F} \subset \binom{[n]}{k}$, $\nu(\mathcal{F}) = s$ and define $\mathcal{F}_0 = \big\{F \cap [ks+k-1] : F \in \mathcal{F}\big\}$. Note that $\mathcal{F}_0$ is not k-uniform in general.

Proposition 2.1. $\nu(\mathcal{F}_0) = s$.

Proof. Suppose for contradiction that $G_1, \dots, G_{s+1} \in \mathcal{F}_0$ are pairwise disjoint and $F_1, \dots, F_{s+1} \in \mathcal{F}$ are such that $F_i \cap [ks+k-1] = G_i$, $1 \le i \le s+1$. Suppose further that $F_1, \dots, F_{s+1}$ are chosen subject to the above condition to minimize

$$\sum_{1 \le i < j \le s+1} |F_i \cap F_j| \tag{2.1}$$

Since $\nu(\mathcal{F}) < s+1$, the above minimum is positive. We establish the contradiction by showing that one can diminish it.

Choose some $x \in F_i \cap F_j$. Since $G_i \cap G_j = \emptyset$, $x \geq k(s+1)$. Consequently, $|G_1| + \cdots + |G_{s+1}| \leq k(s+1) - 2 < ks + k - 1$. Thus we can choose $y \in [ks+k-1]$ with $y \notin G_\ell$ for $1 \leq \ell \leq s+1$. Now replace F_i by $F_i' = (F_i - \{x\}) \cup \{y\}$. Then $F_i' \ll F_i$, implying $F_i' \in \mathcal{F}$.

The intersections $F_j \cap [ks+k-1]$, $j = 1, \ldots, s+1$, $j \neq i$ and $F_i' \cap [ks+k-1]$ are still disjoint but the value of (2.1) is smaller. □

From now on we shall assume that $\mathcal{F} \subset \binom{[n]}{k}$ satisfies $\nu(\mathcal{F}) = s$ and it is maximal, i.e., it cannot be extended without increasing $\nu(\mathcal{F})$. Then the following formula is evident from Proposition 2.1.

$$|\mathcal{F}| = \sum_{H \in \mathcal{F}_0} \binom{n - ks - k + 1}{k - |H|}. \tag{2.2}$$

Formula (2.2) shows that for a fixed k and s, determining $\max |\mathcal{F}|$ is a finite problem, i.e., it is sufficient to compare all families $\mathcal{F}_0 \subset 2^{[ks+k-1]}$ with $\max\limits_{H \in \mathcal{F}_0} |H| \leq k$ and $\nu(\mathcal{F}_0) = s$.

However, this finiteness is only theoretical. There are too many families to check. Let us consider the following families, first defined in the author's Ph.D. dissertation in 1976.

$$\mathcal{A}_\ell(n) = \left\{ F \in \binom{[n]}{k} : \big|F \cap [\ell s + \ell - 1]\big| \geq \ell \right\}.$$

Then $\nu(\mathcal{A}_\ell(n)) = s$ holds for $n \geq ks$.

Unless the next proposition holds, we get a counterexample to Conjecture 1.1.

Proposition 2.2. *For all* $1 \leq \ell \leq k$,

$$|\mathcal{A}_\ell(n)| \leq \max\{|\mathcal{A}_1(n)|, |\mathcal{A}_k|\}. \tag{2.3}$$

In the present paper we only need the validity of Proposition 2.2 for the case $k = 3$. In that case it is not hard to check it by direct calculation.

3 Preliminaries

For a family $\mathcal{F} \subset \binom{[n]}{k}$, $\nu(\mathcal{F}) = s$, $n \geq ks + k - 1$ we want to define a specific partition

$$F_0 \cup F_1 \cup \cdots \cup F_s = [ks + k - 1] \quad \text{where } F_1, \ldots, F_s \in \mathcal{F}. \tag{3.1}$$

Since $\nu(\mathcal{F}) = s$, we can choose $F_1, \dots, F_s \in \mathcal{F}$ with $F_1 \cup \dots \cup F_s = [ks]$. Then $F_0 = [ks+1, ks+k-1]$. However, we fix F_0 to be the lexicographically first $(k-1)$-element subset of $[ks+k-1]$ for which a partition of type (3.1) is possible. Note that $F_0 \notin \mathcal{F}_0$. Once $F_0 = \{d_1, d_2, \dots, d_{k-1}\}$ is fixed we choose $F_i = \big(a_1(i), \dots, a_k(i)\big)$ such that $\sum_{1 \le i \le s} a_1(i)$ is minimal. Once this minimum value is attained we minimize $\sum_{1 \le i \le s} a_2(i)$ and so on. Note that we tacitly assume $a_1(i) < a_2(i) < \dots < a_k(i)$ for all $1 \leqq i \leqq s$.

Proposition 3.1. *For every $1 \le \ell < k$ and every $(e_1, \dots, e_\ell)$ which precedes $(d_1, \dots, d_\ell)$ lexicographically, $(e_1, \dots, e_\ell) \in \mathcal{F}_0$ holds.*

Proof. Since $\mathcal{F}$ is maximal, the contrary would mean that there exist pairwise disjoint sets $F_1, \dots, F_s \in \mathcal{F}$ which are disjoint to $(e_1, \dots, e_\ell)$ as well. However, then $(e_1, \dots, e_\ell)$ can be extended to a $(k-1)$-element set D, which is still disjoint to $F_1, \dots, F_s$ and precedes F_0 lexicographically, a contradiction. □

The following statement is rather simple to prove, but it is extremely useful.

Claim 3.1. *Let h, $1 \le h < k$ be the smallest number, ℓ, such that $a_\ell(i) < d_\ell$ holds, and let $h = k$ if no such ℓ exists. Then*

$$D \overset{def}{=} (d_1, \dots, d_{h-1}, a_h(i), d_h, \dots, d_{k-1}) \in \mathcal{F}$$

holds.

Proof. If $h < k$ then $(d_1, \dots, d_{h-1}, a_h^{(i)}) \in \mathcal{F}_0$ from Proposition 3.1. Thus all k-sets containing it are in $\mathcal{F}$.

If $h = k$ then $D \ll F_i$ implies the claim. □

The next claim can be easily verified using the definitions.

Claim 3.2. *For $\mathcal{F} = \mathcal{A}_\ell(n)$,*

$$F_0 = (1, \dots, \ell-1, \ell s+\ell, \ell s+\ell+1, \dots, \ell s+k-1).$$

Let $R = (r_1, \dots, r_p) \subset [s]$ be a p-tuple (we assume $k \ge p$ here). Define the set $X(R)$ by $X(R) = F_0 \cup F_{r_1} \cup \dots \cup F_{r_p}$. Note that $|X(R)| = kp+k-1$. Define the restriction $\mathcal{H}(R) = \{H \in \mathcal{F}_0 : H \subset X(R)\}$.

Definition 3.1. *The* <u>*width*</u> $v(H)$ *of* $H \in \mathcal{H}(R)$ *is defined by*

$$v(H) = \big|\{i : H \cap F_{r_i} \neq \emptyset\}\big|.$$

Note that $F_0 \notin \mathcal{F}_0$ implies $v(H) > 0$. Next we define the weight of H.

Definition 3.2. *The* <u>*weight*</u> $w(H)$ *of* $H \in \mathcal{H}(R)$ *is defined by*

$$w(H) = \frac{\binom{n-ks-k+1}{k-|H|}}{\binom{s-v(H)}{k-v(H)}}.$$

The weight $w(R)$ of a k-tuple $R = (r_1, \ldots, r_k) \subset [s]$ is defined by

$$w(R) = \sum_{H \in \mathcal{H}(R)} w(H) \tag{3.2}$$

These definitions are justified by:

Lemma 3.1 (Counting Lemma)**.** *For* $s \geq k$,

$$|\mathcal{F}| = \sum_{R \in \binom{[s]}{k}} \sum_{H \in \mathcal{H}(R)} w(H)$$

Proof. In view of (2.2) it is sufficient to note that each $H \in \mathcal{F}_0$ is contained in $\mathcal{H}(R)$ for exactly $\binom{s-v(H)}{k-v(H)}$ k-tuples R. □

It is easy to check that for $\mathcal{F} = \mathcal{A}_\ell(n)$

$$\bigcup_{1 \leq i \leq s} (a_1(i), a_2(i), \ldots, a_\ell(i)) = [\ell, \ell s + \ell - 1]$$

holds. Consequently, the value of (3.2) is independent of the particular choice of $R \subset [s]$. Let $f(\ell)$ denote this common value.

Conjecture 3.1. *If* $\mathcal{F} \subset \binom{[n]}{k}$, $\nu(\mathcal{F}) = s$, $\nu(\mathcal{F}(\bar{1})) = s$, $s \geq k$, *then*

$$\sum_{H \in \mathcal{H}(R)} w(H) \leq \max_{1 \leq \ell \leq k} f(\ell) \tag{3.3}$$

holds.

One can show that Conjecture 3.1 would imply Erdős' Conjecture 1.1 for $s \geq k$. We prove Theorem 1.1 by establishing Conjecture 3.1 for $k = 3$, and certain values of n.

The paper is organized as follows. In Section 4 we prove some easy results, and consider $\mathcal{H}(R)$ with $|R| = 1$. Section 5 provides the foundation for induction. In Section 6 we consider $\mathcal{H}(R)$ with $|R| = 2$, $k = 3$. In Section 7 we prove some general results.

In the later sections we concentrate on the case $k = 3$. In Section 8 we show that Conjecture 1.1 holds for $s = 2$. In Sections 9, 10 and 11 we establish the validity of (3.3) in the necessary range settling Conjecture 1.1 for $s \geq 4$. Section 12 handles the last remaining case, $s = 3$.

4 Some easy facts

The property of $\mathcal{H}(R)$ that we use most is

Fact 4.1. $\nu(\mathcal{H}(R)) = |R|$.

Proof. For $R = (r_1, \ldots, r_p)$ the family $\mathcal{H}(R)$ contains $F_{r_1}, \ldots, F_{r_p}$ showing $|\nu(\mathcal{H}(R))| \geq |R|$. On the other hand, for $1 \leq i \leq s$, $i \notin R$ the edges $F_i \in \mathcal{F}$ are pairwise disjoint and disjoint to the vertex set of $\mathcal{H}(R)$ as well showing $\nu(\mathcal{H}(R)) + s - |R| \leq \nu(\mathcal{F}) = s$, proving $\nu(\mathcal{H}(R)) \leq |R|$. □

Let now $k = 3$ and $R = \{i\}$, $F_i = (a_i, b_i, c_i)$.

Fact 4.2. *If* $d_1 = 1$ *then* $(a_i, c_i) \notin \mathcal{F}_0$, $(b_i, c_i) \notin \mathcal{F}_0$. *Moreover, if* $(a_i, b_i) \in \mathcal{F}_0$ *then* $(1, c_i) \notin \mathcal{F}_0$.

Proof. Since $(1, b_i) \ll (a_i, c_i)$, $(a_i, c_i) \in \mathcal{F}_0$ would imply $(1, b_i) \in \mathcal{F}_0$. This would contradict $\nu(\mathcal{H}(\{i\})) = 1$. Now $(a_i, c_i) \ll (b_i, c_i)$ implies $(b_i, c_i) \notin \mathcal{F}_0$. The last statement is a direct consequence of $\nu(\mathcal{H}(\{i\})) = 1$. □

Fact 4.3. *If* $(d_1, x_i) \in \mathcal{F}_0$ *then* $(F_i - \{x_i\}) \cup \{d_2\}$ *is not in* $\mathcal{F}_0$. □

The following easy fact will prove extremely useful in the sequel.

Fact 4.4. *For every* $1 \leq i \leq s$,

$$\{1, d_2, b_i\} \in \mathcal{H}(\{i\}).$$

Proof. We apply Claim 3.1. If $b_i < d_2$, then $h = 2$, and $(1, b_i, d_2) \in \mathcal{H}(\{i\})$ is a direct consequence of Claim 3.1. If $d_2 < b_i$ then Claim 3.1 yields $(1, d_2, c_i) \in \mathcal{H}(\{i\})$. The statement follows from $(1, d_2, b_i) \ll (1, d_2, c_i)$. □

Fact 4.5. *For any two edges F_u, F_v of the special matching $a_1(u) < a_k(v)$ holds.*

Proof. The contrary means

$$a_1(v) < a_2(v) < \cdots < a_k(v) < a_1(u) < \cdots < a_k(u).$$

By stability, $(a_1(v), \ldots, a_{k-1}(v), a_1(u))$ and $(a_k(v), a_2(u), \ldots, a_k(u))$ are in $\mathcal{F}$. Using these two sets instead of F_u, F_v in the special matching decreases $a_1(1) + \cdots + a_s(1)$, a contradiction. □

In later sections we are going to compare the total weight

$$\sum_{H \in \mathcal{H}(R)} w(H)$$

for $R \in \binom{[s]}{3}$ with the corresponding weights for $\mathcal{A}_3$ and $\mathcal{A}_2(n)$, (possibly adding a constant).

Suppose $d_1 = 1$ and set $d = d_2$. For $\mathcal{A}_3$, the corresponding hypergraph $\mathcal{H}^{(3)}(\{i\})$ is the complete 3-graph $\binom{F_i \cup (1,d)}{3}$. For $\mathcal{A}_2(n)$ one has

$$\mathcal{H}^{(2)}(\{i\}) = \binom{(1, a_i, b_i)}{2} \cup \left\{ H \in \binom{F_i \cup (1, d)}{3} : \left|H \cap (1, a_i, b_i)\right| \geq 2 \right\},$$

it consists of 3 sets of size 2 and 7 of size 3. We are always fixing $\mathcal{A}_3$ or $\mathcal{A}_2(n)$ as our reference, and consider an edge in $\mathcal{H}(R)$ that is not in the reference hypergraph a loss, and an edge in the reference hypergraph that is not in $\mathcal{H}(R)$ a gain. Adding with weights the losses and subtracting the weighted sum of gains is called the balance.

In the case $k = 3$, we define $\mathcal{G} = \{G \in \mathcal{F}_0 : |G| = 2\}$.

Convention 4.1. *For $G \in \mathcal{G}$ with width 1, i.e., $G \in \mathcal{H}(\{i\})$ for some i, we always consider G together with its complement $(1, d) \cup F_i - G$. Since $\nu(\{i\}) = 1$, not both can be in $\mathcal{H}(\{i\})$.*

Corollary 4.1. *The balance (real loss) coming from an extra $G \in \mathcal{H}(i)$, $|G| = 2$ is never more than*

$$\frac{n - 3s - 2}{\binom{s-1}{2}} - \frac{1}{\binom{s-1}{2}} = \frac{n - 3s - 3}{\binom{s-1}{2}}.$$

5 Why induction would work

For $n \geq ks + k - 1$ let $m(n, k, s)$ denote the maximum possible size of $|\mathcal{F}|$ over all $\mathcal{F} \subset \binom{[n]}{k}$ with $\nu(\mathcal{F}) = s$.

Note the obvious inequality $\nu(\mathcal{F}_1 \cup \mathcal{F}_2) \leq \nu(F_1) + \nu(\mathcal{F}_2)$. Let us use it to prove:

Fact 5.1. $m(n, k, s) \geq m(n-1, k, s-1) + \binom{n-1}{k-1}$.

Proof. Let $\mathcal{F}_1 \subset \binom{[2,n]}{k}$ satisfy $\nu(\mathcal{F}_1) = s - 1$ and $|\mathcal{F}_1| = m(n-1, k, s-1)$. Define $\mathcal{F}_2 = \left\{F \in \binom{[n]}{k} : 1 \in F\right\}$. Now $|\mathcal{F}_1 \cup \mathcal{F}_2| = m(n-1, k, s-1) + \binom{n-1}{k-1}$ and $\nu(\mathcal{F}_1 \cup \mathcal{F}_2) \leq s - 1 + 1 = s$. □

Fact 5.1 would provide us with a counterexample to Conjecture 1.1, should the following be false. Fortunately, it is true.

Proposition 5.1.

$$\max\left\{\binom{ks+k-1}{k}, \binom{n}{k} - \binom{n-s}{k}\right\} \geq \max\left\{\binom{ks-1}{k}, \binom{n-1}{k} - \binom{n-s}{k}\right\} + \binom{n-1}{k-1}. \tag{5.1}$$

Proof. If the maximum on the RHS is given by $\binom{n-1}{k} - \binom{n-s}{k}$ then (5.1) follows from

$$\binom{n-1}{k} + \binom{n-1}{k-1} - \binom{n-s}{k} = \binom{n}{k} - \binom{n-s}{k}.$$

Assume $\binom{n-1}{k} - \binom{n-s}{k} < \binom{ks-1}{k}$. Equivalently,

$$\binom{n-2}{k-1} + \binom{n-3}{k-1} + \cdots + \binom{n-s}{k-1} < (s-1)\binom{ks-1}{k-1}.$$

This implies $n - s < ks - 1$. Using that both are integers, $n \leqq (k+1)s - 2$ follows.

If $s \leqq k + 1$ then $(k+1)s - 2 \leqq k(s+1) - 1$ and

$$\binom{ks-1}{k} + \binom{n-1}{k-1} \leqq \binom{ks-1}{k} + \binom{k(s+1)-2}{k-1} < \binom{k(s+1)-1}{k}$$

follow. If $s \geqq k + 2$ then one needs a different argument.

Using the monotonicity of $\binom{n-1}{k-1}$ it is sufficient to prove:

$$\binom{ks-1}{k} + \binom{(k+1)s-3}{k-1} \leqq \binom{ks+k-1}{k}$$

or equivalently,

$$\binom{(k+1)s-3}{k-1} \leqq \binom{ks-1}{k-1} + \binom{ks}{k-1} + \cdots + \binom{ks+k-2}{k-1}.$$

There are $(s-1)$ terms on the RHS and even for the smallest, $\binom{ks-1}{k-1}$ we have

$$\binom{ks}{s-1} \Big/ \binom{(k+1)s-3}{k-1} = \prod_{0 \leqq i \leqq k-2} \frac{ks-i}{(k+1)s-i-3} > \left(\frac{k}{k+1}\right)^{k-1} > \frac{1}{e}.$$

Since $s \geqq k+2 \geqq 4$, the desired inequality is proved. □

Corollary 5.1. *If for a given k, $\mathcal{F}$ is a minimal counterexample to Conjecture 1.1, then $\nu(\mathcal{F}(\bar{1})) = s$ must hold.*

Proof. Suppose $\nu(\mathcal{F}(\bar{1})) = s-1$. By minimality, $\mathcal{F}(\bar{1}) = \{F \in \mathcal{F} : 1 \notin F\}$ is not a counterexample to Conjecture 1.1. Also, for $\mathcal{F}_2 = \{F \in \mathcal{F} : 1 \in F\}$, $|\mathcal{F}_2| \leq \binom{n-1}{k-1}$ is evident. By Proposition 5.1, $\mathcal{F}$ is not a counterexample. □

We have showed now that in an inductive proof of Conjecture 1.1, one can always assume that $\nu(\mathcal{F}(\bar{1})) = s$. Reformulating and elaborating:

Fact 5.2.

(i) $|F| \geq 2$ *for all* $F \in \mathcal{F}_0$

(ii) For $F_0 = (d_1, \ldots, d_{k-1})$, $d_1 = 1$ *holds.*

Proof. Should (i) fail then by stability $\{1\} \in \mathcal{F}_0$. Since $\nu(\mathcal{F}(\bar{1})) = s$, we can find $H_1, \ldots, H_s \in \mathcal{F}(\bar{1})$, that are pairwise disjoint. Now the $s+1$ sets $\{1\}$, $H_i \cap [ks+k-1], i = 1, \ldots, s$ form a matching of size $s+1$ in $\mathcal{F}_0$, contradicting Proposition 2.1. □

Proposition 5.2. *Suppose that Conjecture 1.1 holds for $(n-1, k-1, s)$ and $(n-1, k, s)$. Moreover, for $(n-1, k, s)$ the maximum is given by $\mathcal{A}_1(n-1)$. Then Conjecture 1.1 holds for (n, k, s) and the maximum is given by $\mathcal{A}_1(n)$.*

Proof. Consider the two families $\mathcal{F}(n)$ and $\mathcal{F}(\bar{n})$. By Proposition 2.1, $\nu(\mathcal{F}(n)) \le s$ holds. For $\mathcal{F}(\bar{n})$, $\nu(\mathcal{F}(\bar{n})) \le \nu(\mathcal{F}) \le s$ is evident. By the hypothesis $|\mathcal{F}(\bar{n})| \le \binom{n-1}{k} - \binom{n-s-1}{k}$.

On the other hand, we showed above that for $n \ge ks$, $|\mathcal{A}_1(n-1,k-1)| > \binom{(k-1)(s+1)-1}{k}$, thus $|\mathcal{F}(n)| \le \binom{n-1}{k-1} - \binom{n-s-1}{k-1}$.

Now $|\mathcal{F}| = |\mathcal{F}(n)| + |\mathcal{F}(\bar{n})|$ yields $|\mathcal{F}| \le \binom{n}{k} - \binom{n-s}{k}$. □

Definition 5.1. *For k and s fixed let $n_1(s,k)$ be the minimum integer n, such that $|\mathcal{A}_k| \le |\mathcal{A}_1(n)|$ holds. Then $n_1(s,k)$ is called the pivotal number for k and s.*

Above we showed $n_1(s,k) < (k+1)s$.

Proposition 5.3. $n_1(s,k) \le \left(k + \frac{1}{2}\right)s + k$

Proof. First note that setting $m = \lfloor (k+\frac{1}{2})s+k \rfloor$ we have $m \ge (k+\frac{1}{2})s+k-\frac{1}{2}$. We have to show,

$$\binom{m}{k} - \binom{m-s}{k} \ge \binom{k(s+1)-1}{k}.$$

The right hand side is $s\binom{k(s+1)-1}{k-1}$. The left hand side can be estimated using the convexity of $\binom{x}{k-1}$ by Jensen's inequality.

$$\binom{m}{k} - \binom{m-s}{k} = \sum_{i=1}^{s} \binom{m-i}{k-1} > s\binom{m-\frac{s}{2}-\frac{1}{2}}{k-1}.$$

Since $(k+\frac{1}{2})s + k - \frac{1}{2} - \frac{s}{2} - \frac{1}{2} = k(s+1)-1$, the statement follows. □

Noting that $\mathcal{A}_3 = \binom{[ks+k-1]}{k}$ does not depend on n, we see that proving $m(n,k,s) \le \binom{ks+k-1}{s}$ for $n = n_1(s,k)$ implies the same for all $n < n_1(s,k)$ as well. Since $m(n,2,s) = \binom{n}{2} - \binom{n-s}{2}$ is an old theorem of Erdős and Gallai [3] for $n \ge 3s$, we infer

Fact 5.3. *In order to prove Conjecture 1.1 for $k=3$, it is sufficient to show it for $n = n_1(s,3)$ and $n = n_1(s,3)-1$.* □

6 The structure of $\mathcal{H}(i,j)$

In this section we let $k = 3$ and $R = (i, j)$. Let

$$\mathcal{H}_\ell = \{H \in \mathcal{H}(i,j) : |H| = 2, v(H) = \ell\}, \ \ell = 1, 2.$$

In the previous section we proved $1 \in F_0$. To simplify notation we set $d = d_2$, i.e., $F_0 = (1, d)$.

Proposition 6.1. *If $|\mathcal{H}_2| \geq 3$ then one of the following holds.*

(i) $\mathcal{H}_2 = \{(a_i, a_j), (a_i, b_j), \{b_i, a_j\}, \{b_i, b_j\}\}$,

(ii) $\mathcal{H}_2 = \{(a_i, a_j), (a_i, b_j), \{b_i, a_j\}\}$,

(iii) $\mathcal{H}_2 = \{(a_i, a_j), (a_i, b_j), (a_i, c_j)\}$.

Proof. First of all $(a_i, c_i) \ll (a_j, c_i)$ and Fact 4.2 imply $(a_j, c_i) \notin \mathcal{F}_0$.

If $(a_i, c_j) \notin \mathcal{F}_0$, then stability implies that (i) or (ii) hold.

If $(a_i, c_j) \in \mathcal{F}_0$ then $(a_i, b_j), (a_i, a_j) \in \mathcal{F}_0$ follow by stability. We claim that $\{b_i, a_j\} \notin \mathcal{F}_0$. Indeed, otherwise using $(1, b_j) \ll (a_i, c_j)$ we find three pairwise disjoint sets $\{b_i, a_j\}, (1, b_j), (a_i, c_j) \in \mathcal{H}(i,j)$, contradicting $\nu(\mathcal{H}(i,j)) = 2$. By stability, (iii) holds. □

Fact 6.1. *In cases (i) and (ii) neither $\{1, c_i, c_j\}$ nor $(1, c_i)$, nor $(1, c_j)$ is in $\mathcal{F}_0$. Also neither $\{a_i, d, c_j\}$ nor $\{a_j, d, c_i\}$ is in $\mathcal{F}_0$.*

Proof. Since (a_i, b_j) and $\{b_i, a_j\}$ are in $\mathcal{H}(i,j)$, $\{1, c_i, c_j\} \notin \mathcal{F}_0$, $(1, c_i) \notin \mathcal{H}_1$ and $(1, c_j) \notin \mathcal{H}_1$ are direct consequences of $\nu(\mathcal{H}(i,j)) = 2$. $(a_i, d, c_j), (a_j, d, c_i) \notin \mathcal{F}_0$ follow similarly, using $(1, b_j) \in \mathcal{H}_1$ and $(1, b_i) \in \mathcal{H}_1$. □

Corollary 6.1. *In cases (i) and (ii) the five sets of width 2, $\{x_i, d, c_j\} : x_i \in F_i$, $\{a_j, d, c_i\}, \{b_j, d, c_i\}$ are all missing from $\mathcal{H}(i,j)$.*

Proof. Evident by stability. □

Corollary 6.2. *In case (iii) the six sets $\{x_i, y_j, d\}$ of width 2, $x_i = b_i$ or c_i, $y_j \in F_j$ are missing from $\mathcal{H}(i,j)$.*

Proof. By stability it is sufficient to prove $\{b_i, a_j, d\} \notin \mathcal{H}(i,j)$. This follows from $(1, b_j) \in \mathcal{H}_1$ and $(a_i, c_j) \in \mathcal{H}_2$ using $\nu(\mathcal{H}(i,j)) = 2$. □

Remark 6.1. *There were 9 candidates both for $G \in \mathcal{H}_2$ and also for sets of width 2 containing d in $\mathcal{H}(i,j)$. We proved that not even half are actually in $\mathcal{H}(i,j)$. This will be of great help in proving Conjecture 1.1.*

7 Some important special cases

We consider $\mathcal{H}(R)$ for $R = (i_1, i_2, \dots, i_k)$. To simplify notation we set $F_{i_\ell} = \big(a_1(\ell), \dots, a_k(\ell)\big)$, $\ell = 1, \dots, k$. $F_0 = (1, d_2, \dots, d_{k-1})$.

Let us define the partition $T_1 \cup \dots \cup T_k$ of $F_{i_1} \cup \dots \cup F_{i_k}$ by $T_q = \big\{a_q(1), \dots, a_q(k)\big\}$.

Definition 7.1. *A set D is called a partial diagonal if $D \subset F_{i_1} \cup \dots \cup F_{i_k}$, $v(D) = |D|$ and $\big|D \cap T_q\big| \le 1$ for all $1 \le q \le k$. If further $|D| = k$, then it is called a diagonal.*

Definition 7.2. *If a set T, $|T| = k$ satisfies $\big|T \cap F_{i_\ell}\big| = 1$ for all $1 \le \ell \le k$, (or equivalently, $v(T) = k$) then T is called a transversal.*

Fact 7.1. *There are k^k transversals, $k!$ diagonals and for every diagonal D there are $k!$ transversals T satisfying $D \ll T$.* □

Corollary 7.1. *If there is a diagonal which is not in $\mathcal{H}(R)$ then there are at least $k!$ transversals that are not in $\mathcal{H}(R)$ either.* □

Definition 7.3. *The k-tuple R is called normal if $1 \le q < q' \le k$ and $a \in T_q$, $a' \in T_{q'}$ imply $a < a'$.*

The notion of normality means that in $F_{i_1} \cup \dots \cup F_{i_k}$, the smallest elements are in T_1, the next smallest in T_2 and so on. It is a rather strong property, which cannot be enforced in general. However, in some cases yes.

Proposition 7.1. *If all $k!$ diagonals are in $\mathcal{H}(R)$, then R is normal.*

Proof. Suppose for contradiction that for some $1 \le q < q' \le k$, $a \in T_q$, $a' \in T_{q'}$, $a > a'$ holds.

Since $q \ne q'$, there exists a diagonal D_1 with $(a', a) \subset D_1$. Take $(k-1)$ more diagonals $D_2, \dots, D_k$ such that $D_1, D_2, \dots, D_k$ form a partition of $F_{i_1} \cup \dots \cup F_{i_k}$. Replace $F_{i_1}, \dots, F_{i_k}$ by $D_1, D_2, \dots, D_k$. Should the elements of D_i be listed in the order as in F_{i_ℓ}, that is, the h^{th} element is in T_h, then $\sum_{1 \le p \le k} a_h(p)$ would be unchanged. However, they are reordered in increasing order. The assumption $a > a'$ implies that some are really changed. It is easy to see that the smallest h for which there is a change in $\sum_{1 \le p \le h} a_h(p)$, it is decreasing. That contradicts the minimal choice of $F_1, \dots, F_s$. □

Definition 7.4. *The k-tuple R is called* <u>*fat*</u> *if there exist pairwise disjoint k-sets $H_1, \dots, H_{k-1} \in \mathcal{H}(R)$ such that $H_1 \cup \cdots \cup H_{k-1} = T_2 \cup \cdots \cup T_k$.*

This is also a very strong property.

Proposition 7.2. *If $\mathcal{H}(R)$ is not fat then there are at least $(k-1)^{k-1}$ transversals T with $T \notin \mathcal{H}(R)$.*

Proof. There are $(k-1)^k$ transversals in $T_2 \cup \cdots \cup T_k$. It is easy to partition them into $(k-1)^{k-1}$ groups so that each group consists of $k-1$ transversals, forming a partition of $T_2 \cup \cdots \cup T_k$. Since $\mathcal{H}(R)$ is not fat, at least one transversal is missing from $\mathcal{H}(R)$ for each group. □

The following lemma shows the strength of the above properties.

Lemma 7.1. *If R is both fat and normal then $|H| = k$ holds for every $H \in \mathcal{H}(R)$ with $H \subset F_{i_1} \cup \cdots \cup F_{i_k}$.*

Proof. Suppose that H contradicts the conclusion. Let $|H| = h < k$. Normality implies $(a_1(1), \dots, a_1(h)) \ll H$. By stability, $(a_1(1), \dots, a_1(h)) \in \mathcal{H}(R)$.

From stability and Claim 3.1 we infer $\{a_1(k), d_1, d_2, \dots, d_{k-1}\} \in \mathcal{H}(R)$. Together with the $k-1$ pairwise disjoint sets $H_1, \dots, H_{k-1}$ we obtain a contradiction with $\nu(\mathcal{H}(R)) = k$. □

Remark 7.1. *Using $d_1 = 1$, $F_0 \cup \{a_2(k)\} \in \mathcal{F}$ follows from Claim 3.1. Therefore one can slightly relax the condition of fatness in the lemma and require only that $\big(T_2 \cup \cdots \cup T_k - \{a_2(k)\}\big) \cup \{a_1(k)\}$ can be obtained as the union of $(k-1)$ members of $\mathcal{H}(R)$.*

Definition 7.5. *We say that R is* <u>*slightly fat*</u> *if there are $k-1$ transversals $H_1, \dots, H_{k-1} \in \mathcal{H}(R)$ whose union is $T_1 \cup T_3 \cup T_4 \cup \cdots \cup T_k$.*

One can prove in the above way

Fact 7.2. *If R is slightly fat, $H \in \mathcal{H}(R)$ then H is not a proper subset of T_2.* □

Let us consider now $\mathcal{H}(R)$ with plenty of $H \in \mathcal{H}(R)$ with $|H| = k-1$.

Definition 7.6. *We say that R is* <u>*robust*</u> *if there exist k pairwise disjoint sets $H_1, \dots, H_k \in \mathcal{H}(R)$, each of size $k-1$.*

Claim 7.1. *If* $H \in \mathcal{H}(R)$ *then* $\big|H \cap (\{1\} \cup H_1 \cup \cdots \cup H_k)\big| \geq 2$ *holds.*

Proof. Suppose the contrary. Then we can find H_0 with $H_0 \in \mathcal{H}(R)$, $\big|H_0 \cap \big(\{1\} \cup H_1 \cup \cdots \cup H_k\big)\big| = 1$. If $1 \in H_0$, then $H_0, H_1, \ldots, H_k$ are $k+1$ pairwise disjoint sets, contradicting $\nu(\mathcal{H}(R)) = k$. However, if the intersection is some $x \in H_1 \cup \cdots \cup H_k$, then by stability $(H_0 \setminus \{x\}) \cup \{1\}$ is also in $\mathcal{H}(R)$. Again we get $k+1$ pairwise disjoint sets. □

Let now R be robust and $k = 3$. Set $X = F_{i_1} \cup F_{i_2} \cup F_{i_3} \cup \{1, d\}$ and $Y = H_1 \cup H_2 \cup H_3 \cup \{1\}$. Define $B(X,Y) = \big\{F \in \binom{X}{3} : |F \cap Y| \geq 2\big\} \cup \binom{Y}{2}$. Claim 7.1 implies that $\mathcal{H}(R) \subseteq B(X,Y)$.

Since $B(X,Y)$ corresponds to $\big\{F \cap [3s+2] : F \in \mathcal{A}_2(n)\big\}$, as we will show, $\sum_{H \in \mathcal{H}(R)} w(H) \leq f(2)$ holds almost automatically for $\mathcal{H}(R)$ if R is robust.

Claim 7.2. *For* $k = 3$*, if* R *is robust then* $H_1 \cup \cdots \cup H_k = T_1 \cup T_2$ *holds.*

Proof. In the contrary case we can choose an ℓ, $1 \leq \ell \leq k$ and an element $a \in (a_1(\ell), a_2(\ell))$ such that $a \notin H_1 \cup H_2 \cup \cdots \cup H_k$.

Using Claim 3.1 and $d_1 = 1$, we infer $\{1, a, d_2, \ldots, d_{k-1}\} \in \mathcal{F}$. Together with $H_1, \ldots, H_k$ these sets contradict $\nu(\mathcal{H}(R)) = k$. □

Proposition 7.3. *For* $k = 3$*, if* R *is robust then*

$$\sum_{H \in \mathcal{H}(R)} w(H) \leq f(2)$$

holds.

Proof. If for $\mathcal{A}_2(n)$ and all $R \in \binom{[s]}{k}$ one defines $\mathcal{H}_{\mathcal{A}_2(n)}(R)$ analogously then one has $\mathcal{H}_{\mathcal{A}_2(n)}(R) = \big\{H : |H \cap (T_1 \cup T_2 \cup \{1\})| \geq 2\big\}$, Claims 7.1 and 7.2 imply $\mathcal{H}(R) \subseteq \mathcal{H}_{\mathcal{A}_2(n)}(R)$ and the statement follows. □

Now let us prove a statement restricting the number of 2-sets in $\mathcal{H}(R)$ for the case that R is not robust. Let g_2 denote the number of 2-element sets of width 2 in $\mathcal{H}(R)$. For $\{u, v\}$ let $g(u, v)$ denote the number of 2-element sets of width 2 in $\mathcal{H}(\{u, v\})$. For $R = \{u, v, z\}$,

$$g_2 = g(u, v) + g(u, z) + g(v, z) \tag{7.1}$$

is obvious. For notational convenience we assume $g(u, v) \geq g(u, z) \geq g(v, z)$.

Proposition 7.4. *If $R = \{u, v, z\}$ and R is not robust then $g_2 \leq 9$ holds.*

Proof. For contradiction we assume $g_2 \geq 10$. Using (7.1) and Proposition 6.1 we distinguish two cases.

(a) $g(u, v) = g(u, z) = 4$, $g(v, z) \geq 2$.

In view of Proposition 6.1, all four sets $\{a_u, b_v\}, \{a_v, b_u\}, \{a_u, b_z\}, \{a_z, b_u\}$ are in $\mathcal{H}(R)$. Also, $g(v, z) \geq 2$ implies that either $\{a_v, b_z\}$ or $\{a_z, b_v\}$ is in $\mathcal{H}(R)$. By symmetry assume $\{a_v, b_z\} \in \mathcal{H}(R)$. Together with $\{a_z, b_u\}$ and $\{a_u, b_v\}$ these 3 sets show that R is robust, a contradiction.

(b) $g(u, v) = 4$, $g(u, z) = g(v, z) = 3$.

If both $\{a_u, b_z\}$ and $\{a_z, b_u\}$ are in $\mathcal{H}(R)$, the preceding proof works. Consequently, we may assume that for $\{u, z\}$ one has case (iii) in Proposition 6.1. That is, either (a_u, c_z) or (a_z, c_u) is in $\mathcal{H}(R)$. If $(a_u, c_z) \in \mathcal{H}(R)$, take $\{b_u, b_v\}$ and $\{a_v, a_z\}$ to show that R is robust.

If $(a_z, c_u) \in \mathcal{H}(R)$ then take $\{a_u, a_v\}$ and $\{b_u, b_v\}$ to get the same contradiction. □

8 The case $s = 2$

Let us use the results from Section 6 to show that the Matching Conjecture is true for $s = 2$.

Since in this case $\mathcal{A}_3 = \binom{[8]}{3}$ has 56 elements and $\mathcal{A}_1(10) = \left\{F \in \binom{[10]}{3} : F \cap [2] \neq \emptyset\right\}$ has 64 elements, all we have to show is:

$$|\mathcal{F}| \leq 64 \quad \text{for } n = 10,\ \mathcal{F} \subset \binom{[n]}{3},\ \nu(\mathcal{F}) = 2.$$

(Recall that for $n = 9$ and more generally $n = 3(s + 1)$, the bound $\binom{n-1}{3}$ is true for all $s \geq 2$.)

As we showed before, $\nu(\mathcal{F}(\bar{1})) = 2$ can be assumed WLOG. Now $R = (1, 2)$. Let us write $\mathcal{H}(1, 2)$ instead of $(\mathcal{H}((1, 2))$. Define $\mathcal{G}_i = \{H \in \mathcal{H}(1, 2) : |H| = 2,\ v(H) = i\}$ for $i = 1, 2$, $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$. Let us recall (2.2) and prove

Proposition 8.1. *If $\mathcal{F} \subset \binom{[10]}{3}$ satisfies $\nu(\mathcal{F}) = 2$, $\nu(\mathcal{F}(\bar{1})) = 2$, then*

$$|\mathcal{F}| = \left|\mathcal{F} \cap \binom{[8]}{3}\right| + 2\left|\mathcal{G}\right| \leq 63 \tag{8.1}$$

holds.

Proof. Set $g_i = \left|\mathcal{G}_i\right|$ for $i = 1, 2$. Suppose for contradiction that $|\mathcal{F}| \geq 64$. From (8.1) we infer $g_1 + g_2 \geq 4$. In particular, $(1, a_1) \in \mathcal{G}$.

Now $\nu(\mathcal{F}) = \nu(\mathcal{H}(1,2)) = 2$ implies that $\mathcal{P} \overset{\text{def}}{=} \{H \in \mathcal{H}(R) : H \subset ([8] - (1, a_1))\}$ is an intersecting family. In particular, at least 10 of the 20 subsets of size 3 in $\binom{[8]-(1,a_1)}{3}$ are missing from $\mathcal{F} \cap \binom{[8]}{3}$. Consequently, the first term on the RHS of (8.1) is at most 46, proving $g_1 + g_2 \geq 9$.

Since not both $(1, x_i)$ and $F_i - \{x_i\}$ are in $\mathcal{G}$, for $i = 1, 2$, and $x_i \in F_i$ (cf. Facts 4.2, 4.3), $g_1 \leq 6$. Consequently, $g_2 \geq 3$ follows.

Now we can apply Proposition 6.1 and distinguish the following two cases

(a) (a_1, b_2) and $\{b_1, a_2\}$ are both in $\mathcal{H}(1, 2)$.

Claim 8.1.

$$|F \cap \{1, a_1, a_2, b_1, b_2\}| \geq 2 \quad \textit{for all } F \in \mathcal{F}. \tag{8.2}$$

Indeed, if $\left|F \cap \{1, a_1, a_2, b_1, b_2\}\right| \leq 1$ then by stability there exists some $F' \in \mathcal{F}$ with $F' \cap \{a_1, a_2, b_1, b_2\} = \emptyset$. Using (a_1, b_2) and $\{b_1, a_2\}$ one concludes $\nu(\mathcal{H}(1, 2)) \geq 3$, a contradiction.

The family $\mathcal{F}$ of all $F \in \binom{[10]}{3}$ satisfying (8.2) is exactly $\mathcal{A}_2(10)$ and it has size

$$\binom{5}{3} + 5\binom{5}{2} = 60 < 63$$

(b) $\mathcal{G}_2 \cap \mathcal{H}(1, 2) = \left\{(a_1, a_2), (a_1, b_2), (a_1, c_2)\right\}$.

Now $g_1 + g_2 \geq 9$ and $g_2 = 3$ imply $g_1 \geq 6$. In particular $(1, b_1) \in \mathcal{G}_1$ and one of $(1, c_1), (a_1, b_1)$ is in $\mathcal{G}_1$ too.

However, using Facts 4.1 and 4.3, $(1, c_1) \in \mathcal{G}_1$ implies $\{a_1, b_1, d\} \notin \mathcal{F}$ and $(a_1, b_1) \in \mathcal{G}_1$ implies $\{1, c_1, d\} \notin \mathcal{F}$. In both cases we found a missing set from $\binom{[8]}{3}$ that is not contained in $[8] - (1, a_1)$. Thus we proved $\left|\mathcal{F} \cap \binom{[8]}{3}\right| \leq \binom{8}{3} - 10 - 1 = 45$. Now (8.1) and $g_1 + g_2 = 9$ imply

$$|\mathcal{F}| \leq 45 + 2 \cdot 9 = 63$$

as desired. □

9 Fat and sufficiently fat triples

Let us suppose that R is a fat triple. With notation $A = (a_i, a_j, a_k)$, $B = \{b_i, b_j, b_k\}$, $C = \{c_i, c_j, c_k\}$ this means that there are $F, F' \in \mathcal{H}(R)$ with $F \cup F' = B \cup C$.

Recall the definition of $\mathcal{G}_1$ and $\mathcal{G}_2$. We use freely the results from Sections 3, 4 and 5.

Proposition 9.1. *If R is fat then* (3.3) *holds.*

Proof. We claim that $\mathcal{G}_2 \cap \mathcal{H}(R) = \emptyset$. Indeed, the contrary and stability would imply $(a_i, a_j) \in \mathcal{G}_2$. Since $\{1, a_k, d\} \in \mathcal{F}$, together with F and F' we have 4 pairwise disjoint sets, a contradiction

Comparing with $\mathcal{A}_3$ we see that our maximum surplus is nine sets in $\mathcal{G}_1$. However, the existence of a set $(1, a_u)$ in $\mathcal{H}(R)$, together with F, F' imply that $\{a_v, a_z, d\} \notin \mathcal{H}(R)$. By stability, the 9 sets $\{x_v, x_z, d\}$, $x_v \in F_v$, $x_z \in F_z$ are all missing. Thus for a loss of a maximum of 3 sets ($(1, a_u), (1, b_u)$ and one of $(1, c_u), (a_u, b_u)$) we have a gain of 9 sets of width 2. Comparing weights (using Convention 4.1),

$$\begin{aligned} \frac{3(n-3s-3)}{\binom{s-1}{2}} &< \frac{9}{s-2} \quad \text{is equivalent to} \\ 2(n-3s-3) &< 3s-3, \quad \text{using } n-3s-3 \le \frac{s}{2}. \\ s &< 3s-3, \quad \text{true for } s \ge 3. \end{aligned}$$

□

Fact 9.1. *If R is not fat then in $\mathcal{H}(R)$*

(i) at least 4 sets of width 3 are missing from $\binom{B\cup C}{3}$.

(ii) at least 6 sets of width 2 are missing from $\binom{B\cup C}{3}$.

Proof. Let us look at the 10 unordered partitions of $B \cup C$ into 2 sets of size 3. ($10 = \frac{1}{2}\binom{6}{3}$). Since R is not fat, at least one set from each pair is missing from $\mathcal{H}(R)$. Now 4 partitions use sets of width 3, 6 use sets of width 2. □

We are going to compare $\mathcal{H}(R)$ with $\mathcal{A}_3$, that is, the complete 3-graph on the same 11 vertices. Fact 9.1 provides us with a gain of $4 + \frac{6}{s-2}$.

Proposition 9.2. *If $(1, a_k) \notin \mathcal{H}(R)$ then* (3.3) *holds.*

Proof. First note that $(1, a_k) \notin \mathcal{H}(R)$ implies $\left|\mathcal{G}_1 \cap \mathcal{H}(R)\right| \leq 6$ and $\mathcal{G}_2 \cap \mathcal{H}(u, k) = \emptyset$ for $u \in (i, j)$. Consequently, $\mathcal{G}_2 \cap \mathcal{H}(R) \subset \mathcal{H}(i, j)$. Set $g_2 = \left|\mathcal{G}_2 \cap \mathcal{H}(R)\right|$.

Let us first prove (3.3) for the case $g_2 \leq 2$. Let $n = n_1(s, 3)$. Since by Proposition 5.3 $n_1(s, 3) \leq 3.5s + 3$, we have $n - 3s - 2 \leq \frac{s}{2} + 1 = \frac{s-2}{2} + 2$.

Consequently our losses are at most

$$2\frac{\frac{s-2}{2} + 2}{s-2} + 6\frac{\frac{s-1}{2} + \frac{1}{2}}{\binom{s-1}{2}} = 1 + \frac{10}{s-2} + \frac{6}{(s-1)(s-2)}.$$

Let us compare it with our gains, $4 + \frac{6}{s-2}$.

$$\begin{aligned} 1 + \frac{10}{s-2} + \frac{6}{(s-1)(s-2)} &\leq 4 + \frac{6}{s-2}, \quad \text{equivalently,} \\ \frac{4}{s-2} + \frac{6}{(s-1)(s-2)} &\leq 3. \end{aligned} \tag{9.1}$$

For $s = 4$ both sides are equal. Since the LHS is a decreasing function of s, (9.1) holds for $s \geq 4$. For $s = 3$ we use $n_1(s, 3) = 13$, $n_1(s, 3) - 3s - 2 = 2$ and check directly

$$2 \cdot \frac{2}{1} + 6 \cdot \frac{1}{1} \leq 4 + \frac{6}{1}.$$

Now let $g_2 \geq 3$. Using Corollaries 6.1 and 6.2, we get an extra gain of $\frac{5}{s-2}$. Moreover, if $g_2 = 4$, then $\{1, c_i, c_j\} \notin \mathcal{F}$ (because $(a_i, a_j), \{b_i, b_j\} \in \mathcal{H}(R)$). Consequently, $\{a_u, c_i, c_j\} \notin \mathcal{F}$ for $u \in R$. These 4 sets provide us with an extra gain of $1 + \frac{3}{s-2}$.

Thus the inequalities to check in the two cases are:

$$\frac{3}{2} + \frac{12}{s-2} + \frac{6}{(s-1)(s-2)} \leq 4 + \frac{11}{s-2} \quad (g_2 = 3)$$
$$2 + \frac{14}{s-2} + \frac{6}{(s-1)(s-2)} \leq 5 + \frac{14}{s-2} \quad (g_2 = 4)$$

Rearranging gives

$$\frac{1}{s-2} + \frac{6}{(s-1)(s-2)} \leq \frac{5}{2}$$
$$\frac{6}{(s-1)(s-2)} \leq 3.$$

The first holds for $s \geq 4$, the second for $s \geq 3$. If $g_2 = 3$ and $s = 3$ then using $n - 3s - 2 = 2$ one checks directly

$$3 \cdot \frac{2}{1} + 6 \cdot \frac{1}{1} < 4 + \frac{11}{1}. \qquad \square$$

From now on R is not fat and $(1, a_u) \in \mathcal{H}(R)$ for all $u \in R$.

For a non-fat triple R some slightly weaker properties might hold.

Definition 9.1. *We measure the fatness of R by the set $Q \subseteq (i, j, k)$ by defining $Q = Q(R)$ through: $u \in Q$ if and only if there exist pairwise disjoint $F, F' \in \mathcal{F}$ with $F \cup F' = \{a_u, b_v, b_z, c_u, c_v, c_z\}$. If $Q \neq \emptyset$, R is called sufficiently fat.*

Proposition 9.3. *If $u \in Q$ then $\{a_v, a_z\} \notin \mathcal{H}(R)$.*

Proof. It follows from $\nu(\mathcal{H}(R)) = 3$ since the 4 sets $F, F', \{1, b_u, d\}$ and $\{a_v, a_z\}$ are pairwise disjoint. (The fact that $\{1, b_u, d\} \in \mathcal{F}$ follows by Claim 3.1.) $\square$

Corollary 9.1.

(i) If $|Q| = 3$ then $(a_i, a_j) \notin \mathcal{H}(R)$.

(ii) If $|Q| = 2$ then $(a_i, a_k) \notin \mathcal{H}(R)$.

(iii) If $|Q| = 1$ then $(a_j, a_k) \notin \mathcal{H}(R)$ hold.

Proof. Immediate from Proposition 9.3 and $(a_i, a_j) \ll (a_i, a_k) \ll (a_j, a_k)$. $\square$

Define $\mathcal{F}_\ell = \{F \in \mathcal{F}_0 : |F| = 3,\ v(F) = \ell\}$, $\ell = 2, 3$. Define further $\mathcal{T} = \mathcal{F}_3 \cap \mathcal{H}(R)$. Let us show that, for not sufficiently fat triples, $\mathcal{T}$ is relatively small.

Proposition 9.4. *Suppose that $R = (i, j, k)$ is not sufficiently fat. Then*

(i) $|\mathcal{T}| \leq 20$, and

(ii) there are at least 12 edges missing from $\mathcal{F}_2 \cap \mathcal{H}(R)$.

Proof. (i) There are 8 transversals in $U \stackrel{\text{def}}{=} B \cup C$. If $\{b_i, b_j, b_k\}$ is missing then by stability all are missing. The next smallest in the shifting partial order are $\{b_u, b_v, c_z\}$, $z \in (i,j,k) : (u,v) = (i,j,k) - \{z\}$. Supposing indirectly $|\mathcal{T}| \geq 21$, we may assume that, for one fixed z, $\{b_u, b_v, c_z\} \in \mathcal{T}$ holds.

Since (i,j,k) is not sufficiently fat, $\{c_u, c_v, a_z\} \notin \mathcal{T}$. Consider two more similar 3-sets: $\{c_u, a_v, c_z\}$ and $\{a_u, c_v, c_z\}$. If both are missing from $\mathcal{T}$, then by stability we obtain 7 missing sets and $|\mathcal{T}| \leq 27 - 7 = 20$. Thus one or both are in $\mathcal{T}$. We distinguish two cases accordingly.

(a) $\{c_u, a_v, c_z\}, \{a_u, c_v, c_z\} \in \mathcal{F}$

Since (i,j,k) is not sufficiently fat, neither $\{b_u, c_v, b_z\}$ nor $\{c_u, b_v, b_z\}$ are in $\mathcal{F}$. By stability, out of the 8 transversals of U, only $\{b_u, b_v, b_z\}$ and $\{b_u, b_v, c_z\}$ are in $\mathcal{F}$. Together with $\{c_u, c_v, a_z\}$, we have 7 missing sets proving $\left|\mathcal{F}_3 \cap \mathcal{H}(R)\right| \leq 20$.

(b) $\{c_u, a_v, c_z\} \notin \mathcal{F}$, $\{a_u, c_v, c_z\} \in \mathcal{F}$.

Now $\{a_u, c_v, c_z\} \in \mathcal{F}$ implies $\{c_u, b_v, b_z\} \notin \mathcal{F}$. Thus by stability, $\{c_u, x_v, x_z\} \notin \mathcal{F}$ for $x_v \in (b_v, c_v)$, $x_z \in (b_z, c_z)$. Together with $\{c_u, c_v, a_z\}$ and $\{c_u, a_v, a_z\}$ these are already 6 missing sets. If no more are missing, $\{b_u, c_v, c_z\}$ and $\{c_u, b_v, a_z\}$ would be in $\mathcal{F}$. However that would show that (i,j,k) is sufficiently fat, a contradiction.

(ii) Consider the following 12 disjoint pairs.

$$\begin{array}{ll} \{b_u, c_u, c_v\},\ \{b_v, a_z, c_z\} & \text{and} \\ \{b_u, c_u, b_v\},\ \{c_v, a_z, c_z\}, & u, v, z \text{ is a permutation of } (i,j,k) \end{array}$$

Since (i,j,k) is not sufficiently fat, at least one set of each pair is missing. These are distinct sets of width 2, concluding the proof. □

Even if R is sufficiently fat, but $|Q| = 1$, we can prove bounds slightly worse than (i) and (ii).

Proposition 9.5. *If* $|Q| = 1$ *then (i), (ii) hold.*

(i) $|\mathcal{T}| \leq 21$.

(ii) There are at least 10 missing edges from $\mathcal{F}_2 \cap \mathcal{H}(R)$.

Proof. Let $Q = \{z\}$. Let us define the two six element sets $P(x) = (B \cup C - \{b_x\}) \cup \{a_x\}$, $x = u, v$. By the definition of $Q = Q(R)$, there are no $F, F' \in \mathcal{F}$ with $F \cup F' = P(x)$. Therefore – just as in the proof of Fact 9.1 – if $F \cup F' = P(x)$ is a partition of $P(x)$, then at least one of F, F' is not in $\mathcal{H}(R)$.

Let us list the 4 partitions of $P(u)$ into sets of width 3:

$$\{a_u, b_v, b_z\},\ \{c_u, c_v, c_z\}$$
$$\{a_u, b_v, c_z\},\ \{c_u, c_v, b_z\}$$
$$\{a_u, c_v, b_z\},\ \{c_u, b_v, c_z\}$$
$$\{a_u, c_v, c_z\},\ \{c_u, b_v, b_z\}$$

Let us list further 2 of the partitions of $P(v)$ into 2 sets of width 3:

$$\{c_u, a_v, b_z\},\ \{b_u, c_v, c_z\}$$
$$\{c_u, a_v, c_z\},\ \{b_u, c_v, b_z\}$$

These are altogether 6 partitions using 12 distinct sets, proving (i).

To prove (ii), we make the corresponding list of 10 partitions into sets of width 2.

$$\{a_u, c_u, b_v\},\ \{c_v, b_z, c_z\}$$
$$\{a_u, c_u, c_v\},\ \{b_v, b_z, c_z\}$$
$$\{a_u, c_u, b_z\},\ \{b_v, c_v, c_z\}$$
$$\{a_u, c_u, c_z\},\ \{b_v, c_v, b_z\}$$
$$\{a_u, b_v, c_v\},\ \{c_u, b_z, c_z\}$$
$$\{a_u, b_z, c_z\},\ \{c_u, b_v, c_v\}$$

$$\{b_u, c_u, c_v\},\ \{a_v, b_z, c_z\}$$
$$\{b_u, c_u, b_z\},\ \{a_v, c_v, c_z\}$$
$$\{b_u, c_u, c_z\},\ \{a_v, c_v, b_z\}$$
$$\{b_u, b_z, c_z\},\ \{c_u, a_v, c_v\}$$

□

Remark 9.1. *The proof might look like trial and error, but it is not. There is the underlying idea that $P(u) - P(v) = \{a_u, b_v\}$. Thus if $F \cup F' = P(u)$ is a partition with $a_u \in F$, $b_v \in F'$ then neither F, nor F' is a subset of $P(v)$. This also implies that in case of equality in (i) or (ii) for those partitions where F contains both a_u and b_v, $F \in \mathcal{H}(R)$, $F' \notin \mathcal{H}(R)$ must hold.*

10 Sufficiently fat is sufficient

Let us prove (3.3) with $\mathcal{A}_3$ as a reference for triples R that are sufficiently fat. We distinguish cases according to $|Q|$.

Recall the notation $g_\ell = |\mathcal{G}_\ell \cap \mathcal{H}(R)|$, $\ell = 1, 2$. Our maximal losses can be estimated from above as

$$\frac{g_2\lfloor\frac{s+2}{2}\rfloor}{s-2} + \frac{g_1\lfloor\frac{s}{2}\rfloor}{\binom{s-1}{2}} \tag{10.1}$$

As to our gains, since R is not fat, using Fact 9.1 and Proposition 9.5, we have at least

$$4 \;+\; \frac{6}{s-2} \quad (|Q| \geq 2), \text{ and} \tag{10.2}$$
$$6 \;+\; \frac{10}{s-2} \quad (|Q| = 1). \tag{10.3}$$

These are the "basic" gains. That is, we can use Corollaries 6.1 and 6.2 for some additional gains in case that $|\mathcal{G}_2 \cap \mathcal{H}(u,v)| \geq 3$.

Proposition 10.1. *If* $|Q| = 3$ *then* (3.3) *holds.*

Proof. In view of Proposition 9.3, $g_2 = 0$. Thus we have to prove

$$\frac{9\lfloor\frac{s}{2}\rfloor}{\binom{s-1}{2}} \leq 4 + \frac{6}{s-2}. \tag{10.4}$$

For $s = 3$, it is true. Let $s \geq 4$ and use $\lfloor\frac{s}{2}\rfloor \leq \frac{s-1}{2} + \frac{1}{2}$. Then (10.4) reduces to

$$\frac{3}{s-2} + \frac{9}{(s-1)(s-2)} \leq 4.$$

For $s = 4$, we have $3 < 4$, and the LHS is a decreasing function of s. □

Proposition 10.2. *If* $|Q| = 2$*, then* (3.3) *holds unless* $s = 3,\ n = 13$.

Proof. Stability and Proposition 9.3 imply $(a_i, a_k), (a_j, a_k) \notin \mathcal{G}$. Thus $g_2 = |\mathcal{G}_2 \cap \mathcal{H}(i,j)|$. We distinguish 2 cases: $g_2 \leq 2$ and $g_2 = 3$ or 4.

(a) $g_2 \leq 2$

First let $s \geq 6$. Use $\frac{s+2}{2} = \frac{s-2}{2} + 2$ to get the upper bound for (10.1):

$$2\frac{\frac{s-2}{2}+2}{s-2} + \frac{9}{s-2} + \frac{9}{(s-1)(s-2)} = 1 + \frac{13}{s-2} + \frac{9}{(s-1)(s-2)}$$

Thus it is sufficient to have

$$\frac{7}{s-2} + \frac{9}{(s-1)(s-2)} \leq 3.$$

For $s = 6$, $\frac{7}{4} + \frac{9}{20} < 3$, and the LHS is monotone decreasing with s.

For $s = 5$, $\lfloor \frac{s+2}{2} \rfloor = 3$, $\lfloor \frac{s}{2} \rfloor = 2$ and

$$\frac{2 \cdot 3}{3} + \frac{9 \cdot 2}{6} = 5 < 4 + \frac{6}{3} \quad \text{holds.}$$

For the cases $s = 3$ or 4, let first $n = n_1(s, 3) - 1$. Then $n - 3s - 2$ is 1 for $s = 3$ and 2 for $s = 4$. It can be checked that (10.1) is less than (10.2) in both cases.

For $s = 4$, $n = n_0(4, 3) = 17$ one has $|\mathcal{A}_1(17)| - \binom{14}{3} = 30$. Thus it is sufficient to prove (using $f(1) = f(3) + \frac{30}{\binom{4}{3}}$) that (10.1) is less than (10.2) plus 7.5, which holds largely. However, for $s = 3$, $n = 13$ one has

$$2 \cdot 3 + 9 = 15 > 4 + 6.$$

We shall take care of the $s = 3$, $n = 13$ case separately in Section 12.

(b) $g_2 \geq 3$.

From Proposition 6.1 it follows that $g_2 = 3$ or 4. From Corollaries 6.1 and 6.2 we can replace (10.2) by $4 + \frac{11}{s-2}$. Moreover, in the case $g_2 = 4$, $\{1, c_i, c_j\} \notin \mathcal{F}$ and stability provide us with 4 previously not excluded missing sets $\{1, c_i, c_j\}, \{a_i, c_i, c_j\}, \{a_j, c_i, c_j\}$ and $\{a_k, c_i, c_j\}$. Among them 3 are of width 2 and 1 is of width 3, providing for an extra gain of $1 + \frac{3}{s-2}$.

Consequently, the inequalities needed for $g_2 = 3, 4$ are the following.

$$\frac{3 \cdot \frac{s+2}{2}}{s-2} + \frac{9 \cdot \frac{s}{2}}{\binom{s-1}{2}} \leq 4 + \frac{11}{s-2}, \quad \text{and} \tag{10.5}$$

$$\frac{4 \cdot \frac{s+2}{2}}{s-2} + \frac{9 \cdot \frac{s}{2}}{\binom{s-1}{2}} \leq 5 + \frac{14}{s-2}$$

The second one holds with equality for $s = 4$. The first one holds strictly for $s = 5$. Collecting the terms with $\frac{1}{s-2}$ on the LHS and using monotonicity, both inequalities follow unless $s = 4$ in the first one. However, even in this case the LHS is only 1 larger than the RHS. Consequently, (3.3) holds easily with $f(3)$ replaced by $f(1) = f(3)+7.5$. In the case $s = 4$, $n = n_0(4,3) - 1 = 16$, instead of (10.5) we need (cf. Corollary 4.1)

$$\frac{3 \cdot \frac{4}{2}}{2} + \frac{9 \cdot \frac{2}{2}}{3} = 3 + 3 < 4 + \frac{11}{2}$$

which is true by large. □

Proposition 10.3. (3.3) *holds for* $|Q| = 1$ *and* $s \geqq 4$.

Proof. In view of Corollary 9.1, $(a_j, a_k) \notin \mathcal{G}$. Thus

$$g_2 = |\mathcal{G}_2 \cap \mathcal{H}(i,j)| + |\mathcal{G}_2 \cap \mathcal{H}(i,k)|. \tag{10.6}$$

Using Proposition 9.5 provides us with a gain of $6 + \frac{10}{s-2}$.

Claim 10.1. *For* $s \geq 5$ *one has*

$$5 \cdot \frac{\lfloor \frac{s+2}{2} \rfloor}{s-2} + \frac{9 \cdot \lfloor \frac{s}{2} \rfloor}{\binom{s-1}{2}} \leq 6 + \frac{10}{s-2} \tag{10.7}$$

Proof. (10.7) is easily checked to hold for both $s = 5$ and 6. For $s > 6$ monotonicity considerations yield (10.7).

For $s = 4$ the LHS of (10.7) is $\frac{15}{2} + 6 = 13.5$, the RHS is 11. Since the difference is less than 7.5, we are alright.

In the case $s = 4$, $n = 16$ one can replace $\frac{s+2}{2}$ by $\frac{s}{2}$, $\frac{s}{2}$ by $\frac{s-2}{2}$ and the corresponding version of (10.7) holds in the stronger form

$$8 \cdot \frac{2}{2} + \frac{9}{3} \leq 6 + \frac{10}{2},$$

that is for $g_2 = 8$. Consequently, in view of Proposition 6.1 in the sequel we do not need to consider the case $s = 4$, $n = 16$.

In view of Claim 10.1, we can assume $g_2 \geq 6$. Let us use (10.6). For $g_2 = 8$, $|\mathcal{G}_2 \cap \mathcal{H}(i,j)| = |\mathcal{G}_2 \cap \mathcal{H}(i,k)| = 4$. For $g_2 = 7$, one of them is 4, the other is 3. For $g_2 = 6$, $6 = 4 + 2$, or $6 = 3 + 3$ hold.

Let us first check the case $g_2 = 6$. Now Corollaries 6.1, 6.2 provide us with an extra gain of $\frac{5}{s-2}$. Thus we need

$$\frac{6 \cdot \lfloor \frac{s+2}{2} \rfloor}{s-2} + \frac{9 \cdot \lfloor \frac{s}{2} \rfloor}{\binom{s-1}{2}} \leq 6 + \frac{15}{s-2}$$

This inequality is true for both $s = 5$ and 6. By monotonicity it holds for all $s \geq 5$. For $s = 4$ the two sides are 15 and 13.5 showing that the extra 7.5 is more than sufficient.

In the cases of $g_2 = 7, 8$ we can use the extra gains from Corollaries 6.1, 6.2. These amount to $\frac{10}{s-2}$, for missing sets containing d. For the extra gains from Fact 6.1, that is the 4 sets $\{1, c_i, c_x\}, \{a_i, c_i, c_x\}, \{a_x, c_i, c_x\}$ and $\{a_y, c_i, c_x\}$, where $x = j$ or k and $\{y\} = \{j, k\} - \{x\}$, we have to be more careful to avoid counting the same missing set twice. The problem is coming from the fact that we are already using Proposition 9.5. The sets containing 1 are safe as there is no such set in Proposition 9.5.

Let us sort it out a little. Note that from Proposition 9.3 we infer $Q = \{i\}$. That is, the u, v in Proposition 9.5 are j and k. Consequently, the sets containing a_i do not occur there either. Thus along with $\{1, c_i, c_j\}, \{1, c_i, c_k\}$, the two sets $\{a_i, c_i, c_k\}$ and $\{a_i, c_i, c_j\}$ provide us with extra gains of $\frac{4}{s-2}$. However the same cannot be said about the other sets. For our purpose it is enough already. We have now gains of

$$6 + \frac{10}{s-2} + \frac{2 \cdot 5}{s-2} + \frac{4}{s-2} = 6 + \frac{24}{s-2}. \qquad \square$$

Claim 10.2. *For $s \geq 4$*

$$\frac{8 \cdot \lfloor \frac{s+2}{2} \rfloor}{s-2} + \frac{9 \cdot \lfloor \frac{s}{2} \rfloor}{\binom{s-1}{2}} \leq 6 + \frac{24}{s-2}.$$

Proof. For $s = 5$ we have

$$\frac{8 \cdot 3}{3} + \frac{9 \cdot 2}{6} = 11 < 6 + \frac{24}{3} = 14.$$

For $s = 4$ we have

$$\frac{8 \cdot 3}{2} + \frac{9 \cdot 2}{3} = 18 = 6 + \frac{24}{2}.$$

The rest follows from monotonicity. $\square$

This concludes the proof of Proposition 10.3. $\square$

11 Not sufficiently fat is sufficient

In view of Section 10, we may suppose that R is not sufficiently fat. By Proposition 9.4 we have an initial gain of

$$7+\frac{12}{s-2}. \tag{11.1}$$

For each $(u,v)\subset R$ satisfying $|\mathcal{G}_2\cap\mathcal{H}(u,v)|\ge 3$ we have an additional gain of $\frac{5}{s-2}$. Moreover, if $|\mathcal{G}_2\cap\mathcal{H}(u,v)|=4$, then we can add to this $\frac{1}{s-2}$ for the missing set $\{1,c_i,c_j\}$.

Let us compare our maximal loss with (11.1)

$$\frac{g_2\lfloor\frac{s+2}{2}\rfloor}{s-2}+\frac{9\cdot\lfloor\frac{s}{2}\rfloor}{\binom{s-1}{2}}\le 7+\frac{12}{s-2}. \tag{11.2}$$

For $s=5$ we have

$$g_2+3\le 7+4$$

which is true even for $g_2=8$. For $g_2=9$, that is, increasing g_2 by 1, increases the LHS by 1. However, adding $\frac{5}{s-2}$ to the RHS, it increases by $\frac{5}{3}$, which is more than 1, proving (3.3) for $s=5$.

For $s\ge 6$ we use $\frac{s+2}{2}=\frac{s-2}{2}+2$, $\frac{s}{2}=\frac{s-1}{2}+\frac{1}{2}$ to rewrite the LHS of (11.2) as

$$\frac{g_2}{2}+\frac{2g_2}{s-2}+\frac{9}{s-2}+\frac{9}{(s-1)(s-2)}$$

and use it to rewrite (11.2) as

$$\frac{2g_2-3}{s-2}+\frac{9}{(s-1)(s-2)}\le 7-\frac{g_2}{2}. \tag{11.3}$$

In this form, for g_2 fixed, the RHS is constant and the LHS is a decreasing function of s. If it holds for $s=6$, it holds for all $s\ge 6$. For $g_2=6$, the inequality (11.3) reduces to

$$\frac{9}{4}+\frac{9}{20}\le 4,$$

which is true.

For $g_2\ge 7$, at least one $\mathcal{G}_2\cap\mathcal{H}(u,v)$ has to contain at least 3 elements. Thus our gains increase by $\frac{5}{s-2}$ leading to the adjusted version of (11.3):

$$\frac{2g_2-8}{s-2}+\frac{9}{(s-1)(s-2)}\le 7-\frac{g_2}{2}.$$

For $g_2 = 8$, plugging in $s = 6$ gives

$$2 + \frac{9}{20} \leq 3$$

which is true, and the case $s \geq 6$ follows by monotonicity.

For $g_2 = 9$, $9 > 4 + 2 + 2$ implies that we can add $2 \cdot \frac{5}{s-2}$ to increase our gains. Consequently, the inequality that we have to prove reduces to

$$\frac{2g_2 - 13}{s - 2} + \frac{9}{(s-1)(s-2)} \leq 7 - \frac{g_2}{2}.$$

Plugging in $g_2 = 9,\ s = 6$ gives

$$\frac{5}{4} + \frac{9}{20} \leq \frac{5}{2}$$

which is true. Thus we have proved the next proposition except for $s = 4$.

Proposition 11.1. *If R is not sufficiently fat and $g_2 \leq 9$ then* (3.3) *holds for $s \geq 4$.*

Proof. We only have to deal with the case of $s = 4$. There are 2 sub-cases: $n = 16$ and $n = 17$. In the first case our losses can be written as

$$g_2 + \frac{9}{6} \leq 10.5 < 7 + \frac{12}{2} = 13.$$

For the case $n = 17,\ n - 3s - 2 = \frac{s+2}{2}$. We can bound our losses as:

$$\frac{3g_2}{2} + 6 \tag{11.4}$$

Since our gains are $7 + \frac{12}{s-2} = 13$, we need only that (11.4) is less than 20.5. Fortunately, even for $g_2 = 9$ one has

$$\frac{3g_2}{2} + 6 = \frac{27}{2} + 6 = 19.5$$

concluding the proof. □

By Propositions 7.3, 7.4 and 11.1 our proof is complete except for $s = 3$, $n = n_1(3, 3) = 13$. We are going to handle this case directly in Section 12. One might think that our whole proof, which in its initial parts used double induction, might collapse without this case. It is not the case. Applying induction for some particular s, we always have $n \geq n_1(s,3) - 1 \geq n_1(s-1,3) + 2$. Therefore, to support the induction, it is sufficient to prove that the maximum size of a 3-graph on $n = n_1(s-1,3) + 2$ vertices is at most $|\mathcal{A}_1(n)|$. In particular, in our "missing" case, $n = 16$, $s - 1 = 3$, using $\binom{s-1-v(H)}{k-v(H)} = \binom{3-v(H)}{3-v(H)} = 1$ we need to give a bound of the form

$$\sum_{H \in \mathcal{H}(R)} w(H) \leq |\mathcal{A}_1(15)| = \binom{15}{3} - \binom{12}{3} = 235 = |\mathcal{A}_3| + 70.$$

That is, we do not have to struggle to get $f(3)$ or $f(3)+1$ as an upper bound, $f(3)+70$ is sufficient. That is too easy, the bounds we have proven so far are much stronger. Anyway, this discussion is only philosophical. We are going to handle the quite tedious case $n = 13$, $s = 3$ below.

12 The last case

Let $n = 13,\ s = 3,\ \mathcal{F} \subseteq \binom{[13]}{3},\ \nu(\mathcal{F}) = 3$. Since for $s = 3,\ s - 2 = 1 = \binom{s-1}{2}$, computation is easier. With previous notation let $2 \leq d \leq 11$ and let

$$F_1 \cup F_2 \cup F_3 = [11] - (1,d), \quad \text{where } F_i = (a_i, b_i, c_i).$$

Set $\mathcal{G}_i = \left\{G \in \binom{[11]}{2} : v(G) = i,\ \exists F \in \mathcal{F} : F \cap [11] = G\right\}$, and $g_i = |\mathcal{G}_i|$ for $i = 1, 2$. Set further $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$ and $\mathcal{F}_1 = \{F \in \mathcal{F} : F \subset [11]\}$. Now the formula for $|\mathcal{F}|$ is simple

$$|\mathcal{F}| = |\mathcal{F}_1| + 2|\mathcal{G}| = |\mathcal{F}_1| + 2(g_1 + g_2). \tag{12.1}$$

Proposition 12.1. $|\mathcal{F}| \leq |\mathcal{A}_3| = \binom{11}{3} = 165.$

Arguing indirectly we assume $|\mathcal{F}| \geq 166 = |\mathcal{A}_1(13)|$. We are going to prove Proposition 12.1 as an end result of a series of claims.

Claim 12.1. $(1,2) \in \mathcal{G}$.

Proof. Otherwise $|\mathcal{G}| = 0$ by stability and (12.1) implies $|\mathcal{F}| \leq 165$. □

Claim 12.2.

$$|\mathcal{F}_1| \le \binom{11}{3} - \binom{8}{2} = 137. \tag{12.2}$$

Proof. Consider $\tilde{\mathcal{F}} \stackrel{\text{def}}{=} \{F \in \mathcal{F}_1 : F \subset [3,11]\}$. Now $\nu(\tilde{\mathcal{F}}) \le 2$ follows from $(1,2) \in \mathcal{G}$. Since $\big|[3,11]\big| = 9$, from the $s = 2$ case we infer $|\tilde{\mathcal{F}}| \le \binom{8}{3} = \binom{9}{3} - \binom{8}{2}$. That is, we showed that at least $\binom{8}{2}$ sets are missing already from $\binom{[3,11]}{3}$. It can not be less on $\binom{[11]}{3}$, proving (12.2). □

Corollary 12.1. $g_1 + g_2 \ge 15$.

Proof. If $g_1 + g_2 \le 14$ then combining it with (12.2) and using (12.1) gives

$$|\mathcal{F}| \le 137 + 2 \cdot 14 = 165.$$ □

In Section 7 we proved Conjecture 1.1 for robust triples. Since we are arguing indirectly, WLOG [3] is not robust. Thus Proposition 7.4 gives $g_2 \le 9$. We showed also (the much easier inequality) $g_1 \le 9$. Along the lines of Proposition 7.4 let us prove:

Claim 12.3.

$$g_1 + g_2 \le 17 \tag{12.3}$$

Proof. Arguing indirectly we assume $g_1 = g_2 = 9$. For $(u,v) \subset [3]$ let $\mathcal{G}(u,v)$ denote the family of those $G \in \mathcal{G}_2$ that satisfy $G \subset F_u \cup F_v$. In Proposition 6.1 we characterized $\mathcal{G}(u,v)$ for $|\mathcal{G}(u,v)| \ge 3$. Let us show that possibilities (i) and (iii) cannot occur simultaneously. Indeed if $|\mathcal{G}(u,v)| = 4$ for some $\{u,v\} \subset [3]$, and either (a_u, c_z) or (a_z, c_u) is in $\mathcal{G}$, then we can take $(a_u, c_z), \{b_u, b_v\}$ and $\{a_v, a_z\}$ or the 3 sets $(a_z, c_u), \{a_u, a_v\}, \{b_u, b_v\}$ to show that [3] is robust, a contradiction.

Should no (a_u, c_z) be in $\mathcal{G}$, then there are only $3 \cdot 4 = 12$ possibilities for $G \in \mathcal{G}_2$. These 12 sets can be partitioned into 4 groups of 3 sets each, where each group gives a partition of $A \cup B$. Since [3] is not a robust triple, at most 2 sets from each group are in $\mathcal{G}_2$. Thus $|\mathcal{G}_2| \le 4 \cdot 2 = 8 < 9$.

Until now we showed that there is at least one (u,v) with $(a_u, c_v) \in \mathcal{G}$ and there is no (u,v) with $|\mathcal{G}(u,v)| = 4$. Hence by $g_2 = 9$ and Proposition 6.1 $|\mathcal{G}(u,v)| = 3$ for each $(u,v) \subset [3]$.

Let us show that possibility (ii) cannot hold for two choices of $(u,v) \subset [3]$. Indeed, if it held for, say, $\{u,z\}$ and $\{v,z\}$ and $(a_u, c_v) \in \mathcal{G}$, then we could use (a_u, c_v), $\{b_u, a_z\}$ and $\{b_z, a_v\}$ to show that [3] is robust.

Note that if $(a_2, c_3) \in \mathcal{G}$ then by stability $(a_1, c_3) \in \mathcal{G}$ holds as well. Consequently, we are left with only two possibilities.

(a) $(a_1, c_2) \in \mathcal{G},\ (a_1, c_3) \in \mathcal{G},\ \mathcal{G}(2,3)$ is of type (ii).

(b) $(a_1, c_2),\ (a_1, c_3),\ (a_2, c_3) \in \mathcal{G}$.

Let us consider these separately.

(a) $(a_1, c_2),\ (a_2, b_3),\ \{a_3, b_2\}$ show that $[3]$ is robust.

(b) In this case we are going to prove $|\mathcal{F}| \leq 165$.

First let us show that $\mathcal{F}_1 \cap \binom{F_3 \cup \{b_2, c_2\}}{3} = \{F_3\}$. Let $H \subset (F_3 \cup \{b_2, c_2\})$ and $H \neq F_3$ satisfy $H \in \mathcal{F}_1$. By stability, we may assume that either $H = \{b_2, c_2, a_3\}$, or $H = \{b_2, a_3, b_3\}$.

In the first case look at the 4 sets $\{b_2, c_2, a_3\}$, (a_1, c_3), (a_2, b_3) and $(1, b_1)$ to obtain the contradiction $\nu(\mathcal{H}([3])) \geq 4$.

In the second case look at the 4 sets $\{b_2, a_3, b_3\}$, (a_1, c_2), (a_2, c_3) and $(1, b_1)$ to get the same contradiction. (Let us remark that $(1, b_1) \in \mathcal{G}$ follows from $g_1 = 9$.)

Basically the same argument shows that none of the remaining subsets of $F_3 \cup \{b_2, c_2\} \cup \{b_1, c_1, d\}$ are in $\mathcal{F}_1$. This provides us with $\binom{8}{3} - 1 = 55$ sets missing from $\mathcal{F}_1$. Using (12.1) gives

$$|\mathcal{F}| \leq (165 - 55) + 18 \cdot 2 = 146 < 165. \qquad \square$$

What we showed is that either $g_2 \leq 8$ or $(1, b_1) \notin \mathcal{G}_1$ holds.

Plugging $g_1 + g_2 \leq 17$ back into (12.1) and using the indirect assumption $|\mathcal{F}| \geq 166$ gives

$$|\mathcal{F}_1| \geq 166 - 2 \cdot 17 = 132 \tag{12.4}$$

Claim 12.4. *$(4, 5) \notin \mathcal{G}$.*

Proof. Suppose for contradiction that $(4,5) \in \mathcal{G}$. Let us define $\mathcal{H} = \Big\{H \in \binom{[11]}{3} : \big|H \cap [3]\big| = 1, \big|H \cap [6,11]\big| = 2\Big\}$. Note that $|\mathcal{H}| = 3 \times \binom{6}{2} = 45$. Note also that $\mathcal{H} \cap \mathcal{F}_1$ contains no three pairwise disjoint sets. That is, for every choice of $U, V, W \in \binom{[6,11]}{2}$, U, V, W forming a partition of $[6, 11]$, at least one of the sets $\{1\} \cup U$, $\{2\} \cup V$ and $\{3\} \cup W$ is missing from $\mathcal{F}_1$.

It is both well-known and easily proved that the 15 edges of the complete graph on the vertex set $[6, 11]$ can be partitioned into 5 perfect matchings: (U_i, V_i, W_i), $1 \leqq i \leqq 5$. For each i we consider three triples of 3-sets:

$$\big(U_i \cup \{1\}, V_i \cup \{2\}, W_i \cup \{3\}\big), \quad \big(U_i \cup \{2\}, V_i \cup \{3\}, W_i \cup \{1\}\big),$$
$$\big(U_i \cup \{3\}, V_i \cup \{1\}, W_i \cup \{2\}\big).$$

These are altogether 15 disjoint triples, each giving rise to at least one 3-set in $\mathcal{H} \setminus \mathcal{F}_1$. By averaging we may choose j, $1 \leqq j \leqq 3$ such that at least five sets of the form (j, u, v) are in $\mathcal{H} \setminus \mathcal{F}_1$. By stability, neither $(4, u, v)$ nor $(5, u, v)$ are in $\mathcal{F}_1$.

Thus we found ten 3-sets containing 4 or 5 and missing from $\mathcal{F}_1$.

On the other hand by $\nu\left(\mathcal{F}_1 \cap \binom{[3]\cup[6,11]}{3}\right) \leqq 2$, at least $\frac{1}{3}\binom{9}{3} = 28$ sets are missing here too. Consequently $|\mathcal{F}_1| \leqq \binom{11}{3} - (28+10) = 127$, contradicting (12.4). □

Note that Claim 12.4 shows that $G \cap [3] \neq \emptyset$ for all $G \in \mathcal{G}$. This brings $\mathcal{F}$ pretty close to $\mathcal{A}_1(13)$. Next we show that, except for $(3,4)$ and $(3,5)$, there are no sets starting with 3.

Claim 12.5. $(3,6) \notin \mathcal{G}$.

Proof. Assume $(3,6) \in \mathcal{G}$. Consider now the family $\mathcal{P}$ of missing 3-sets in $\binom{[11]-(3,6)}{3}$. Just as in Claim 12.2, $|\mathcal{P}| \geq 28$ holds. Since $(1,5) \ll (2,5) \ll (3,6)$, both $(1,5)$ and $(2,5)$ are in $\mathcal{G}$. Thus there is no $P \in \mathcal{P}$ with $|P \cap ((1,2) \cup (4,5))| \geq 2$, except possibly if $P \cap ((1,2)\cup(4,5)) = (4,5)$. There can be at most $\big|[7,11]\big| = 5$ sets of the latter type. There can be $\left|\binom{[7,11]}{3}\right| = 10$ sets in $\mathcal{P}$ that do not intersect $(1,2) \cup (4,5)$. For the remaining at least $28 - 15 = 13$ sets $P \in \mathcal{P}$ one has $\big|P \cap ((1,2)\cup(4,5))\big| = 1$.

For a set of the form $(i,p,q) \in \mathcal{P}$ with $i \in (1,2)$, $(p,q) \subset [7,11]$, note that $(3,p,q) \notin \mathcal{F}_1$ holds by stability. Similarly if $i \in (4,5)$ then $(6,p,q) \notin \mathcal{F}_1$ follows. This way we associate the same missing new set with at most 2 sets in $\mathcal{P}$. Thus we obtain at least $\lceil \frac{13}{2} \rceil = 7$ extra missing sets. This brings the total to at least $28 + 7 = 35$, i.e., $|\mathcal{F}_1| \leq 165 - 35 = 130$, contradicting (12.4). □

Inequality (12.4) shows that at most $165 - 132 = 33$ sets are missing from $\binom{[11]}{3}$. On the other hand, in Claim 12.2 we showed that at least 28 sets are missing from $\binom{[3,11]}{3}$. This implies

Claim 12.6. *There are at most five* 3*-element sets containing* 1 *or* 2 *that are missing from* $\mathcal{F}_1$*.*

Corollary 12.2. $(2,8,9) \in \mathcal{F}_1$ *and* $(2,8,10) \in \mathcal{F}_1$ *unless* <u>*all*</u> 3*-sets containing* 1 *are in* $\mathcal{F}_1$*.*

Proof. There are $\binom{4}{2} = 6$ sets of the form $(2,a,b) : (a,b) \subset (8,9,10,11)$. Using stability the statement follows. □

Claim 12.7. $(5,6,7)$, $(5,6,8) \in \mathcal{F}_1$.

Proof. Since $\left|\binom{[5,11]}{3}\right| = 35$, at least 2 of these sets have to be in $\mathcal{F}_1$. The statement follows by stability. $\square$

Claim 12.8. $(3,4) \in \mathcal{G}$.

Proof. Suppose the contrary. Since $F_0 = (1,d) \notin \mathcal{F}$, we have $2 = a_1$. We infer that all edges in $\mathcal{G}$ contain either 1 or a_1. In particular, $\mathcal{G}(2,3) = \emptyset$. For $\mathcal{G}_2(1,2)$ and $\mathcal{G}_2(1,3)$ also, there can be a maximum of 3 edges, namely the ones containing a_1. Thus $g_2 \leq 6$. Using Corollary 12.1, $g_2 = 6$, $g_1 = 9$ follow. In particular, (a_1, c_2) and (a_1, c_3) are in $\mathcal{G}$. Consequently, $(a_1, x) \notin \mathcal{G}$ might be possible only for $x = b_1, c_1$ and d.

Moreover, using $g_1 = 9$ and Fact 4.1 (with $R = (1)$), either (a_1, b_1) or $(1, c_1)$ is in $\mathcal{G}$. Now the 15 edges in $\mathcal{G}$ can be listed:

$$\{(1,x) : 2 \leq x \leq 9\} \cup \{(2,y) : 3 \leq y \leq 8\}$$

along with either $(1,10)$ or $(2,9)$. Plugging $g_1 + g_2 = 15$ once again into (12.1) gives:

$$|\mathcal{F}_1| \geq 166 - 2 \cdot 15 = 136 = \binom{11}{3} - 29.$$

That is, except for the, at least 28, elements of $\binom{[3,11]}{3}$ there is at most 1 missing 3-set from $\mathcal{F}_1$. By stability, only $(2,10,11)$ could be missing. Thus $(1,10,11)$ and $(2,9,11)$ are in $\mathcal{F}_1$. By stability, $\{1, c_1, d\} \in \mathcal{F}$ and $\{2, b_1, d\} \in \mathcal{F}$ follow.

Now we can get easily 4 pairwise disjoint sets:

F_2, F_3, $\{1, c_1, d\}$, (a_1, b_1) or
F_2, F_3, (a_1, b_1, d), $(1, c_1)$, a contradiction. $\square$

Claim 12.9. $(1,7) \in \mathcal{G}$.

Proof. Otherwise $\mathcal{G} \subset \binom{[6]}{2}$. Using $(4,5) \notin \mathcal{G}$, $|\mathcal{G}| \leq 14$ follows, a contradiction. $\square$

Claim 12.10. *$(2,x,y) \notin \mathcal{F}$ for $(x,y) \subset (9,10,11)$.*

Proof. $(1,7)$, $(3,4)$, $(5,6,8)$ and $(2,x,y)$ are 4 pairwise disjoint sets. $\square$

Corollary 12.3. $(1, 9, 10) \in \mathcal{F}_1$.

Proof. Otherwise, by stability, all 3 sets $(1, x, y)$ are missing from $\mathcal{F}_1$, $(x, y) \subset (9, 10, 11)$. Together we find six, that is more than five, missing 3-sets containing 1 or 2, a contradiction □

Corollary 12.4. $(2, 7) \notin \mathcal{G}$.

Proof. The 4 sets $(1, 9, 10)$, $(2, 7)$, $(3, 4)$ and $(5, 6, 8)$ are pairwise disjoint. □

Claim 12.11. $\mathcal{G} = \big\{(3, x) : x = 4, 5\big\} \cup \big\{(2, y) : 3 \le y \le 6\big\} \cup \big\{(1, z) : 2 \le z \le 10\big\}$.

Proof. The above $\mathcal{G}$ has 15 elements. Now the statement follows from $|\mathcal{G}| \ge 15$ and $(4, 5) \notin \mathcal{G}$, $(3, 6) \notin \mathcal{G}$, $(2, 7) \notin \mathcal{G}$, $(1, d) \notin \mathcal{G}$. □

Claim 12.12. $(5, 7, 8) \notin \mathcal{F}_1$.

Proof. The 4 sets $(1, 9)$, $(2, 6)$, $(3, 4)$ and $(5, 7, 8)$ are pairwise disjoint. □

Corollary 12.5. *The following* 30 *sets are missing from* $\mathcal{F}_1$*:*

$$\binom{[7, 11]}{3}, \ \{(i, x, y) : i = 5, 6; \ (x, y) \subset [7, 11]\}.$$

Proof. By $(5, 7, 8) \notin \mathcal{F}_1$ and stability. □

Finally, we can get the contradiction. Corollary 12.5 and Claim 12.10 provide us with 33 missing sets. Now (12.1) and $|\mathcal{G}| = 15$ imply

$$|\mathcal{F}| \le (165 - 33) + 2 \cdot 15 = 162 < 166$$

13 Uniqueness and beyond

We did not explicitly state it, but the case of stable families, the proof yields that $|\mathcal{F}| = \max\big\{|\mathcal{A}_3|, \ |\mathcal{A}_1(n)|\big\}$ is only possible if $\mathcal{F} = \mathcal{A}_3$ or $\mathcal{F} = \mathcal{A}_1(n)$ holds. Then it is not hard to show that even without assuming stability, the families of maximal size are unique up to isomorphism. For stable families our proof yields much more.

Theorem 13.1. *Let $\mathcal{F} \subset \binom{[n]}{3}$ be a stable family with $\nu(\mathcal{F}) = \nu(\mathcal{F}(\bar{1})) = s,\ s \geq 5$. Then*

$$|\mathcal{F}| \leq \max\Big\{\big|\mathcal{F}_3\big|,\ \big|\mathcal{F}_2(n)\big|\Big\} \tag{13.1}$$

holds and in case of equality $\mathcal{F} = \mathcal{F}_3$ or $\mathcal{F} = \mathcal{F}_2(n)$.

For the cases $s = 2, 3$ and 4 the same result holds, but one has to do an even more detailed case analysis (or find a different proof).

In this paper we prove some results for general k but did not even come close to giving a full proof of the Matching Conjecture. Let us announce two results which will appear in a forthcoming paper.

Theorem 13.2. *For $k = 4$ and $s > s_0$ the Matching Conjecture is true.*

For the second we need a definition.

Let $(x_0, x_1, \ldots, x_{s-1}) \subset [n]$ and let $F_1, \ldots, F_s$ be pairwise disjoint sets, $x_i \in F_i,\ 1 \leq i < s$ but $x_0 \notin F_1 \cup \cdots \cup F_s \subset [n]$. Define a graph $\mathcal{G}$ with edge set consisting of all $\{x_i, y_i\}$ satisfying $y_i \in F_{i+1} \cup \cdots F_s,\ 0 \leq i < s$. Finally define the k-graph $\mathcal{F}(\mathcal{G})$ by

$$\mathcal{F}(\mathcal{G}) = \left\{F \in \binom{[n]}{k} : E \subset F \text{ holds for some edge } E \in \mathcal{G}\right\} \cup \{F_1, F_2, \ldots, F_s\}.$$

Theorem 13.3 ([12])**.** *Let $k \geq 4$, $n \geq n_1(k, s)$ and let $\mathcal{F} \subset \binom{[n]}{k}$ be a stable family with $\nu(\mathcal{F}) = \nu(\mathcal{F}(\bar{1})) = s$. Then $\big|\mathcal{F}\big| \leq \big|\mathcal{F}(\mathcal{G})\big|$ and in case of equality $\mathcal{F}$ is isomorphic to $\mathcal{F}(\mathcal{G})$.*

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