New inequalities for cross-intersecting families

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Abstract

Two families $\mathcal{F}$ and $\mathcal{G}$ are called cross-intersecting if for all $F \in \mathcal{F}$, $G \in \mathcal{G}$ the intersection $F \cap G$ is non-empty. Under some additional conditions we prove best possible bounds on $|\mathcal{F}| + |\mathcal{G}|$. These results were recently applied to obtain a stability result for a classical theorem of Katona [Ka].

1 Introduction

Let $[n] = \{1, \ldots, n\}$ be the standard n-element set. For an integer k, $0 \leqq k \leqq n$, $\binom{[n]}{k}$ is the collection of k-subsets of $[n]$. A family $\mathcal{F} \subset \binom{[n]}{k}$ is called *t-intersecting* if $|F \cap F'| \geqq t$ for all $F, F' \in \mathcal{F}$. Also, if $\mathcal{F} \subset \binom{[n]}{k}$ and $\mathcal{G} \subset \binom{[n]}{\ell}$ are two families satisfying $F \cap G \neq \emptyset$ for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$ then they are called *cross-intersecting.*

The study of possible maximum sizes of t-intersecting families and pairs of cross-intersecting families is one of the central problems of extremal set theory. It goes back to the papers of Erdős–Ko–Rado [EKR], Katona [Ka] and Hilton–Milner [HM].

Since this is a short note we are unable to give an overview of the known results. We restrict ourselves to mentioning some results that are closely related to our results and/or which are used in the proof. First of all let us state our results.

Theorem. *Let n, k, t be non-negative integers, $n \geqq k + 2t$. Suppose that $\mathcal{F} \subset \binom{[n]}{k+t}$, $\mathcal{G} \subset \binom{[n]}{k}$ are cross-intersecting. Then* (i) *and* (ii) *hold.*

(i) *If $\mathcal{F}$ is t-intersecting then*

$$|\mathcal{F}| + |\mathcal{G}| \leqq \binom{n}{k}. \tag{1}$$

(ii) *If $\mathcal{F}$ is $(t+1)$-intersecting and non-empty then*

$$|\mathcal{F}| + |\mathcal{G}| \leqq 1 + \binom{n}{k} - \binom{n-t-k}{k}. \tag{2}$$

2 Tools and proofs

Our main tool is the following classical result.

For a family $\mathcal{H}$ and an integer ℓ let $\Delta_\ell(\mathcal{H}) \subset \binom{[n]}{\ell}$ be the *ℓ-shadow* of $\mathcal{H}$:

$$\Delta_\ell(\mathcal{H}) = \left\{ D \in \binom{[n]}{\ell} : \exists\, H \in \mathcal{H} \text{ satisfying } D \subset H \right\}.$$

Katona Intersecting Shadow Theorem ([Ka]). *Suppose that $\mathcal{H} \subset \binom{[n]}{h}$ is r-intersecting. Then*

$$\left|\Delta_{h-r}(\mathcal{H})\right| \geqq |\mathcal{H}| \quad \textit{holds.} \tag{3}$$

First let us prove (i). Define $\mathcal{H} = \{[n] - F : F \in \mathcal{F}\} \subset \binom{[n]}{n-k-t}$ and note that $n \geqq 2k+t$ implies $n-k-t \geqq k$.

For $F, F' \in \mathcal{F}$ the t-intersecting property implies $|F \cup F'| = |F| + |F'| - |F \cap F'| \leqq 2(k+t) - t = 2k+t$. Equivalently, $\left|([n]-F) \cap ([n]-F')\right| = n - |F \cup F'| \geqq n - 2k - t$, i.e., $\mathcal{H}$ is $(n-2k-t)$-intersecting.

Let us apply (3) with $h = n-k-t$, $r = n-2k-t$ to obtain

$$\left|\Delta_k(\mathcal{H})\right| \geqq |\mathcal{H}|.$$

On the other hand $\Delta_k(\mathcal{H})$ and $\mathcal{G}$ are disjoint by the cross-intersecting property. Indeed $G \in \mathcal{G} \cap \Delta_k(\mathcal{H})$ would mean $G \subset ([n]-F)$ for some $F \in \mathcal{F}$, i.e., $G \cap F = \emptyset$.

Since both $\mathcal{G}$ and $\mathcal{H}$ are subsets of $\binom{[n]}{k}$, $|\mathcal{G}| + |\mathcal{H}| = |\mathcal{G}| + |\mathcal{F}| \leqq \binom{n}{k}$ follows. $\square$

To prove (2) is more difficult. Let us recall that a family $\mathcal{F} \subset 2^{[n]}$ is called *shifted* if for all $1 \leqq i < j \leqq n$, $F \cap \{i,j\} = \{j\}$ implies for $F \in \mathcal{F}$ that $(F - \{j\}) \cup \{i\}$ is also in $\mathcal{F}$.

The following statement which is implicitly contained in [EKR] and explicitly used in [Ka] and [HM] is very useful in obtaining inequalities concerning t-intersecting families and pairs of cross-intersecting families.

Lemma. *Let $\mathcal{F} \subset \binom{[n]}{\ell}$ and $\mathcal{G} \subset \binom{[n]}{k}$ be cross-intersecting families and let $\mathcal{F}$ be t-intersecting as well. Then there exist shifted families $\widetilde{\mathcal{F}} \subset \binom{[n]}{\ell}$, $\widetilde{\mathcal{G}} \subset \binom{[n]}{k}$ such that $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ are cross-intersecting, $\widetilde{\mathcal{F}}$ is t-intersecting and $|\widetilde{\mathcal{F}}| = |\mathcal{F}|$, $|\widetilde{\mathcal{G}}| = |\mathcal{G}|$ hold.*

In view of the Lemma, upon proving (ii) we may assume that both $\mathcal{F}$ and $\mathcal{G}$ are shifted.

Proof of (ii). We want to apply double induction on n and k. Therefore we first check the validity of (2) in the cases $k = 0$ and 1 and also for $n = 2k+t$. If $k = 0$ or 1 then $\mathcal{F} \neq \emptyset$ implies by shiftedness $\mathcal{F} = \big\{\{1, 2, \dots, k+t\}\big\}$. Since $\mathcal{F}$ and $\mathcal{G}$ are cross-intersecting $\mathcal{G} \cap \binom{[n]-\{1,2,\dots,k+t\}}{k} = \emptyset$ must hold. Thus

$$|\mathcal{G}| \leqq \binom{n}{k} - \binom{n-k-t}{k}, \quad \text{proving (2).}$$

In the case $n = 2k+t$ the formula on the RHS of (2) reduces simply to $\binom{n}{k}$ and it readily follows from (1). Or also from $\mathcal{G} \cap \big\{[n] - F : F \in \mathcal{F}\big\} = \emptyset$, using the cross-intersecting property.

From now on $k \geqq 2$, $n > 2k+t$. For $\mathcal{H} \subset 2^{[n]}$ define the following two families:

$$\mathcal{H}(n) = \big\{H - \{n\} : n \in H \in \mathcal{H}\big\},$$
$$\mathcal{H}(\overline{n}) = \{H : n \notin H \in \mathcal{H}\}.$$

Note that

$$|\mathcal{H}| = |\mathcal{H}(n)| + |\mathcal{H}(\overline{n})| \ \text{ holds.}$$

Claim 1. *$\mathcal{F}(n) \subset \binom{[n-1]}{k+t-1}$ is $t+1$-intersecting.*

Proof. If $F, F' \in \mathcal{F}$ then $|F \cap F'| \geqq t+1$ implies $|F \cup F'| = |F| + |F'| - |F \cap F'| \leqq 2(k+t) - (t+1) = 2k+t-1 \leqq n-2$. Therefore one can find $i \in [n-1]$ with $i \notin F \cup F'$. Suppose now $n \in F \cap F'$. We need to prove

$$\big|(F - \{n\}) \cap (F' - \{n\})\big| \geqq t+1. \tag{4}$$

Since $|F \cap F'| \geqq t+1$, the only way that (4) can fail is

$$\big|(F - \{n\}) \cap (F' - \{n\})\big| = t.$$

Suppose for contradiction that this is the case. By shiftedness

$$F'' \overset{\text{def}}{=} (F - \{n\}) \cup \{i\} \ \text{ is in } \ \mathcal{F}.$$

However,

$$F'' \cap F' = F \cap F' - \{n\}, \quad \text{i.e.,} \quad |F'' \cap F'| = t, \quad \text{a contradiction.} \qquad \square$$

Claim 2. *$\mathcal{F}(n)$ and $\mathcal{G}(n)$ are cross-intersecting.*

Proof. The proof is similar to Claim 1, therefore we shall be somewhat sketchy. The only way that Claim 2 can be false is if there are $F \in \mathcal{F}$, $G \in \mathcal{G}$ such that $F \cap G = \{n\}$ holds.

$|F| + |G| = 2k + t < n$ implies the existence of $i \in [n-1]$, $i \notin F \cup G$. By shiftedness, $F' \overset{\text{def}}{=} (F - \{n\}) \cup \{i\}$ is in $\mathcal{F}$. However, $F' \cap G = \emptyset$, a contradiction. $\square$

Now we are ready to do the induction step.

Since $\mathcal{F} \neq \emptyset$ is shifted, $\{1, 2, \ldots, k+t\} \in \mathcal{F}(\overline{n})$ holds. Let us apply the induction hypothesis to $\mathcal{F}(\overline{n}) \subset \mathcal{F}$ and $\mathcal{G}(\overline{n}) \subset \mathcal{G}$.

We infer

$$|\mathcal{F}(\overline{n})| + |\mathcal{G}(\overline{n})| \leqq 1 + \binom{n-1}{k} + \binom{n-1-k-t}{k}. \tag{5}$$

Note that $\{1, \ldots, k+t\} \in \mathcal{F}$ and the cross-intersecting property imply

$$|\mathcal{G}(n)| \leqq \binom{n-1}{k-1} - \binom{n-1-k-t}{k-1}. \tag{6}$$

If $\mathcal{F}(n) = \emptyset$, then adding (5) and (6) yields (2). Suppose $\mathcal{F}(n) \neq \emptyset$ then by the induction hypothesis we have

$$\begin{aligned} |\mathcal{F}(n)| + |\mathcal{G}(n)| &\leqq 1 + \binom{n-1}{k-1} - \binom{n-1-(k+t-1)}{k-1} \\ &= \binom{n-1}{k-1} - \binom{n-1-k-t}{k-1} + \left(1 - \binom{n-1-k-t}{k-2}\right). \end{aligned} \tag{7}$$

Since $k \geqq 2$ and $n > 2k+t$ the term in the bracket is non-positive, adding (5) and (7) concludes the proof. $\square$ $\square$

To see that the bound (2) is best possible one takes $\mathcal{F} = \{[k+t]\}$ and $\mathcal{G} = \left\{G \subset \binom{[n]}{k} : G \cap [k+t] \neq \emptyset\right\}$. In the case of (1) equality holds for $\mathcal{F} = \emptyset$ and $\mathcal{G} = \binom{[n]}{k}$ and if $n = 2k+t$ then also for $\mathcal{F} = \binom{[2k+t]}{k+t}$, $\mathcal{G} = \emptyset$. One can show that in all other cases the inequalities are strict.

3 Concluding remarks

Let us mention that Hilton and Milner [HM] proved (2) in the case $t = 0$, with the much weaker assumption that both $\mathcal{F}$ and $\mathcal{G}$ are non-empty. However, one can easily show that for $t \geqq 1$ it is not sufficient to assume the t-intersecting property instead of $\mathcal{F}$ being $(t+1)$-intersecting.

Let us close this paper by a conjecture. Let k, t, s be positive integers, $k \geqq s + 1$. Consider the two families

$$\mathcal{F} = \binom{[k+t+s]}{k+t}, \quad \mathcal{G} = \left\{ G \in \binom{[n]}{k} : \big| G \cap [k+t+s] \big| \geqq s+1 \right\}.$$

Note that $\mathcal{F}$ is $(t+1)$-intersecting, $\mathcal{F}$ and $\mathcal{G}$ are cross-intersecting and $|\mathcal{G}| = \binom{n}{k} - \sum\limits_{0 \leqq i \leqq s} \binom{k+t+s}{i} \binom{n-k-t-s}{k-i}$ hold.

Conjecture. *Let k, t, s be positive integers, $k \geqq s + 1$. Suppose that $\mathcal{F} \subset \binom{[n]}{k+t}$ is $(t+1)$-intersecting and $\binom{[k+t+s]}{t} \subset \mathcal{F}$. Let $\mathcal{G} \subset \binom{[n]}{k}$ and suppose that $\mathcal{F}$ and $\mathcal{G}$ are cross-intersecting. Then for $n \geqq 2k + t$,*

$$(8) \quad |\mathcal{F}| + |\mathcal{G}| \leqq \binom{k+t+s}{k+t} + \binom{n}{k} - \sum_{0 \leqq i \leqq s} \binom{k+t+s}{i} \binom{n-k-t-s}{k-i}$$

holds.

Note that setting $s = 0$ we get back (2). We can prove (8) for some small values of k and also for $n > ck^2$, however to prove it in the full range appears to be difficult. Let us mention that it is shown in [FT] that in the case $s = t = 1$ assuming $|\mathcal{F}| \geqq k + 2$ instead of $\binom{[k+2]}{k+1} \subset \mathcal{F}$ is not sufficient to guarantee (8).

References

[EKR] P. Erdős, C. Ko, R. Rado, Intersection theorems for systems of finite sets, *Quart. J. Math. Oxford* **12** (1961), 313–320.

[FT] P. Frankl, N. Tokushige, Some inequalities concerning cross-intersecting families, *Combinatorics, Prob. and Computing* **7** (1998), 247–260.

[HM] A. J. W. Hilton and E. C. Milner, Some intersection theorems for systems of finite sets, *Quart. J. Math. Oxford* **18** (1967), 369–384.

[Ka] G. O. H. Katona, Intersection theorems for systems of finite sets, *Acta Math. Hungar.* **15** (1964), 329–337.