An Erdős–Ko–Rado theorem for cross t-intersecting families

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A R T I C L E   I N F O

A B S T R A C T

Two families $\mathcal{A}$ and $\mathcal{B}$, of k-subsets of an n-set, are *cross t-intersecting* if for every choice of subsets $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have $|A \cap B| \geq t$. We address the following conjectured cross t-intersecting version of the Erdős–Ko–Rado theorem: For all $n \geq (t+1)(k-t+1)$ the maximum value of $|\mathcal{A}||\mathcal{B}|$ for two cross t-intersecting families $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$ is $\binom{n-t}{k-t}^2$. We verify this for all $t \geq 14$ except finitely many n and k for each fixed t. Further, we prove uniqueness and stability results in these cases, showing, for instance, that the families reaching this bound are unique up to isomorphism. We also consider a *p-weight* version of the problem, which comes from the product measure on the power set of an n-set.

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1. Introduction

Let $[n] = \{1, 2, \ldots, n\}$ and let $2^{[n]}$ denote the power set of $[n]$. A family $\mathcal{F} \subset 2^{[n]}$ is called *t-intersecting* if $|F \cap F'| \geq t$ for all $F, F' \in \mathcal{F}$. Let $\binom{[n]}{k}$ denote the set of all k-subsets of $[n]$. A family in $\binom{[n]}{k}$ is called *k-uniform*. For example,

$$\mathcal{F}_0 = \left\{ F \in \binom{[n]}{k} \colon [t] \subset F \right\}$$

is a k-uniform t-intersecting family of size $|\mathcal{F}_0| = \binom{n-t}{k-t}$. Erdős, Ko and Rado [6] proved that there exists some $n_0(k, t)$ such that if $n \geq n_0(k, t)$ and $\mathcal{F} \subset \binom{[n]}{k}$ is t-intersecting, then $|\mathcal{F}| \leq \binom{n-t}{k-t}$. The smallest possible such $n_0(k, t)$ is $(t+1)(k-t+1)$. This was proved by Frankl [7] for $t \geq 15$, and then completed by Wilson [28] for all t. These proofs are very different, the former uses combinatorial tools while the later is based on the eigenvalue method. If $n < (t+1)(k-t+1)$ then $\binom{n-t}{k-t}$ is no longer the maximum size. In fact we can construct a t-intersecting family

$$\mathcal{F}_i^t(n, k) := \left\{ F \in \binom{[n]}{k} \colon |F \cap [t + 2i]| \geq t + i \right\}$$

for $0 \leq i \leq k - t$, and it can be shown that $|\mathcal{F}_0^t(n, k)| \geq |\mathcal{F}_1^t(n, k)|$ iff $n \geq (t+1)(k-t+1)$. Frankl conjectured in [7] that if $\mathcal{F} \subset \binom{[n]}{k}$ is t-intersecting, then

$$|\mathcal{F}| \leq \max_i |\mathcal{F}_i^t(n, k)|. \tag{1}$$

This conjecture was proved partially by Frankl and Füredi [9], and then settled completely by Ahlswede and Khachatrian [1]. This result is one of the highlights of extremal set theory.

We will use the proof technique used in [9], in another direction, to deal with cross t-intersecting families. Two families $\mathcal{A}, \mathcal{B} \subset 2^{[n]}$ are called *cross t-intersecting* if $|A \cap B| \geq t$ holds for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Pyber [20] considered the case $t = 1$, and proved that if $n \geq 2k$ and $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$ are cross 1-intersecting, then $|\mathcal{A}||\mathcal{B}| \leq \binom{n-1}{k-1}^2$. It was then proved in [18] that if $n \geq \max\{2a, 2b\}$, and $\mathcal{A} \subset \binom{[n]}{a}$ and $\mathcal{B} \subset \binom{[n]}{b}$ are cross 1-intersecting, then $|\mathcal{A}||\mathcal{B}| \leq \binom{n-1}{a-1}\binom{n-1}{b-1}$. See Borg [4] for a corresponding cross t-intersecting result for $n > n_0(k, t)$. Gromov [14] found an application of these inequalities to geometry. For the general cross t-intersecting case, it is natural to expect that if $n \geq (t+1)(k-t+1)$ and $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$ are cross t-intersecting, then $|\mathcal{A}||\mathcal{B}| \leq \binom{n-t}{k-t}^2$. This conjecture was verified for $n > 2tk$ in [24] using a combinatorial approach, and for $\frac{k}{n} < 1 - \frac{1}{\sqrt[t]{2}}$, or more simply, $n > 1.443(t+1)k$ in [25] using the eigenvalue method. We also mention that Suda and Tanaka [21] obtained a similar result concerning cross 1-intersecting families of vector subspaces based on semidefinite programming.

In this paper we prove the following result which almost reaches the conjectured lower bound for n. We say that two families $\mathcal{A}$ and $\mathcal{B}$ in $2^{[n]}$ are *isomorphic* if there is a

permutation f on $[n]$ such that $\mathcal{A} = \{\{f(b)\colon b \in B\}\colon B \in \mathcal{B}\}$, and in this case we write $\mathcal{A} \cong \mathcal{B}$.

Theorem 1.1. *For every $k \geq t \geq 14$ and $n \geq (t+1)k$ we have the following. If $\mathcal{A} \subset \binom{[n]}{k}$ and $\mathcal{B} \subset \binom{[n]}{k}$ are cross t-intersecting, then*

$$|\mathcal{A}||\mathcal{B}| \leq \binom{n-t}{k-t}^2$$

with equality holding iff $\mathcal{A} = \mathcal{B} \cong \mathcal{F}_0^t(n,k)$.

The extremal configuration has a *stability*; if $|\mathcal{A}||\mathcal{B}|$ is very close to $\binom{n-t}{k-t}^2$, then both families are very close to $\mathcal{F}_0^t(n,k)$. By saying $\mathcal{A}$ is close to $\mathcal{F}$ we mean that the symmetric difference $\mathcal{A} \triangle \mathcal{F} = (\mathcal{A} \setminus \mathcal{F}) \cup (\mathcal{F} \setminus \mathcal{A})$ is of small size. A family $\mathcal{A} \subset 2^{[n]}$ is called *shifted* if $(A \setminus \{j\}) \cup \{i\} \in \mathcal{A}$ whenever $1 \leq i < j \leq n$, $A \in \mathcal{A}$, and $A \cap \{i,j\} = \{j\}$. (We will explain more about shifting operations in the next section.)

Theorem 1.2. *For every $k \geq t \geq 14$, $\delta > 0$, $n \geq (t+1+\delta)k$, and $\eta \in (0,1]$, we have the following. If $\mathcal{A}$ and $\mathcal{B}$ are shifted cross t-intersecting families in $\binom{[n]}{k}$, then one of the following holds.*

(i) $\sqrt{|\mathcal{A}||\mathcal{B}|} < (1-\gamma\eta)\binom{n-t}{k-t}$, *where $\gamma \in (0,1]$ depends only on t and δ.*
(ii) $|\mathcal{A} \triangle \mathcal{F}_0^t(n,k)| + |\mathcal{B} \triangle \mathcal{F}_0^t(n,k)| < \eta\binom{n-t}{k-t}$.

We also consider the so-called *p-weight version* or *measure version* (see e.g. [11–13]) of the above result concerning k-uniform families. Let $p \in (0,1)$ be a fixed real, and let μ_p be the product measure on $2^{[n]}$ defined by

$$\mu_p(F) := p^{|F|}(1-p)^{n-|F|}.$$

For a family $\mathcal{F} \subset 2^{[n]}$ let us define its p-weight (or measure) by

$$\mu_p(\mathcal{F}) := \sum_{F \in \mathcal{F}} \mu_p(F).$$

Ahlswede and Khachatrian [2] proved that if $\mathcal{F} \subset 2^{[n]}$ is t-intersecting, then

$$\mu_p(\mathcal{F}) \leq \max_i \mu_p\big(\mathcal{F}_i^t(n)\big) \tag{2}$$

where

$$\mathcal{F}_i^t(n) := \big\{F \subset [n]\colon \big|F \cap [t+2i]\big| \geq t+i\big\}.$$

It is not difficult to derive (2) from (1), see [22,5]. In particular, if $p \leq \frac{1}{t+1}$ then $\max_i \mu_p(\mathcal{F}_i^t(n)) = \mu_p(\mathcal{F}_0^t(n)) = p^t$. In [13], Friedgut gave a proof of (2), in the case $p \leq \frac{1}{t+1}$, using the eigenvalue method, which is the p-weight version of Wilson's proof [28]. Friedgut's proof can easily be extended to cross t-intersecting families if $p < \frac{0.69}{t+1}$ as in [26]. More precisely, if $t \geq 1$, $p \leq 1 - \frac{1}{\sqrt[t]{2}}$, and two families $\mathcal{A}, \mathcal{B} \subset 2^{[n]}$ are cross t-intersecting, then we have $\mu_p(\mathcal{A})\mu_p(\mathcal{B}) \leq p^{2t}$. In the present paper, we prove the same inequality for all $t \geq 14$ and $p \leq \frac{1}{t+1}$.

Theorem 1.3. *For every $t \geq 14$, $n \geq t$, and p with $0 < p \leq \frac{1}{t+1}$, we have the following. If $\mathcal{A} \subset 2^{[n]}$ and $\mathcal{B} \subset 2^{[n]}$ are cross t-intersecting, then*

$$\mu_p(\mathcal{A})\mu_p(\mathcal{B}) \leq p^{2t}.$$

Equality holds iff either $\mathcal{A} = \mathcal{B} \cong \mathcal{F}_0^t(n)$, or $p = \frac{1}{t+1}$ and $\mathcal{A} = \mathcal{B} \cong \mathcal{F}_1^t(n)$.

We have the following stability result as well. A family $\mathcal{G} \subset 2^{[n]}$ is called *inclusion maximal* if $G \in \mathcal{G}$ and $G \subset G'$ imply $G' \in \mathcal{G}$.

Theorem 1.4. *For every $t \geq 14$, $n \geq t$, $\epsilon > 0$, $\eta \in (0,1]$, and p with $0 < p \leq \frac{1}{t+1} - \epsilon$, we have the following. If $\mathcal{A}$ and $\mathcal{B}$ are shifted, inclusion maximal cross t-intersecting families in $2^{[n]}$, then one of the following holds.*

(i) $\sqrt{\mu_p(\mathcal{A})\mu_p(\mathcal{B})} < (1 - \gamma\eta)p^t$, *where $\gamma \in (0,1]$ depends only on t and ϵ.*
(ii) $\mu_p(\mathcal{A} \triangle \mathcal{F}_0^t(n)) + \mu_p(\mathcal{B} \triangle \mathcal{F}_0^t(n)) < \eta p^t$.

In [13], Friedgut obtained similar stability results for (not necessarily shifted) t-intersecting families. He used a result due to Kindler and Safra [16], which states that Boolean functions whose Fourier transforms are concentrated on small sets, essentially depend only on a few variables.

We cannot replace condition (ii) of Theorem 1.2 with the condition $\mathcal{A}, \mathcal{B} \subset \mathcal{F}_0^t(n,k)$ which is sometimes sought in such stability results. Indeed, we can construct a shifted t-intersecting family $\mathcal{A} \subset \binom{[n]}{k}$ such that $|\mathcal{A}| = \binom{n-t}{k-t}(1 - o(1))$ where $o(1) \to 0$ as $n, k \to \infty$ with $n > (t+1)k$, but $\mathcal{A} \not\subset \mathcal{F}$ for any $\mathcal{F} \cong \mathcal{F}_0^t(n,k)$. For this, let $\mathcal{T} = \{F \in \mathcal{F}_0^t(n,k): F \cap [t+1,k+1] = \emptyset\}$, $\mathcal{H} = \{[k+1] \setminus \{i\}: 1 \leq i \leq t\}$, and let $\mathcal{A} = (\mathcal{F}_0^t(n,k) \setminus \mathcal{T}) \cup \mathcal{H}$. Then it is easy to see that $\mathcal{A}$ fulfills the prescribed properties.

Similarly, we cannot replace condition (ii) of Theorem 1.4 with the condition $\mathcal{A}, \mathcal{B} \subset \mathcal{F}_0^t(n)$. In fact there is a t-intersecting family $\mathcal{G} \subset 2^{[n]}$ such that $\mu_p(\mathcal{G})$ is arbitrarily close to p^t, but $\mathcal{G}$ is not a subfamily of any isomorphic copy of $\mathcal{F}_0^t(n)$. For example, let $T = [t]$, $\mathcal{H} = \{[n] \setminus \{i\}: 1 \leq i \leq t\}$, and let $\mathcal{G} = (\mathcal{F}_0^t(n) \setminus \{T\}) \cup \mathcal{H}$. Then $\mathcal{G}$ is a shifted, inclusion maximal t-intersecting family with $\mathcal{G} \not\subset \mathcal{F}_0^t(n)$ and $\mathcal{G} \triangle \mathcal{F}_0^t(n) = \mathcal{H} \cup \{T\}$. Moreover we have $\mu_p(\mathcal{G}) = p^t - p^t q^{n-t} + t p^{n-1} q = (1 - o(1))p^t$ where $o(1) \to 0$ as $n \to \infty$ for fixed t and p.

We conjecture that Theorem 1.1 holds for all n, k, and t such that $k \geq t \geq 1$ and $n > (t+1)(k-t+1)$, and Theorem 1.3 holds for all $t \geq 1$. We also conjecture that Theorems 1.2 and 1.4 are valid for families that are not necessarily shifted as well. We mention that one can show Theorem 1.1 for $t \geq 14$, $k > k_0(t)$, and $n > (t+1)(k-t+1)$ as well, see Theorem 4.8 in Section 4.

The approach in this paper follows that used in [9,24]. We relate subsets in the cross t-intersecting families with walks in the plane. After a normalizing process called shifting, these families will have the property that the corresponding walks all hit certain lines. In the p-weight version, the measure of such families is bounded by the probability that a certain random walk hits the same lines. Results for k-uniform cross t-intersecting families can often be inferred by corresponding p-weight results applied to the families obtained by taking all supersets of the original k-uniform families, which will also be cross t-intersecting. Indeed, using Theorem 1.4 it is relatively easy to prove results similar to Theorems 1.1 and 1.2 but with somewhat weaker bounds for n and k. However, to get our k-uniform results in full strength, we need to prove them directly instead of relying on our p-weight results. Nevertheless understanding the proof of p-weight results is very helpful for the proof of k-uniform results. They have a similar proof with corresponding steps, though the actual computations appearing in the proof of the p-weight version are usually much easier than those of the k-uniform version.

The paper is organized as follows. In Section 2 we present tools that we will use throughout the paper. In Section 3 we prove Proposition 3.1, our main result about the p-weight version of the problem, from which Theorems 1.3 and 1.4 easily follow. In Section 4 we prove Proposition 4.1, our main result about the k-uniform version of the problem, from which Theorems 1.1 and 1.2 follow. In Section 5 we present an application to families of t-intersecting integer sequences.

2. Tools

In this section we present some standard tools. The proofs are also standard (see, e.g. [9,12,23]), but we include them for completeness.

Throughout this paper let $p \in (0,1)$ be a real number, let $q = 1 - p$, and let $\alpha = p/q$. The *walk* associated to a set $F \subset [n]$ is an n-step walk on the integer grid $\mathbb{Z}^2$ starting at the origin $(0,0)$ whose i-th step is *up* (going from (x,y) to $(x, y+1)$) if $i \in F$, and is *right* (going from (x,y) to $(x+1,y)$) if $i \notin F$. We thus refer to $F \in 2^{[n]}$ as either a set or a walk, depending on which point of view is more convenient. Correspondingly, consider an n-step random walk $W_{n,p}$ whose i-th step is a random variable, independent of other steps, going 'up' with probability p and 'right' otherwise. Since μ_p is a probability measure on $2^{[n]}$, the p-weight of a family $\mu_p(\mathcal{F})$, where $\mathcal{F} \subset 2^{[n]}$ consists of all walks that satisfy a given property P, is exactly the probability that $W_{n,p}$ satisfies P.

Example 2.1. The p-weight of the family of all walks in $2^{[n]}$ that hit the point $(0, t)$ is the probability that $W_{n,p}$ hits $(0, t)$, which is p^t. The p-weight of the family of all walks

in $2^{[n]}$ that hit $(1, t)$ but not $(0, t)$ is $tp^t q$. Indeed for a walk to hit $(1, t)$ but not $(0, t)$, it must move up $t - 1$ of its first t steps, this can be done in t ways, and then must move up on the $(t + 1)$-th step. So the probability is $\binom{t}{1} p^{t-1} q \cdot p$, as needed.

Lemma 2.2. *Let $\mathcal{F} \subset 2^{[n]}$, and let t be a positive integer.*

(i) *If all walks in $\mathcal{F}$ hit the line $y = x + t$, then $\mu_p(\mathcal{F}) \leq \alpha^t$.*
(ii) *For every ϵ there is an n_0 such that if $n > n_0$ and no walk in $\mathcal{F}$ hits the line $y = x + t$, then $\mu_p(\mathcal{F}) < 1 - \alpha^t + \epsilon$.*
(iii) *If all walks in $\mathcal{F}$ hit the line $y = x + t$ at least twice, but do not hit the line $y = x + (t + 1)$, then $\mu_p(\mathcal{F}) \leq \alpha^{t+1}$.*

Proof. We notice that, for fixed p, the probability $P_n := \mathrm{Prob}(W_{n,p}$ hits $y = x + t)$ is monotone increasing and bounded, and hence $\lim_{n \to \infty} P_n$ exists. In fact this limit is known to be exactly $\alpha^t = (p/q)^t$, see e.g., [23]. This gives (i) and (ii).

There is an injection from (I) the family of walks that hit the line $y = x + t$ at least twice but do not hit $y = x + (t + 1)$ to (II) the family of walks that hit $y = x + (t + 1)$. Indeed for a walk F in (I) that hits $y = x + t$ for the first time at $(x_1, x_1 + t)$ and for the second time at $(x_2, x_2 + t)$, we get a walk in (II) by reflecting the portion of F between $(x_1, x_1 + t)$ and $(x_2, x_2 + t)$ across the line $y = x + t$. Further, these walks have the same p-weight. Thus we have (iii). □

For $1 \leq i < j \leq n$ we define the *shifting operation* $s_{ij} : 2^{[n]} \to 2^{[n]}$ by

$$s_{ij}(\mathcal{F}) := \big\{ s_{ij}(F) \colon F \in \mathcal{F} \big\}$$

where $\mathcal{F} \subset 2^{[n]}$ and

$$s_{ij}(F) := \begin{cases} (F \setminus \{j\}) \cup \{i\} & \text{if } F \cap \{i, j\} = \{j\} \text{ and } (F \setminus \{j\}) \cup \{i\} \notin \mathcal{F}, \\ F & \text{otherwise.} \end{cases}$$

A family $\mathcal{F}$ is called *shifted* if $s_{ij}(\mathcal{F}) = \mathcal{F}$ for all $1 \leq i < j \leq n$. Here we list some basic properties concerning shifting operations.

Lemma 2.3. *Let $1 \leq i < j \leq n$ and let $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$.*

(i) *Shifting operations preserve the p-weight of a family, that is, $\mu_p(s_{ij}(\mathcal{G})) = \mu_p(\mathcal{G})$.*
(ii) *If $\mathcal{G}_1$ and $\mathcal{G}_2$ in $2^{[n]}$ are cross t-intersecting families, then $s_{ij}(\mathcal{F})$ and $s_{ij}(\mathcal{G})$ are cross t-intersecting families as well.*
(iii) *For a pair of families we can always obtain a pair of shifted families by repeatedly shifting families simultaneously finitely many times.*
(iv) *If $\mathcal{G}$ is inclusion maximal, and $s_{ij}(\mathcal{G}) = \mathcal{F}_\ell^t(n)$, then $\mathcal{G} \cong \mathcal{F}_\ell^t(n)$ for $\ell = 0, 1$.*

Proof. Since $|s_{ij}(G)| = |G|$ for $G \subset [n]$ we have $\mu_p(s_{ij}(G)) = p^{|s_{ij}(G)|}q^{n-|s_{ij}(G)|} = p^{|G|}q^{n-|G|} = \mu_p(G)$. Thus $\mu_p(s_{ij}(\mathcal{G})) = \sum_{G \in \mathcal{G}} \mu_p(s_{ij}(G)) = \sum_{G \in \mathcal{G}} \mu_p(G) = \mu_p(\mathcal{G})$. This gives (i).

Let $\mathcal{F}' = s_{ij}(\mathcal{F})$ and $\mathcal{G}' = s_{ij}(\mathcal{G})$. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are cross t-intersecting, but $\mathcal{F}'$ and $\mathcal{G}'$ are not. Then there are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $|F \cap G| \geq t$ but $|F' \cap G'| < t$, where $F' = s_{ij}(F)$ and $G' = s_{ij}(G)$. Consider the case when $F \cap \{i, j\} = \{j\}$ and $G \cap \{i, j\} = \{j\}$. (The other cases are ruled out easily.) By symmetry we may assume that $F' \cap \{i, j\} = \{j\}$ and $G' \cap \{i, j\} = \{i\}$. This means that $F' = F$ and this happens because $F_1 := (F \setminus \{j\}) \cup \{i\}$ is already in $\mathcal{F}$. Then $|F_1 \cap G| = |F' \cap G'| < t$, which contradicts the cross t-intersecting property of $\mathcal{F}$ and $\mathcal{G}$. This shows (ii).

Next we show (iii). Let $\mathcal{G}_1, \mathcal{G}_2 \subset 2^{[n]}$. Suppose that at least one of these families, say, $\mathcal{G}_1$ is not shifted. Then there is a shifting s_{ij} such that $s_{ij}(\mathcal{G}_1) \neq \mathcal{G}_1$. Let $f(\mathcal{G})$ be the total sum of elements in the subsets of $\mathcal{G}$, that is, $f(\mathcal{G}) := \sum_{G \in \mathcal{G}} \sum_{x \in G} x$. Then $f(s_{ij}(\mathcal{G}_1)) \leq f(\mathcal{G}_1) - j + i \leq f(\mathcal{G}_1) - 1$. Namely, we can decrease the value $f(\mathcal{G}_1) + f(\mathcal{G}_2)$ at least 1 by applying a shifting operation unless both of the families are already shifted. On the other hand $f(\mathcal{G}_1) + f(\mathcal{G}_2) \geq 0$ for all $\mathcal{G}_1, \mathcal{G}_2$. Thus we get (iii).

Finally we prove (iv). Let $\mathcal{G}' = s_{ij}(\mathcal{G}) = \mathcal{F}_\ell^t(n)$. Observe that $\binom{[t+2\ell]}{t+\ell}$ is a 'generating set' of $\mathcal{G}'$, namely,

$$\mathcal{G}' = \left\{ G \subset [n]: \ F \subset G \text{ for some } F \in \binom{[t+2\ell]}{t+\ell} \right\}.$$

First let $\ell = 0$. Then since $\mathcal{G}' = \mathcal{F}_0^t(n)$ we have $[t] \in \mathcal{G}'$. Thus $\mathcal{G}$ must contain some t-element set G_0, and so as $\mathcal{G}$ is inclusion maximal, $\mathcal{G}$ contains $\{G \subset [n]: G_0 \subset G\} \cong \mathcal{F}_0^t(n)$. On the other hand, by (i) and our assumption, we have $\mu_p(\mathcal{G}) = \mu_p(s_{ij}(\mathcal{G})) = \mu_p(\mathcal{F}_0^t(n))$. Thus we indeed have $\mathcal{G} \cong \mathcal{F}_0^t(n)$.

Next let $\ell = 1$. If $|\{i, j\} \cap [t + 2]| = 0$ or 2, then it is easy to see that $s_{ij}(\mathcal{G}) = \mathcal{G}$ and we are done, so we may assume that $i = t + 2$ and $j = t + 3$. Since $\mathcal{G}' = \mathcal{F}_1^t(n)$ we have $\binom{[t+2]}{t+1} \subset \mathcal{G}'$. If $\binom{[t+2]}{t+1} \subset \mathcal{G}$ then $\mathcal{G} = \mathcal{F}_1^t(n)$, too. If $\binom{[t+2]}{t+1} \not\subset \mathcal{G}$ then there is some $G' \in \binom{[t+2]}{t+1}$ such that $G' \notin \mathcal{G}$. In this case we have $G' = A \cup \{t + 2\} \in \mathcal{G}' \setminus \mathcal{G}$ for some $A \in \binom{[t+1]}{t}$, and $G = A \cup \{t + 3\} \in \mathcal{G} \setminus \mathcal{G}'$. This means $s_{ij}(G) = G'$. For $x \in \{t + 2, t + 3\}$ let $\mathcal{G}(x) := \{A \in \binom{[t+1]}{t}: A \cup \{x\} \in \mathcal{G}\}$. Then $\mathcal{G}(t + 2) \cup \mathcal{G}(t + 3)$ is a partition of $\binom{[t+1]}{t}$. It follows from $G \in \mathcal{G}(t + 3)$ that $\mathcal{G}(t + 3) \neq \emptyset$. If there is some $G'' \in \mathcal{G}(t + 2)$ then $|G \cap G''| \geq t$ implies that $G'' = G'$, which is a contradiction because $G' \notin \mathcal{G}$. Thus $\mathcal{G}(t + 2) = \emptyset$ must hold. Consequently we have $\mathcal{G} = \{G \subset [n]: |G \cap T| \geq t + 1\}$ where $T = [t + 3] \setminus \{t + 2\}$, and $\mathcal{G} \cong \mathcal{F}_1^t(n)$. $\quad\square$

The following two simple facts are used only to prove Lemma 2.6 below. For $n > 2k$ we define a Kneser graph $K(n, k) = (V, E)$ on the vertex set $V = \binom{[n]}{k}$ by $(F, F') \in E$ iff $F \cap F' = \emptyset$.

Fact 2.4. *If $n > 2k$, then the Kneser graph $K(n, k)$ is connected and non-bipartite.*

Proof. We use Katona's cyclic permutation method [15]. Observe that $K(2k + 1, k)$ contains C_{2k+1} (a cycle of length $2k + 1$). To see this, let $F_i = \{i, i + 1, \ldots, i + k - 1\}$ (indices are read modulo $2k + 1$), then $F_0, F_k, F_{2k}, \ldots, F_{(k-1)k}$ give the cycle. Moreover any two vertices $F, F' \subset [2k + 1]$ are on some C_{2k+1}, because one can choose a cyclic ordering $i_1, i_2, \ldots, i_{2k+1}$ such that both F and F' consist of consecutive elements in this ordering. Since $K(n, k)$ $(n > 2k)$ contains $K(2k+1, k)$ as an induced subgraph, it follows that $K(n, k)$ is connected and non-bipartite. □

For two graphs G and H we define the direct product $G \otimes H = (V, E)$ on $V = V(G) \times V(H)$ by $((u, v), (u', v')) \in E$ iff $uu' \in E(G)$ and $vv' \in E(H)$.

Fact 2.5. *Let G and H be connected and non-bipartite graphs.*

(i) *$G \otimes H$ is connected and non-bipartite.*
(ii) *$G \otimes K_2$ is connected, where K_2 is the complete graph of order 2.*

Proof. By a closed trail of length n in G we mean a sequence of vertices $x_0 x_1 \ldots x_{n-1} x_0$ such that $x_i x_{i+1} \in E(G)$ for all i (indices are read modulo n). Since G is connected and non-bipartite, one can find a closed trail of odd length containing any given two vertices. Now let $(x, y), (x', y') \in G \otimes H$ be given. Choose a closed trail $x_0 x_1 \ldots x_{n-1} x_0$ containing x, x' in G, and a closed trail $y_0 y_1 \ldots y_{m-1} y_0$ containing y, y' in H, where both n and m are odd. Then (x_i, y_i), $i = 0, 1, \ldots, mn - 1$, give a closed trail of length mn in $G \otimes H$, where indices of x_i are read modulo n, while indices of y_i are read modulo m. This closed odd trail contains both (x, y) and (x', y'), so there is a path from (x, y) to (x', y') and there is an odd cycle in this closed trail. Thus we get (i).

One can prove (ii) directly, but this is a special case of Weichsel's result [27] which states that if G and H are connected, then $G \otimes H$ is connected iff G or H contains an odd cycle. □

Lemma 2.6. *Let $k - t \geq \ell \geq 0$ and $\mathcal{F} := \mathcal{F}_\ell^t(n, k)$.*

(i) *If $\mathcal{F}$ and $\mathcal{B} \subset \binom{[n]}{k}$ are cross t-intersecting, and $|\mathcal{F}| = |\mathcal{B}|$, then $\mathcal{F} = \mathcal{B}$.*
(ii) *Let $n \geq 2k - t + 2$ and $t \geq 2$. If $\mathcal{A}$ and $\mathcal{B}$ are cross t-intersecting families in $\binom{[n]}{k}$, and $s_{ij}(\mathcal{A}) = s_{ij}(\mathcal{B}) = \mathcal{F}$, then $\mathcal{A} = \mathcal{B} \cong \mathcal{F}$.*

Proof. To prove (i) we notice that $\mathcal{F}$ is a maximal t-intersecting family in the sense that adding any k-subset (not contained in $\mathcal{F}$) to $\mathcal{F}$ would destroy the t-intersecting property. Since $\mathcal{F}$ and $\mathcal{B}$ are cross t-intersecting, for any $B \in \mathcal{B}$, $\mathcal{F} \cup \{B\}$ is still t-intersecting. This with the maximality of $\mathcal{F}$ forces $B \in \mathcal{F}$, namely, $\mathcal{B} \subset \mathcal{F}$. Then $|\mathcal{F}| = |\mathcal{B}|$ gives $\mathcal{F} = \mathcal{B}$.

Next we prove (ii) following [1]. For $1 \leq i < j \leq n$ and a family $\mathcal{G} \subset 2^{[n]}$, let

$$\mathcal{G}[\bar{i}j] := \big\{ G \in \mathcal{G} \colon i \notin G, \; j \in G, \; (G \cup \{i\}) \setminus \{j\} \notin \mathcal{G} \big\},$$

$$\mathcal{G}[i\bar{j}] := \big\{ G \in \mathcal{G} \colon j \notin G, \; i \in G, \; (G \cup \{j\}) \setminus \{i\} \notin \mathcal{G} \big\},$$

and let $\tilde{\mathcal{G}}$ be the family obtained from $\mathcal{G}$ by exchanging the coordinates i and j. We list some basic properties about these families.

- By definition, $\mathcal{G} \cong \tilde{\mathcal{G}}$. Also $\mathcal{G}[i\bar{j}] \cong \tilde{\mathcal{G}}[\bar{i}j]$, $\mathcal{G}[\bar{i}j] \cong \tilde{\mathcal{G}}[i\bar{j}]$.
- It follows that $\mathcal{G} \setminus s_{ij}(\mathcal{G}) = \mathcal{G}[\bar{i}j]$ and $s_{ij}(\mathcal{G}) \setminus \mathcal{G} = \tilde{\mathcal{G}}[i\bar{j}]$.
- If $\mathcal{G}[\bar{i}j] = \emptyset$, then $s_{ij}(\mathcal{G}) = \mathcal{G}$.
- It follows that $\mathcal{G} \setminus \tilde{\mathcal{G}} = \mathcal{G}[\bar{i}j] \cup \mathcal{G}[i\bar{j}]$ and $\tilde{\mathcal{G}} \setminus \mathcal{G} = \tilde{\mathcal{G}}[\bar{i}j] \cup \tilde{\mathcal{G}}[i\bar{j}]$.
- If $\mathcal{G}[i\bar{j}] = \emptyset$, then $\mathcal{G} \cap \tilde{\mathcal{G}} = \mathcal{G} \cap s_{ij}(\mathcal{G}) = \mathcal{G} \setminus \mathcal{G}[\bar{i}j] = \tilde{\mathcal{G}} \setminus \tilde{\mathcal{G}}[i\bar{j}]$.
- If $\mathcal{G}[i\bar{j}] = \emptyset$, then $s_{ij}(\mathcal{G}) = \tilde{\mathcal{G}}$. In fact, if $\mathcal{G}[i\bar{j}] = \emptyset$, then $\tilde{\mathcal{G}}[\bar{i}j] = \emptyset$, and

$$\tilde{\mathcal{G}} = (\mathcal{G} \cap \tilde{\mathcal{G}}) \cup \tilde{\mathcal{G}}[i\bar{j}] = \big(\mathcal{G} \cap s_{ij}(\mathcal{G}) \big) \cup \tilde{\mathcal{G}}[i\bar{j}] = s_{ij}(\mathcal{G}).$$

Now we assume that $\mathcal{A}$ and $\mathcal{B}$ are cross t-intersecting, and $s_{ij}(\mathcal{A}) = s_{ij}(\mathcal{B}) = \mathcal{F}$. Then clearly $|\mathcal{A}| = |\mathcal{B}|$. We will show that $\mathcal{A} = \mathcal{B} = \mathcal{F}$ or $\mathcal{A} = \mathcal{B} = \tilde{\mathcal{F}}$.

If $\mathcal{A}[\bar{i}j] = \emptyset$, then $s_{ij}(\mathcal{A}) = \mathcal{A}$. Thus $\mathcal{A} = \mathcal{F}$, and (i) gives that $\mathcal{A} = \mathcal{B} = \mathcal{F}$, as desired. Similarly, if $\mathcal{A}[i\bar{j}] = \emptyset$, then $s_{ij}(\mathcal{A}) = \tilde{\mathcal{A}}$. Thus $\tilde{\mathcal{A}} = \mathcal{F}$, and (i) gives that $\tilde{\mathcal{A}} = \tilde{\mathcal{B}} = \mathcal{F}$, or equivalently, $\mathcal{A} = \mathcal{B} = \tilde{\mathcal{F}}$.

Thus we may assume that $\mathcal{A}[\bar{i}j] \neq \emptyset$ and $\mathcal{A}[i\bar{j}] \neq \emptyset$. By the same reasoning, we may assume that $\mathcal{B}[\bar{i}j] \neq \emptyset$ and $\mathcal{B}[i\bar{j}] \neq \emptyset$. We will show that this is impossible. Without loss of generality we may also assume that $i = t + 2\ell$ and $j = i + 1$.

Note that $F \in \mathcal{F}[i\bar{j}]$ iff $|F \cap [t + 2\ell - 1]| = t + \ell - 1$. Keeping this in mind, let

$$\mathcal{H} := \left\{ H \in \binom{[n] \setminus \{i, j\}}{k-1} \colon \big| H \cap [t + 2\ell - 1] \big| = t + \ell - 1 \right\}.$$

For every $H \in \mathcal{H}$ we have $H \cup \{i\} \in \mathcal{F}$ and $H \cup \{j\} \notin \mathcal{F}$. Since $\mathcal{F} = s_{ij}(\mathcal{A})$ it follows that

$$\text{either} \quad H \cup \{i\} \in \mathcal{A} \quad \text{or} \quad H \cup \{j\} \in \mathcal{A} \quad \text{(but not both).} \tag{3}$$

(In fact, if both hold, then $s_{ij}(H \cup \{j\}) = H \cup \{j\} \in s_{ij}(\mathcal{A}) = \mathcal{F}$, a contradiction.) If $A \in \mathcal{A}_{ij} := \mathcal{A}[i\bar{j}] \cup \mathcal{A}[\bar{i}j]$, then $|A \cap [t + 2\ell - 1]| = t + \ell - 1$. (In fact if $A \in \mathcal{A}[i\bar{j}]$, then $A' := (A \cup \{j\}) \setminus \{i\} \notin \mathcal{A}$, which means that there is some $B \in \mathcal{B}$ such that $|A \cap B| \geq t$ but $|A' \cap B| < t$, and this happens only when $|A' \cap [t + 2\ell - 1]| = |B \cap [t + 2\ell - 1]| = t + \ell - 1$.) Thus (3) defines a bijection $f : \mathcal{H} \to \mathcal{A}_{ij}$. Similarly we obtain a bijection $g : \mathcal{H} \to \mathcal{B}_{ij}$, where $\mathcal{B}_{ij} := \mathcal{B}[i\bar{j}] \cup \mathcal{B}[\bar{i}j]$.

Here we construct a bipartite graph $G = (V_{\mathcal{A}} \cup V_{\mathcal{B}}, E)$, where both $V_{\mathcal{A}}$ and $V_{\mathcal{B}}$ are copies of $\mathcal{H}$, and $(H_A, H_B) \in E$ iff $|H_A \cap H_B| = t - 1$. We divide $V_{\mathcal{A}}$ into $V_{\mathcal{A}[i\bar{j}]}$ and

$V_{\mathcal{A}[\bar{i}j]}$ according to whether $f(H) \in \mathcal{A}[i\bar{j}]$ or $f(H) \in \mathcal{A}[\bar{i}j]$. In the same way, we also get the partition $V_{\mathcal{B}} = V_{\mathcal{B}[i\bar{j}]} \cup V_{\mathcal{B}[\bar{i}j]}$ using g. It then follows from the cross t-intersecting property that there are no edges between $\mathcal{A}[\bar{i}j]$ and $\mathcal{B}[i\bar{j}]$, and no edges between $\mathcal{A}[i\bar{j}]$ and $\mathcal{B}[\bar{i}j]$. Recall that none of $\mathcal{A}[i\bar{j}]$, $\mathcal{A}[\bar{i}j]$, $\mathcal{B}[i\bar{j}]$, and $\mathcal{B}[\bar{i}j]$ are empty. Thus the graph G is disconnected.

Let $G_0 = (V_0, E_0)$ be a graph such that $V_0 = \mathcal{H}$ and $(F_0, F_0') \in E'$ iff $|F_0 \cap F_0'| = t - 1$. If $n \geq 2k - t + 2$ and $t \geq 2$, then it is readily seen that G_0 is isomorphic to $K(n_1, k_1) \otimes K(n_2, k_2)$, where

$$n_1 = t + 2\ell - 1, \qquad\qquad k_1 = (t + 2\ell - 1) - (t + \ell - 1) = \ell,$$

$$n_2 = (n - 2) - (t + 2\ell - 1), \qquad k_2 = (k - 1) - (t + \ell - 1) = k - t - \ell.$$

(We used $n \geq 2k - t + 2$ and $t \geq 2$ to ensure that $n_1 > 2k_1$ and $n_2 > 2k_2$.) So it follows from Fact 2.4 and Fact 2.5 that G_0 is connected and non-bipartite. By definition, G is isomorphic to $G_0 \otimes K_2$, and by Fact 2.5, G is connected. This is a contradiction. $\quad\square$

Similarly one can show the following, which can be used as an alternative to Lemma 2.3(iv). (For a proof we use that a graph $G = (V, E)$, where $V = 2^{[n]}$ and $(F, F') \in E$ iff $F \cap F' = \emptyset$, is connected and non-bipartite for $n \geq 3$.)

Lemma 2.7. *Let $n \geq 3$, $t \geq 2$, and $k - t \geq \ell \geq 0$. If $\mathcal{A}$ and $\mathcal{B}$ are cross t-intersecting families in $2^{[n]}$, and $s_{ij}(\mathcal{A}) = s_{ij}(\mathcal{B}) = \mathcal{F}_\ell^t(n)$, then $\mathcal{A} = \mathcal{B} \cong \mathcal{F}_\ell^t(n)$.*

For $A \subset [n]$ let $(A)_i$ denote the i-th element of A, where $(A)_1 < (A)_2 < \cdots$. For $A, B \subset [n]$, we say A *shifts to* B, and write

$$A \to B$$

if $|A| \leq |B|$ and $(A)_i \geq (B)_i$ for all $i \leq |A|$. E.g., $\{2, 4, 6, 8\} \to \{1, 2, 4, 8, 9\}$.

We list some easy facts below, which we will use without referring to explicitly.

Fact 2.8. *Let $\mathcal{A} \subset 2^{[n]}$ be shifted.*

(i) *If $A \in \mathcal{A}$, $A \to A'$, and $|A| = |A'|$, then $A' \in \mathcal{A}$.*
(ii) *If $\mathcal{A}$ is inclusion maximal, $A \in \mathcal{A}$, and $A \to A''$, then $A'' \in \mathcal{A}$.*

Let $t \in [n]$ and $A \subset [n]$. We define the *dual* of A with respect to t by

$$\mathrm{dual}_t(A) := \big[(A)_t - 1\big] \cup \big([n] \setminus A\big).$$

Clearly we have $|A \cap \mathrm{dual}_t(A)| = t - 1$. The following simple fact is very useful, and we will use it for some particular choices of A.

Fact 2.9. *If $\mathcal{A}$ and $\mathcal{B}$ are cross t-intersecting and $A \in \mathcal{A}$, then $\mathrm{dual}_t(A) \notin \mathcal{B}$.*

For $\mathcal{F} \subset 2^{[n]}$ let $\lambda(\mathcal{F})$ be the maximum λ such that all walks in $\mathcal{F}$ hit the line $y = x + \lambda$. Let

$$F_{[u]} := [u] \cup \{u + 2i \in [n]: \ i \geq 1\}$$

be the "maximal" walk that does not hit $y = x + (u + 1)$. If $u \leq t$, then

$$\mathrm{dual}_t(F_{[u]}) = F_{[2t-u-1]}.$$

Note also that if $F \subset [n]$ does not hit the line $y = x + (u + 1)$, then $F \to F_{[u]}$. In particular, we have the following.

Fact 2.10. *For a shifted, inclusion maximal family $\mathcal{F} \subset 2^{[n]}$, if $F_{[u]} \notin \mathcal{F}$ then $\lambda(\mathcal{F}) \geq u+1$.*

Lemma 2.11. *Let $\mathcal{A}$ and $\mathcal{B}$ be shifted cross t-intersecting families in $2^{[n]}$.*

(i) *For every $A \in \mathcal{A}$ and $B \in \mathcal{B}$ there is some i such that $|A \cap [i]| + |B \cap [i]| \geq i + t$.*
(ii) *Suppose further that $\mathcal{A}$ and $\mathcal{B}$ are inclusion maximal cross t-intersecting families. Then $\lambda(\mathcal{A}) + \lambda(\mathcal{B}) \geq 2t$.*

Proof. Suppose the contrary to (i). Choose a pair of counterexamples $A \in \mathcal{A}$, $B \in \mathcal{B}$ so that $|A \cap B|$ is minimal. Let $j = (A \cap B)_t$. (Recall that this is the t-th element of $A \cap B$.) Then we have $|A \cap [j]| + |B \cap [j]| < t + j = |A \cap B \cap [j]| + |[j]|$, which is equivalent to $|(A \cup B) \cap [j]| < |[j]|$. Thus we can find some i with $1 \leq i < j$ such that $i \notin A \cup B$, where $i \neq j$ follows from $j = (A \cap B)_t$. Since $\mathcal{B}$ is shifted, we have $B' := (B - \{j\}) \cup \{i\} \in \mathcal{B}$. Then $|B \cap [j]| = |B' \cap [j]|$, and so A and B' are also counterexamples. But we get $|A \cap B'| < |A \cap B|$, which contradicts the minimality. This gives (i).

Let $u = \lambda(\mathcal{A}) \leq \lambda(\mathcal{B})$. We may assume that $u < t$. Since $\mathcal{A}$ is inclusion maximal and $u = \lambda(\mathcal{A})$ we have that $F_{[u]} \in \mathcal{A}$. Then cross t-intersecting property yields $\mathrm{dual}_t(F_{[u]}) = F_{[2t-u-1]} \notin \mathcal{B}$. Since $\mathcal{B}$ is also inclusion maximal this with Fact 2.10 gives $\lambda(\mathcal{B}) \geq 2t - u$, as desired. $\quad\square$

Cross t-intersecting families have the following monotone property. Let $f(n)$ be the maximum of $\mu_p(\mathcal{A})\mu_p(\mathcal{B})$ where $\mathcal{A}$ and $\mathcal{B}$ are cross t-intersecting families in $2^{[n]}$.

Lemma 2.12. $f(n) \leq f(n + 1)$.

Proof. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are cross t-intersecting families in $2^{[n]}$ with $f(n) = \mu_p(\mathcal{A})\mu_p(\mathcal{B})$. Let $\mathcal{A}' = \mathcal{A} \cup \mathcal{A}'' \subset 2^{[n+1]}$ where $\mathcal{A}'' = \{A \cup \{n + 1\}: \ A \in \mathcal{A}\}$. We write μ_p^n for the p-weight to emphasize the size of the ground set. Then we have $\mu_p^{n+1}(\mathcal{A}') = q\mu_p^n(\mathcal{A}) + p\mu_p^n(\mathcal{A}) = \mu_p^n(\mathcal{A})$. Similarly, letting $\mathcal{B}' = \mathcal{B} \cup \{B \cup \{n+1\}: \ B \in \mathcal{B}\}$,

we also have $\mu_p^{n+1}(\mathcal{B}') = \mu_p^n(\mathcal{B})$ and thus $\mu_p^{n+1}(\mathcal{A}')\mu_p^{n+1}(\mathcal{B}') = \mu_p^n(\mathcal{A})\mu_p^n(\mathcal{B})$. Since $\mathcal{A}'$ and $\mathcal{B}'$ are cross t-intersecting families in $2^{[n+1]}$, we have $f(n) \leq f(n+1)$. $\square$

Many of the above results have natural k-uniform versions. For example Lemma 2.2 can be transformed as follows.

Lemma 2.13. *Let x_0, y_0, c be integers with $0 < c < y_0 < x_0 + c$.*

(i) *The number of walks from $(0,0)$ to (x_0, y_0) which hit the line $y = x + c$ is $\binom{x_0+y_0}{y_0-c}$.*
In particular, if all walks in $\mathcal{F} \subset \binom{[n]}{k}$ hit the line $y = x + c$, then $|\mathcal{F}| \leq \binom{n}{k-c}$.

(ii) *The number of walks from $(0,0)$ to (x_0, y_0) which do not hit the line $y = x + c$ is $\binom{x_0+y_0}{x_0} - \binom{x_0+y_0}{y_0-c}$. In particular, if no walk in $\mathcal{F} \subset \binom{[n]}{k}$ hits the line $y = x+c$, then $|\mathcal{F}| \leq \binom{n}{k} - \binom{n}{k-c}$.*

(iii) *If all walks in $\mathcal{F}$ hit the line $y = x + c$ at least twice, but do not hit the line $y = x + (c+1)$, then $|\mathcal{F}| \leq \binom{n}{k-c-1}$.*

Proof. The walks from $O = (0,0)$ to $P = (x_0, y_0)$ that hit the line $L : y = x + c$ are in bijection with the walks from $(-c, c)$ to P; this is seen by reflecting the part of the walk, from O to the first hitting point on the line L, in this line, and the number of such lines is $\binom{x_0+y_0}{y_0-c}$. This gives (i). If $\mathcal{F} \subset \binom{[n]}{k}$, then notice that $x_0 = n - k$ and $y_0 = k$.

There are $\binom{x_0+y_0}{x_0}$ walks from $(0,0)$ to (x_0, y_0), and $\binom{x_0+y_0}{y_0-c}$ of them hit the given line by (i). This gives (ii).

Consider walks from $(0,0)$ to $(n - k, k)$. Then, as in the proof of (iii) of Lemma 2.2, there is an injection from (I) the family of walks that hit the line $y = x + c$ at least twice but do not hit $y = x + (c+1)$ to (II) the family of walks that hit $y = x + (c+1)$. Thus (iii) follows from (i). $\square$

In the k-uniform setting, observe that if $A, B \in \binom{[n]}{k}$, $A \to B$ simply means that $(A)_i \geq (B)_i$ for all $i \leq k$. So Fact 2.8 reads as follows.

Fact 2.14. *Let $\mathcal{A} \subset \binom{[n]}{k}$ be shifted. If $A \in \mathcal{A}$, and $A \to A'$, then $A' \in \mathcal{A}$.*

For $A = \{x_1, x_2, \ldots, x_k, \ldots\}$ with $|A| \geq k$ and $x_1 < x_2 < \cdots$, let $\mathrm{first}_k(A)$ be the first k elements of A, that is,

$$\mathrm{first}_k(A) := \{x_1, x_2, \ldots, x_k\}.$$

For $t \leq k \leq n$ we define the *dual* of $A \in \binom{[n]}{k}$ with respect to t and k by

$$\mathrm{dual}_t^{(k)}(A) := \mathrm{first}_k\big(\mathrm{dual}_t(A)\big).$$

Again, we clearly have that $|A \cap \mathrm{dual}_t^{(k)}(A)| = t - 1$, and Fact 2.9 reads as follows.

Fact 2.15. *If $\mathcal{A}$ and $\mathcal{B}$ are k-uniform cross t-intersecting families and $A \in \mathcal{A}$, then $\mathrm{dual}_t^{(k)}(A) \notin \mathcal{B}$.*

Let $F_{[u]}^{(k)} := \mathrm{first}_k(F_{[u]})$. If $u \leq t$ and $2t - u - 1 \leq k$, then

$$\mathrm{dual}_t^{(k)}\big(F_{[u]}^{(k)}\big) = F_{[2t-u-1]}^{(k)}.$$

If $F \in \binom{[n]}{k}$ does not hit the line $y = x + (u+1)$, then $F \to F_{[u]}^{(k)}$.

Fact 2.16. *For a shifted family $\mathcal{F} \subset \binom{[n]}{k}$, if $F_{[u]}^{(k)} \notin \mathcal{F}$ then $\lambda(\mathcal{F}) \geq u + 1$.*

We remark that (i) of Lemma 2.11 is valid for $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$ as well. As for (ii) we assume that both families are non-empty instead of inclusion maximal, as follows.

Lemma 2.17. *Let $\mathcal{A}$ and $\mathcal{B}$ be shifted cross t-intersecting families in $\binom{[n]}{k}$. If $|\mathcal{A}||\mathcal{B}| > 0$, then $\lambda(\mathcal{A}) + \lambda(\mathcal{B}) \geq 2t$.*

Proof. Let $u = \lambda(\mathcal{A}) \leq \lambda(\mathcal{B})$. We may assume that $u < t$. Since $\mathcal{A}$ is shifted and $u = \lambda(\mathcal{A})$ we have that $F_{[u]}^{(k)} \in \mathcal{A}$. Then cross t-intersecting property yields $\mathrm{dual}_t^{(k)}(F_{[u]}^{(k)}) = F_{[2t-u-1]}^{(k)} \notin \mathcal{B}$. If $2t - u - 1 \leq k$, then this gives $\lambda(\mathcal{B}) \geq 2t - u$, as desired. If $2t - u - 1 > k$, then

$$\big|[k] \cap F_{[u]}^{(k)}\big| \leq u + \frac{k - u}{2} < u + \frac{2t - 2u - 1}{2} < t,$$

and $[k] \notin \mathcal{B}$. Since $\mathcal{B}$ is shifted, this means $\mathcal{B} = \emptyset$. But this contradicts our assumption $|\mathcal{A}||\mathcal{B}| > 0$. $\square$

3. Results about weighted families

In this section we prove the following main result from which Theorems 1.3, 1.4, and 5.1 will follow.

Proposition 3.1. *For every $t \geq 14$, $n \geq t$, $\eta \in (0, 1]$, and p with $0 < p \leq \frac{1}{t+1}$, we have the following. If $\mathcal{A}$ and $\mathcal{B}$ are shifted, inclusion maximal cross t-intersecting families in $2^{[n]}$, then one of the following holds.*

(i) $\sqrt{\mu_p(\mathcal{A})\mu_p(\mathcal{B})} < (1 - \gamma\eta)p^t$, *where $\gamma \in (0, 1]$ depends only on t.*
(ii) $\mu_p(\mathcal{A} \triangle \mathcal{F}_s^t(n)) + \mu_p(\mathcal{B} \triangle \mathcal{F}_s^t(n)) < \eta\mu_p(\mathcal{F}_s^t(n))$, *where $s = 0$ or 1.*

If (ii) *happens then $\sqrt{\mu_p(\mathcal{A})\mu_p(\mathcal{B})} < \mu_p(\mathcal{F}_s^t(n))$ with equality holding iff $\mathcal{A} = \mathcal{B} = \mathcal{F}_s^t(n)$.*

We need some definitions which we will continue to use throughout the proof of the main proposition.

Let $\mathcal{F}^u$ be the family of all walks that hit the line $y = x + u$. We identify a subset and its walk, so formally,

$$\mathcal{F}^u = \big\{F \subset [n]\colon \big|F \cap [j]\big| \geq (j + u)/2 \text{ for some } j\big\}.$$

We partition $\mathcal{F}^u$ into the following three subfamilies:

$$\tilde{\mathcal{F}}^u := \big\{F \in \mathcal{F}^u\colon F \text{ hits } y = x + u + 1\big\},$$
$$\dot{\mathcal{F}}^u := \big\{F \in \mathcal{F}^u\colon F \text{ hits } y = x + u \text{ exactly once, but does not hit } y = x + u + 1\big\},$$
$$\ddot{\mathcal{F}}^u := \big\{F \in \mathcal{F}^u\colon F \text{ hits } y = x + u \text{ at least twice, but does not hit } y = x + u + 1\big\}.$$

We remark that no walk F in $\dot{\mathcal{F}}^u \cup \ddot{\mathcal{F}}^u$ hits the line $y = x + (u + 1)$. This can also be stated as the fact that if $F \in \dot{\mathcal{F}}^u \cup \ddot{\mathcal{F}}^u$ then $|F \cap [i]| \leq (i + u)/2$ for all i.

If $u + 2s \leq n$, then, for simplicity, we also use $\mathcal{F}_s^u$ to mean $\mathcal{F}_s^u(n)$. This is the family of walks that hit the line $y = x + u$ within the first $u + 2s$ steps. So if $F \in \mathcal{F}_s^u$ then there is some $0 \leq i \leq s$ such that the walk corresponding to F hits $(i, u + i)$.

To simplify the notation we write $X <_t Y$ if there is a positive function $\gamma = \gamma(t) > 0$ depending only on t such that $X < (1 - \gamma(t))Y$ for all $t \geq 14$ (or $t \geq t_0$ for some specified value $t_0 \leq 14$). For example, we write $\mu_p(\mathcal{A})\mu_p(\mathcal{B}) <_t p^{2t}$ to mean $\mu_p(\mathcal{A})\mu_p(\mathcal{B}) < (1 - \gamma)p^{2t}$ for some $\gamma = \gamma(t) > 0$, which would give (i) of Proposition 3.1.

3.1. Proof of the main proposition: setup

Let $t \geq 14$, $0 < p \leq \frac{1}{t+1}$, $q = 1 - p$, and $\alpha = p/q$. Here we record some basic inequalities that will be used frequently:

$$p \leq \frac{1}{t + 1}, \qquad q \geq \frac{t}{t + 1}, \qquad q^{-t} \leq \left(1 + \frac{1}{t}\right)^t < e, \qquad \alpha \leq \frac{1}{t}, \qquad pq \leq \frac{t}{(t + 1)^2}. \tag{4}$$

By Lemma 2.12, we may assume that n is sufficiently large. Let $\mathcal{A}$ and $\mathcal{B}$ be shifted, inclusion maximal cross t-intersecting families in $2^{[n]}$.

Let $u = \lambda(\mathcal{A})$ and $v = \lambda(\mathcal{B})$. Recall that by Lemma 2.11(ii) we have $u + v \geq 2t$. If $u + v \geq 2t + 1$, then (i) of Lemma 2.2 with (4) yields

$$\mu_p(\mathcal{A})\mu_p(\mathcal{B}) \leq \alpha^u \alpha^v \leq \alpha^{2t+1} = p^{2t}\frac{p}{q^{2t+1}} < p^{2t}\frac{e^{2+1/t}}{t + 1}.$$

One can check that $\frac{e^{2+1/t}}{t+1} <_t 1$ for $t \geq 8$, which gives $\mu_p(\mathcal{A})\mu_p(\mathcal{B}) <_t p^{2t}$. Thus if $u + v \geq 2t + 1$, then (i) of the proposition holds.

From now on let $u + v = 2t$. By symmetry we may assume that $u \leq v$.

We partition $\mathcal{A} \subset \mathcal{F}^u$ into families $\dot{\mathcal{A}} := \mathcal{A} \cap \dot{\mathcal{F}}^u$, $\ddot{\mathcal{A}} := \mathcal{A} \cap \ddot{\mathcal{F}}^u$, and $\tilde{\mathcal{A}} := \mathcal{A} \cap \tilde{\mathcal{F}}^u$. Similarly, we define $\dot{\mathcal{B}} := \mathcal{B} \cap \dot{\mathcal{F}}^v$, $\ddot{\mathcal{B}} := \mathcal{B} \cap \ddot{\mathcal{F}}^v$, and $\tilde{\mathcal{B}} := \mathcal{B} \cap \tilde{\mathcal{F}}^v$. We remark that if $A \in \dot{\mathcal{A}} \cup \ddot{\mathcal{A}}$, then $|A \cap [i]| \leq (i + u)/2$ for all i. Moreover equality holds exactly once if $A \in \dot{\mathcal{A}}$, and at least twice if $A \in \ddot{\mathcal{A}}$.

If $\dot{\mathcal{A}} = \emptyset$ then $\mathcal{A} = \ddot{\mathcal{A}} \cup \tilde{\mathcal{A}}$ and $\mu_p(\mathcal{A}) = \mu_p(\ddot{\mathcal{A}}) + \mu_p(\tilde{\mathcal{A}})$. Thus by (i) and (iii) of Lemma 2.2 we have

$$\mu_p(\mathcal{A})\mu_p(\mathcal{B}) \leq \left(\alpha^{u+1} + \alpha^{u+1}\right)\alpha^v \leq 2\alpha^{2t+1} <_t p^{2t}, \tag{5}$$

where the last inequality holds for $t \geq 14$. (This is the point where we really need $t \geq 14$.) The same holds for the case when $\dot{\mathcal{B}} = \emptyset$. Thus if $\dot{\mathcal{A}} = \emptyset$ or $\dot{\mathcal{B}} = \emptyset$ then (i) of the proposition holds.

From now on we assume that $\dot{\mathcal{A}} \neq \emptyset$ and $\dot{\mathcal{B}} \neq \emptyset$. Thus there exist s, s' such that $\dot{\mathcal{A}} \cap \mathcal{F}_s^u \neq \emptyset$ and $\dot{\mathcal{B}} \cap \mathcal{F}_{s'}^v \neq \emptyset$. Remarkably, these s and s' are uniquely determined. Extending a result in [9] we show this structural result as follows.

Lemma 3.2. *There exist unique nonnegative integers s and s' such that $s - s' = (v - u)/2$, $\mathcal{A}_s := \dot{\mathcal{A}} \cup \ddot{\mathcal{A}} \subset \mathcal{F}_s^u$, and $\mathcal{B}_{s'} := \dot{\mathcal{B}} \cup \ddot{\mathcal{B}} \subset \mathcal{F}_{s'}^v$.*

Proof. Suppose that $\dot{\mathcal{A}} \cap \mathcal{F}_s^u \neq \emptyset$ and $\dot{\mathcal{B}} \cap \mathcal{F}_{s'}^v \neq \emptyset$. For any $A \in \dot{\mathcal{A}} \cap \mathcal{F}_s^u$ we have

$$\left|A \cap [i]\right| \leq (i + u)/2 \tag{6}$$

for all i, with equality holding iff $i = 2s + u$. Similarly, for any $B \in \dot{\mathcal{B}} \cap \mathcal{F}_{s'}^v$, we have $|B \cap [i]| \leq (i + v)/2$ with equality holding iff $i = 2s' + v$. These two inequalities give

$$\left|A \cap [i]\right| + \left|B \cap [i]\right| \leq i + (u + v)/2 = i + t \tag{7}$$

with equality holding iff $i = 2s + u = 2s' + v$. By Lemma 2.11(i), equality must hold in (7) for this i, which gives $s - s' = (v - u)/2$. If there is some $B' \in \dot{\mathcal{B}} \cap \mathcal{F}_x^v$, then, by considering A and B', we also have $s - x = (v - u)/2$. This gives $x = s'$, and hence $\dot{\mathcal{B}} \subset \mathcal{F}_{s'}^v$. Similarly we have $\dot{\mathcal{A}} \subset \mathcal{F}_s^u$.

We need to show $\ddot{\mathcal{A}} \subset \mathcal{F}_s^u$ and $\ddot{\mathcal{B}} \subset \mathcal{F}_{s'}^v$. Suppose, to the contrary, that $\ddot{\mathcal{B}} \not\subset \mathcal{F}_{s'}^v$. Then there is some $B \in \ddot{\mathcal{B}}$ such that $|B \cap [i]| \leq (i + v)/2$ for all i, and where equality does *not* hold at $i = 2s' + v$. For any $A \in \dot{\mathcal{A}} \subset \mathcal{F}_s^u$, we have (6) with equality holding only at this same $i = 2s + u = 2s' + v$. So equality in (7) never holds for these A and B, contradicting Lemma 2.11(i). One can show $\ddot{\mathcal{A}} \subset \mathcal{F}_s^u$ similarly.   □

Here we record our **setup**.

- $t \geq 14$, $0 < p \leq \frac{1}{t+1}$, $q = 1 - p$, and $\alpha = p/q$.
- $u + v = 2t$, $0 \leq u \leq t \leq v \leq 2t$, $s \geq s' \geq 0$, and $s - s' = (v - u)/2$.

- $u = t - (s - s')$ and $v = t + (s - s')$.
- $\mathcal{A} = \dot{\mathcal{A}} \cup \ddot{\mathcal{A}} \cup \tilde{\mathcal{A}} \subset \mathcal{F}^u$, $\mathcal{B} = \dot{\mathcal{B}} \cup \ddot{\mathcal{B}} \cup \tilde{\mathcal{B}} \subset \mathcal{F}^v$, $\dot{\mathcal{A}} \neq \emptyset$ and $\dot{\mathcal{B}} \neq \emptyset$.
- $\mathcal{A}_s := \dot{\mathcal{A}} \cup \ddot{\mathcal{A}} \subset \mathcal{F}^u_s$ and $\mathcal{B}_{s'} := \dot{\mathcal{B}} \cup \ddot{\mathcal{B}} \subset \mathcal{F}^v_{s'}$.

The rest of the proof of the proposition is divided into three parts, which break over three more subsections. In Section 3.2, we deal with easy cases, namely, all cases but $(s, s') = (1,0), (0,0), (1,1)$, and in Section 3.3 we settle one of the more difficult cases, $(s, s') = (1,0)$. In these cases, only (i) of the proposition happens. Finally in Section 3.4, we consider the last two cases $(s, s') = (0,0), (1,1)$ where the extremal configurations satisfying (ii) appear. Then Theorems 1.3 and 1.4 will be easily proved using the proposition.

3.2. Proof of Proposition 3.1: easy cases

Let $\bar{\mathcal{F}}^r_i := (\dot{\mathcal{F}}^r \cup \ddot{\mathcal{F}}^r) \cap \mathcal{F}^r_i$.

Claim 3.3. *Let $r \geq 1$ and $i \geq 0$ be integers, let $p \leq p_0 < 1/2$, and let $\epsilon = 0.001$. There exists an n_0 such that for all $n \geq n_0$ we have*

$$\mu_p\big(\tilde{\mathcal{F}}^r \cup \bar{\mathcal{F}}^r_i\big) < p^r f(r, i, p)(1 + \epsilon),$$

where

$$f(r, i, p) = \frac{p}{q^{r+1}} + \binom{2i+r}{i} \frac{r+1}{r+i+1}\left(1 - \frac{p}{q}\right)(pq)^i.$$

Proof. Let L be the line $y = x + r + 1$. Then every walk in $\tilde{\mathcal{F}}^r$ hits L, and (i) of Lemma 2.2 yields

$$\mu_p\big(\tilde{\mathcal{F}}^r\big) \leq \alpha^{r+1} = \frac{p^{r+1}}{q^{r+1}} \leq \frac{p^{r+1}}{q^{r+1}}(1 + \epsilon). \tag{8}$$

A walk $W \in \bar{\mathcal{F}}^r_i$ must go from $O = (0,0)$ to $Q = (i, i+r)$ without hitting the line L, and then continue on without hitting L. It follows from (ii) of Lemma 2.13 that the number of walks from O to Q that do not hit L is

$$\binom{i + (i+r)}{i} - \binom{i + (i+r)}{(i+r) - (r+1)} = \binom{2i+r}{i}\frac{r+1}{r+i+1},$$

so the probability of a random walk $W_{2i+r,p}$ hitting Q without hitting L is $\binom{2i+r}{i}\frac{r+1}{r+i+1} \times p^{r+1}q^i$. By Lemma 2.2(ii), the random walk continues on from Q without hitting L, with probability at most $1 - \alpha + \delta \leq (1 - \alpha)(1 + \epsilon)$, where $\delta = \frac{1 - 2p_0}{1 - p_0}\epsilon$. Therefore we have

$$\mu_p\big(\bar{\mathcal{F}}^r_i\big) \leq \binom{2i+r}{i}\frac{r+1}{r+i+1}p^{r+i}q^i(1-\alpha)(1+\epsilon). \tag{9}$$

Combining (8) and (9) completes the proof of Claim 3.3. $\square$

Lemma 3.4. *If $s \geq 2$, then $\mu_p(\mathcal{A})\mu_p(\mathcal{B}) <_t p^{2t}$.*

Proof. Since $\mu_p(\mathcal{A})\mu_p(\mathcal{B}) \leq \mu_p(\tilde{\mathcal{F}}^u \cup \bar{\mathcal{F}}^u_s)\mu_p(\tilde{\mathcal{F}}^v \cup \bar{\mathcal{F}}^v_{s'})$, it suffices to show the RHS is $<_t p^{2t}$. We will show that $f(u,s,p)f(v,s',p) <_t 0.99$ for all $n \geq n_0$. We claim that it implies Lemma 3.4. Indeed, since $0.99(1 + \epsilon)^2 < 0.992$ and Claim 3.3 holds, we have $\mu_p(\mathcal{A})\mu_p(\mathcal{B}) <_t p^{2t}$ for all $n \geq n_0$. By Lemma 2.12, this implies Lemma 3.4 for all n.

Now we show that $f(u,s,p)f(v,s',p) <_t 0.99$ for all $n \geq n_0$. Since $u \leq t$ and $f(u,s,p)$ is an increasing function of u, we have $f(u,s,p) \leq f(t,s,p)$. The first term of $f(t,s,p)$ is clearly increasing in p, and the second term is also an increasing function of p iff $\frac{1}{p} + 4p > 4 + \frac{1}{s}$, which is certainly true for $p \leq \frac{1}{t+1} \leq 1/15$ and $s \geq 1$. Thus we have $f(t,s,p) \leq f(t,s,\frac{1}{t+1}) =: g(s,t)$. By a direct computation we see that $g(s,t) > g(s+1,t)$ iff

$$\frac{(t+1)^2(s+1)(s+t+2)}{t(2s+t+2)(2s+t+1)} > 1,$$

or equivalently,

$$s^2(t-1)^2 + s(t^3 + t^2 + t + 3) + (t^2 + 3t + 2) > 0$$

which is true for $t \geq 1$ and $s \geq 0$. Similarly, noting that $v \leq 2t$, we have

$$f(v,s',p) \leq f(2t,s',p) \leq f\left(2t, s', \frac{1}{t+1}\right) := h(s',t),$$

and $h(s',t) > h(s'+1,t)$ holds for all $t \geq 1$ and $s' \geq 1$. Thus for $t \geq 14$, $s \geq 3$ and $s' \geq 1$ we have

$$f(u,s,p)f(v,s',p) \leq g(s,t)h(s',t) \leq g(3,t)h(1,t) \leq g(3,14)h(1,14) < 0.87.$$

The remaining cases are $s' = 0$ or $s = 2$. If $s' = 0$ and $s \geq 2$ then, observing that $f(v,0,p)$ is (decreasing in p but) bounded by 1 as p goes to 0, we have

$$f(u,s,p)f(v,s',p) \leq g(2,14) \cdot 1 < 0.96.$$

If $s = 2$ and $s' = 1$ then $u = t - 1$, $v = t + 1$, and

$$f(u,s,p)f(v,s',p) \leq f\left(t-1, 2, \frac{1}{t+1}\right) f\left(t+1, 1, \frac{1}{t+1}\right)$$

$$\leq f\left(13, 2, \frac{1}{15}\right) f\left(15, 1, \frac{1}{15}\right) < 0.68.$$
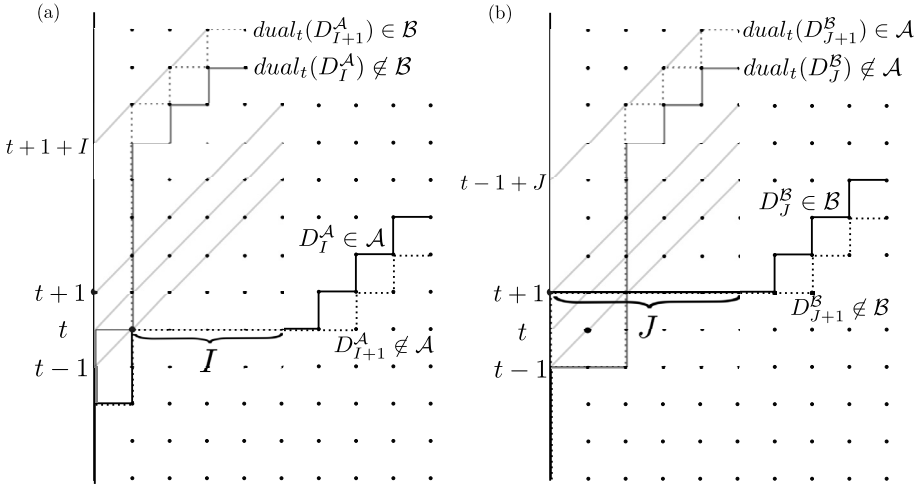
If $s = s' = 2$ then $u = v = t$ and

Fig. 1. (a) $D_I^{\mathcal{A}}$ and (b) $D_J^{\mathcal{B}}$.

$$f(u, s, p) f(v, s', p) \le f\left(t, 2, \frac{1}{t+1}\right)^2 \le f\left(14, 2, \frac{1}{15}\right)^2 < 0.46.$$

This completes the proof of Lemma 3.4. □

We have proved the proposition for the cases $s \ge 2$, and the remaining cases are $s \le 1$, namely, $(s, s') = (0, 0), (1, 1), (1, 0)$. We will discuss these three cases in the next two subsections.

3.3. Proof of Proposition 3.1: a harder case

Here we consider the case $(s, s') = (1, 0)$. In this case, we have $u = t - 1$ and $v = t + 1$.

Lemma 3.5. *We have* $\mu_p(\mathcal{A})\mu_p(\mathcal{B}) <_t p^{2t}$ *for* $(s, s') = (1, 0)$.

Proof. For $1 \le i \le n - t - 2$, let

$$D_i^{\mathcal{A}} = [t - 2] \cup \{t, t + 1\} \cup \{t + 1 + i + 2\ell \in [n]\colon \ell = 1, 2, \ldots\} \in \dot{\mathcal{F}}^{t-1} \cap \mathcal{F}_1^{t-1},$$

and for $1 \le j \le n - t - 2$, let

$$D_j^{\mathcal{B}} = [t + 1] \cup \{t + 1 + j + 2\ell \in [n]\colon \ell = 1, 2, \ldots\} \in \dot{\mathcal{F}}^{t+1} \cap \mathcal{F}_0^{t+1}.$$

(See Fig. 1.)

Let $I := \max\{i\colon D_i^{\mathcal{A}} \in \mathcal{A}\}$. We claim that I is well defined. Indeed, we have assumed that $\emptyset \ne \dot{\mathcal{A}} \subset \mathcal{A}_1$ (see setup in Section 3.1), and $A \to D_1^{\mathcal{A}}$ for any $A \in \dot{\mathcal{A}}$. Thus we have that $D_1^{\mathcal{A}} \in \mathcal{A}$, and so $\{i\colon D_i^{\mathcal{A}} \in \mathcal{A}\} \ne \emptyset$. Similarly, we have $\emptyset \ne \dot{\mathcal{B}} \subset \mathcal{B}_0$ and $B \to D_1^{\mathcal{B}}$ for any $B \in \dot{\mathcal{B}}$. Thus $D_1^{\mathcal{B}} \in \mathcal{B}$ and we can define $J := \max\{j\colon D_j^{\mathcal{B}} \in \mathcal{B}\}$.

We consider several cases separately. During this subsection, ϵ is an arbitrarily small constant, depending only on n, where $\epsilon \to 0$ as $n \to \infty$.

Case 1. When $I \geq 2$ and $J \geq 2$.

First we show

$$\mu_p(\mathcal{A}) < p^t(e\alpha + 1 + tq + t\epsilon). \tag{10}$$

Note that $\mu_p(\mathcal{A}) = \mu_p(\tilde{\mathcal{A}}) + \mu_p(\mathcal{A}_1)$. To estimate $\mu_p(\tilde{\mathcal{A}})$, let $A \in \tilde{\mathcal{A}}$. Then A hits the line L_0: $y = x + t$. At the same time, since the dual walk $\mathrm{dual}_t(D_J^{\mathcal{B}})$ is not in $\mathcal{A}$, where

$$\mathrm{dual}_t\big(D_J^{\mathcal{B}}\big) = [t-1] \cup [t+2, t+J+2] \cup \{t+J+4, t+J+6, \ldots\},$$

we have that $A \not\rightarrow \mathrm{dual}_t(D_J^{\mathcal{B}})$ (see Fig. 1). To satisfy these conditions, it suffices for A that one of the following properties holds:

(1a) A hits the line L_1: $y = x + (t+J-1)$. (Then A hits the line L_0 automatically.)
(1b) A does not hit L_1, but hits $(0, t)$. (Notice that $(0, t)$ is on L_0, and a walk hitting this point cannot shift to $\mathrm{dual}_t(D_J^{\mathcal{B}})$.)
(1c) A does not hit L_1 or $(0, t)$, but hits $(1, t)$ and the line L_0. (Notice that a walk hitting $(1, t)$ cannot shift to $\mathrm{dual}_t(D_J^{\mathcal{B}})$.)

Thus we have

$$\mu_p(\tilde{\mathcal{A}}) \leq \mu_p\big(\text{walks of (1a)}\big) + \mu_p\big(\text{walks of (1b)}\big) + \mu_p\big(\text{walks of (1c)}\big)$$
$$\leq \alpha^{t+J-1} + p^t\big(1 - \alpha^{J-1} + \epsilon\big) + tp^tq\big(\alpha - \alpha^J + \epsilon\big).$$

The last inequality uses Lemma 2.2(i) for the first term. For the second and third terms, we use Lemma 2.2(ii) in combination with Example 2.1. Hence,

$$\mu_p(\tilde{\mathcal{A}}) \leq \alpha^{t+J-1} + p^t + tp^tq\alpha. \tag{11}$$

To bound $\mu_p(\mathcal{A}_1)$ we simply use $\mathcal{A}_1 \subset \bar{\mathcal{F}}_1^{t-1}$ and simply bound $\mu_p(\bar{\mathcal{F}}_1^{t-1})$. Let $F \in \bar{\mathcal{F}}_1^{t-1} = \mathcal{F}_1^{t-1} \cap (\dot{\mathcal{F}}^{t-1} \cup \ddot{\mathcal{F}}^{t-1})$. Then, it follows from $F \in \mathcal{F}_1^{t-1}$ that

$$\big|F \cap \big[(t-1)+2\big]\big| \geq (t-1)+1,$$

that is, F hits $(0, t+1)$ or $(1, t)$. On the other hand, it follows from $F \in \dot{\mathcal{F}}^{t-1} \cup \ddot{\mathcal{F}}^{t-1}$ that F hits the line $y = x + t - 1$, but does not hit $y = x + t$. Combining these things, we have that $F \in \bar{\mathcal{F}}_1^{t-1}$ hits $(1, t)$ without hitting $(0, t)$, and then from $(1, t)$ it will never hit $y = x + 1$. Therefore we have

$$\mu_p(\mathcal{A}_1) \leq \mu_p\big(\bar{\mathcal{F}}_1^{t-1}\big) \leq tp^tq(1 - \alpha + \epsilon) = tp^tq(1 + \epsilon) - tp^tq\alpha, \tag{12}$$

where again we use Lemma 2.2(ii) in combination with Example 2.1.

Combining (11) and (12) implies that

$$\mu_p(\mathcal{A}) = \mu_p(\tilde{\mathcal{A}}) + \mu_p(\mathcal{A}_1) \leq \alpha^{t+J-1} + p^t + tp^t q(1+\epsilon)$$
$$= p^t \left(\frac{1}{q^t} \alpha^{J-1} + 1 + tq(1+\epsilon) \right) < p^t (e\alpha + 1 + tq + t\epsilon),$$

where the last inequality follows from $q^{-t} \leq (1+\frac{1}{t})^t < e$ and $J \geq 2$. This proves (10).

Next we show

$$\mu_p(\mathcal{B}) \leq p^t \left(e\alpha^3 + p + p\epsilon \right). \tag{13}$$

Since $\mu_p(\mathcal{B}) = \mu_p(\tilde{\mathcal{B}}) + \mu_p(\mathcal{B}_0)$ we estimate the p-weights of $\tilde{\mathcal{B}}$ and $\mathcal{B}_0$ separately.

Let $B \in \tilde{\mathcal{B}}$. It follows from $\tilde{\mathcal{B}} \subset \bar{\mathcal{F}}^{t+1}$ that B hits the line $y = x + t + 2$. On the other hand, it follows from $\mathrm{dual}_t(D_I^{\mathcal{A}}) \notin \mathcal{B}$ that B hits $(0, t+1)$ or the line $y = x + (t+1+I)$. Thus we get

$$\mu_p(\tilde{\mathcal{B}}) < \mu_p\big(\text{walks in } \tilde{\mathcal{B}} \text{ hitting } y = x + (t+1+I)\big)$$
$$+ \mu_p\big(\text{walks in } \tilde{\mathcal{B}} \text{ hitting both } (0, t+1) \text{ and } y = x + t + 2\big)$$
$$\leq \alpha^{t+1+I} + p^{t+1}\alpha. \tag{14}$$

As for $\mathcal{B}_0 \subset \bar{\mathcal{F}}_0^{t+1}$, noting that walks in $\bar{\mathcal{F}}_0^{t+1}$ hit $(0, t+1)$ but do not hit the line $y = x + t + 2$, we obtain

$$\mu_p(\mathcal{B}_0) \leq \mu_p\big(\bar{\mathcal{F}}_0^{t+1}\big) \leq p^{t+1}(1 - \alpha + \epsilon) = p^{t+1}(1+\epsilon) - p^{t+1}\alpha. \tag{15}$$

Combining (14) and (15) yields that

$$\mu_p(\mathcal{B}) = \mu_p(\tilde{\mathcal{B}}) + \mu_p(\mathcal{B}_0) \leq \alpha^{t+1+I} + p^{t+1}\alpha + p^{t+1}(1+\epsilon) - p^{t+1}\alpha$$
$$= p^t \left(\frac{1}{q^t} \alpha^{1+I} + p(1+\epsilon) \right) < p^t \left(e\alpha^3 + p + p\epsilon \right),$$

where the last inequality follows from $q^{-t} < e$ and $I \geq 2$. This proves (13).

Now we are ready to show $\mu_p(\mathcal{A})\mu_p(\mathcal{B}) <_t p^{2t}$. By (10) and (13) we have

$$\mu_p(\mathcal{A})\mu_p(\mathcal{B}) < p^{2t}(e\alpha + 1 + tq + t\epsilon)\left(e\alpha^3 + p + p\epsilon \right)$$
$$= p^{2t}(ep\alpha + p + tpq + tp\epsilon)\left(ep^2/q^3 + 1 + \epsilon \right)$$
$$< p^{2t}\left(\frac{e}{t(t+1)} + \frac{1}{t+1} + \frac{t^2}{(t+1)^2} + \epsilon \right)\left(e\frac{t+1}{t^3} + 1 + \epsilon \right)$$
$$< p^{2t}\left(\left(\frac{e}{t(t+1)} + \frac{1}{t+1} + \frac{t^2}{(t+1)^2} \right)\left(e\frac{t+1}{t^3} + 1 \right) + 4\epsilon \right)$$
$$=: p^{2t}\big(g(t) + 4\epsilon \big), \tag{16}$$

where the second inequality follows from (4). Note that $\frac{d}{dt}g(t) < 0$ for $t \leq 13$ while $\frac{d}{dt}g(t) > 0$ for $t \geq 14$. In addition, $g(7) < 0.999$, and $\lim_{t\to\infty} g(t) = 1$. Hence, we have $g(t) <_t 1$ for all $t \geq 7$. Hence we have $\mu_p(\mathcal{A})\mu_p(\mathcal{B}) <_t p^{2t}$ for all $t \geq 7$. This completes the proof for Case 1.

Case 2. When $I = 1$.

We first estimate $\mu_p(\mathcal{A})$. Trivially,

$$\mu_p(\tilde{\mathcal{A}}) \leq \alpha^t = \frac{p^t}{q^t} \leq p^t\left(1 + \frac{1}{t}\right)^t.$$

Next we consider the walks in $\mathcal{A}_1$. These walks hit the line $y = x + t - 1$ but do not hit $y = x + t$, and $\mathcal{A}_1 \subset \mathcal{F}_1^{t-1}$ implies that they hit $(0, t+1)$ or $(1, t)$. Consequently, walks in $\mathcal{A}_1$ hit $(1, t)$ without hitting the $y = x + t$. The weight of these walks is at most $tp^tq(1 - \alpha + \epsilon)$. Among them, we look at the walks that hit all of $(1, t-2)$, $(1, t)$, $(3, t)$, and do not hit $y = x + t - 2$ after hitting $(3, t)$. These walks cannot be in $\mathcal{A}_1$, as they shift to

$$D_2^{\mathcal{A}} = [t-2] \cup \{t, t+1\} \cup \{t+5, t+7, \ldots\};$$

but $D_2^{\mathcal{A}} \notin \mathcal{A}$. The weight of such walks is at least $(t-1)p^tq \cdot q^2(1-\alpha)$. Hence we infer

$$\mu_p(\mathcal{A}_1) \leq tp^tq(1 - \alpha + \epsilon) - (t-1)p^tq \cdot q^2(1-\alpha)$$
$$= p^t\big((1-\alpha)\big(tq - (t-1)q^3\big) + tq\epsilon\big).$$

Then we use the fact that $(1-\alpha)(tq - (t-1)q^3)$ is increasing in p for $0 \leq p \leq \frac{1}{t+1}$, which gives

$$(1-\alpha)\big(tq - (t-1)q^3\big) \leq \frac{t(t-1)(3t+1)}{(t+1)^3}. \tag{17}$$

Thus we have

$$\mu_p(\mathcal{A}) \leq \mu_p(\tilde{\mathcal{A}}) + \mu_p(\mathcal{A}_1) \leq p^t\left(\left(1 + \frac{1}{t}\right)^t + \frac{t(t-1)(3t+1)}{(t+1)^3} + t\epsilon\right). \tag{18}$$

On the other hand, we trivially have

$$\mu_p(\mathcal{B}) \leq \alpha^{t+1} = \frac{p^{t+1}}{q^{t+1}} \leq p^t\left(1 + \frac{1}{t}\right)^{t+1}\frac{1}{t+1} = p^t\left(1 + \frac{1}{t}\right)^t\frac{1}{t}.$$

Combining this with (18) yields that

$$\frac{\mu_p(\mathcal{A})\mu_p(\mathcal{B})}{p^{2t}} \leq g(t) + t\epsilon\left(1 + \frac{1}{t}\right)^t\frac{1}{t} < g(t) + e\epsilon,$$

where

$$g(t) := \left(\left(1 + \frac{1}{t} \right)^t + \frac{t(t-1)(3t+1)}{(t+1)^3} \right) \left(1 + \frac{1}{t} \right)^t \frac{1}{t}. \tag{19}$$

Thus it suffices to show that $g(t) <_t 1$. By direct computation we see that $g(13) < 1$. For $t \geq 14$ we use $(1 + \frac{1}{t})^t < e$ again to get

$$g(t) < \left(\frac{e}{t} + \frac{(t-1)(3t+1)}{(t+1)^3} \right) e =: g_2(t). \tag{20}$$

The RHS is decreasing in t for $t > 0$, and is less than 1 when $t = 14$. This completes the proof for Case 2.

Case 3. When $J = 1$.

Clearly, we have

$$\mu_p(\mathcal{A}) \leq \alpha^{t-1} = p^{t-1} \frac{1}{q^{t-1}},$$

and

$$\mu_p(\tilde{\mathcal{B}}) \leq \alpha^{t+2} = \frac{p^{t+2}}{q^{t+2}}.$$

As for $\mathcal{B}_0$ we count the walks that hit $(0, t+1)$ and do not hit the line $y = x + t + 2$. Among them we delete the walks that hit both $(0, t+1)$ and $(2, t+1)$, and do not hit the line $y = x + t - 1$ after hitting $(2, t+1)$. (If such walk was in $\mathcal{B}$, then this would give $D_2^{\mathcal{B}} \in \mathcal{B}$, which is a contradiction.) Thus we have

$$\mu_p(\mathcal{B}_0) \leq p^{t+1} \left(1 - \alpha + \epsilon - q^2(1-\alpha) \right).$$

Hence we infer

$$\begin{aligned}
\mu_p(\mathcal{A})\mu_p(\mathcal{B}) &\leq p^{2t} \frac{1}{q^{t-1}} \left(\frac{p}{q^{t+2}} + 1 - \alpha + \epsilon - q^2(1-\alpha) \right) \\
&\leq p^{2t} e \left(e \frac{(t+1)^2}{t^2} p + (1-\alpha)(1-q^2) + \epsilon \right) \\
&\leq p^{2t} \left(e^2 \frac{(t+1)}{t^2} + e \frac{(t-1)(2t+1)}{t(t+1)^2} + e\epsilon \right) =: p^{2t} \left(h(t) + e\epsilon \right),
\end{aligned}$$

where the second inequality follows from $\frac{1}{q} \leq \frac{t+1}{t}$ and $\frac{1}{q^t} \leq (1 + \frac{1}{t})^t < e$, and the third inequality follows from $p \leq \frac{1}{t+1}$ and the fact that the function $(1-\alpha)(1-q^2)$ is increasing in p for $p \leq 0.274$. Since $\frac{dh(t)}{dt} < 0$ and $h(13) < 0.96$, we have that $h(t) < 0.96$ for all $t \geq 13$, and hence, for all $t \geq 13$,
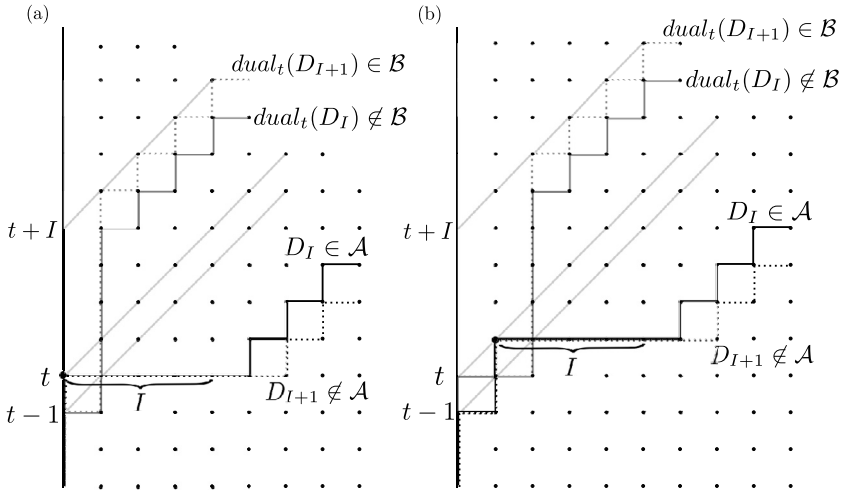
Fig. 2. The walk D_I when (a) $s = 0$ and (b) $s = 1$.

$$\mu_p(\mathcal{A})\mu_p(\mathcal{B}) < 0.97p^{2t}.$$

This completes the proof for Case 3, and so for Lemma 3.5. □

We state now a partial version of Proposition 3.1, recording what we have proved so far.

Corollary 3.6. *For every $t \geq 14$, $n \geq t$, and p with $0 < p \leq \frac{1}{t+1}$, we have the following. If $\mathcal{A}$ and $\mathcal{B}$ are shifted, inclusion maximal cross t-intersecting families in $2^{[n]}$, then one of the following holds.*

  (i) $\sqrt{\mu_p(\mathcal{A})\mu_p(\mathcal{B})} < (1 - \gamma)p^t$, *where $\gamma \in (0, 1]$ depends only on t.*
  (ii) $(s, s') = (0, 0), (1, 1)$.

3.4. Proof of Proposition 3.1: extremal cases

Finally, we consider the cases $(s, s') = (0, 0), (1, 1)$. In these cases $u = v = t$. Recall that $\mathcal{F}_s^t = \{F \subset [n]: |F \cap [t + 2s]| \geq t + s\}$. Let

$$D_i = [1, t - 1] \cup \{t + s, t + 2s\} \cup \{t + 2s + i + 2j \in [n]: j = 1, 2, \ldots\} \in \dot{\mathcal{F}}^t \cap \mathcal{F}_s^t$$

for $1 \leq i \leq n - t - 2s - 1 =: i_{\max}$. (See Fig. 2.) Notice that

$$D_{i_{\max}} = [1, t - 1] \cup \{t + s, t + 2s\}$$

and

$$D_{i_{\max}-1} = [1, t-1] \cup \{t+s, t+2s\} \cup \{n\}.$$

Let $I := \max\{i \colon D_i \in \mathcal{A}\}$. We claim that I exists. Indeed, one can check that $A \to D_1$ for any $A \in \dot{\mathcal{A}} \cap \mathcal{A}_s$, and $D_1 \in \mathcal{A}$. (Recall from the setup at the end of Section 3.1 that we assume $\emptyset \neq \dot{\mathcal{A}} \subset \mathcal{A}_s$.) Similarly, $J := \max\{j \colon D_j \in \mathcal{B}\} \neq \emptyset$.

Claim 3.7. *If $I \neq i_{\max}$, then $\mu_p(\mathcal{B} \setminus \mathcal{F}_s^t) <_t \mu_p(\mathcal{F}_s^t \setminus \mathcal{A})$.*

Proof. We first show that

$$\mu_p\big(\mathcal{F}_s^t \setminus \mathcal{A}\big) \geq \binom{t}{s} p^{t+s} q^{s+I+1} (1 - \alpha) \tag{21}$$

and

$$\mu_p\big(\mathcal{B} \setminus \mathcal{F}_s^t\big) \leq \alpha^{t+I}. \tag{22}$$

Consider a walk W that hits $(s, t+s)$ and shifts to D_{I+1}. Since $D_{I+1} \notin \mathcal{A}$ we have $W \in \mathcal{F}_s^t \setminus \mathcal{A}$. Further, such a walk W must hit $Q_1 = (s, t-s)$ and $Q_2 = (s+I+1, t+s)$. There are $\binom{t}{s}$ ways for W to go from $(0,0)$ to Q_1, then the next $2s+I+1$ steps to Q_2 are unique. A random walk $W_{t+2s+I+1,p}$ has this property with probability $\binom{t}{s} p^{t+s} q^{s+I+1}$. From Q_2, a point on the line $y = x + (t - I - 1)$, the walk must not hit $y = x + (t - I)$. (Otherwise $W \to D_{I+1}$ fails.) This happens, by Lemma 2.2(i), with probability at least $1 - \alpha$, which gives (21).

Next we show (22). Since $\mathrm{dual}_t(D_I) \notin \mathcal{B}$, each walk in $\mathcal{B}$ hits at least one of $(0, t+s)$, $(s, t+s)$, and $y = x + (t + I)$. Since each walk hitting $(0, t+s)$ or $(s, t+s)$ is in $\mathcal{F}_s^t$, each walk in $\mathcal{B} \setminus \mathcal{F}_s^t$ hits $y = x + (t + I)$. This yields (22).

Therefore it suffices, by (21) and (22), to show $\alpha^{t+I} <_t \binom{t}{s} p^{t+s} q^{s+I+1} (1-\alpha)$. We have

$$\frac{\binom{t}{s} p^{t+s} q^{s+I+1} (1-\alpha)}{\alpha^{t+I}} = \binom{t}{s} p^s q^{t+s+1} (1-\alpha) \left(\frac{q^2}{p}\right)^I \geq \binom{t}{s} p^{s-1} q^{t+s+2} (q - p),$$

where the first inequality holds because of $q^2/p > 1$ for $p < 0.38$. Since $p \leq 1/(t+1)$, one can easily check that $\binom{t}{s} p^{s-1} q^{t+s+2} (q - p) >_t 1$ if $s = 0$ and $t \geq 5$, or if $s = 1$ and $t \geq 6$. $\square$

The following part will also be used in proving k-uniform results. To make this reuse easier we introduce some names as follows. Let $f = \mu_p(\mathcal{F}_s^t)$, $a = \mu_p(\mathcal{A})$, $a_0 = \mu_p(\mathcal{A} \cap \mathcal{F}_s^t)$, $a_1 = \mu_p(\mathcal{A} \triangle \mathcal{F}_s^t)$, $a_f = \mu_p(\mathcal{A} \setminus \mathcal{F}_s^t)$, and $f_a = \mu_p(\mathcal{F}_s^t \setminus \mathcal{A})$. (So $f = a_0 + f_a$, $a = a_0 + a_f$, and $a_1 = a_f + f_a$.) Define b, b_0, b_1, b_f, f_b similarly.

Lemma 3.8. *Let $\eta > 0$ be given. If $I \neq i_{\max}$, then one of the following holds.*

(i) $\sqrt{ab} < (1 - \frac{\beta\eta}{4}) f$, where $\beta \in (0, 1]$ depends only on t.
(ii) $a_1 + b_1 < \eta f$ and $\sqrt{ab} < f$.

Proof. We first show that there exists $\beta = \beta(t) > 0$ such that

$$b_f \leq (1 - \beta)f_a \quad \text{and} \quad a_f \leq (1 - \beta)f_b. \tag{23}$$

By Claim 3.7, there is $\beta_1 = \beta_1(t) > 0$ such that $b_f \leq (1 - \beta_1)f_a$. Similarly if $J \neq i_{\max}$, then there is $\beta_2 > 0$ such that $a_f \leq (1 - \beta_2)f_b$. If $J = i_{\max}$, then $\mathcal{A} \subset \mathcal{F}_s^t$, that is, $a_f = 0$, and $a_f \leq (1 - \beta_2)f_b$ holds for any $\beta_2 < 1$. Thus, letting $\beta = \min\{\beta_1, \beta_2\} = \beta(t) > 0$, we have (23).

Now suppose that $a_0 + b_0 < (1 - \frac{\eta}{4})2f$. Then, using (23), we have

$$a + b = a_0 + a_f + b_0 + b_f \leq (1 - \beta)(a_0 + f_a + b_0 + f_b) + \beta(a_0 + b_0)$$

$$\leq (1 - \beta)2f + \beta(a_0 + b_0) < (1 - \beta)2f + \beta\left(1 - \frac{\eta}{4}\right)2f = \left(1 - \frac{\beta\eta}{4}\right)2f.$$

Thus we have $\sqrt{ab} \leq \frac{a+b}{2} < (1 - \frac{\beta\eta}{4})f$.

Next suppose that $a_0 + b_0 \geq (1 - \frac{\eta}{4})2f$. This gives $f_a + f_b \leq \frac{\eta}{2}f$. Thus, using (23), we have $a_1 + b_1 = a_f + f_a + b_f + f_b < 2(f_a + f_b) \leq \eta f$. Also it follows from (23) that $a + b = a_0 + b_0 + a_f + b_f < a_0 + b_0 + f_a + f_b \leq 2f$ which gives $\sqrt{ab} < f$. $\quad\square$

If $I \neq i_{\max}$, then one of (i) or (ii) of Proposition 3.1 holds by Lemma 3.8. (In this case we always have $\sqrt{ab} < f$.) The same holds for the case $J \neq i_{\max}$.

Consequently we may assume that $I = J = i_{\max}$. It follows from $I = i_{\max}$ that $D_{i_{\max}} \in \mathcal{A}$, and hence the dual, $\text{dual}_t(D_{i_{\max}}) = [n] \setminus \{t + s, t + 2s\}$ is not in $\mathcal{B}$. Thus all walks B in $\mathcal{B}$ satisfy $B \not\rightarrow \text{dual}_t(D_{i_{\max}})$, and $\mathcal{B} \subset \mathcal{F}_s^t$ holds. Also, $J = i_{\max}$ yields $\mathcal{A} \subset \mathcal{F}_s^t$. In this situation, we clearly have $\sqrt{ab} \leq f$ with equality holding iff $\mathcal{A} = \mathcal{B} = \mathcal{F}_s^t$. Thus all we need to do is to show that one of (i) or (ii) of Proposition 3.1 holds. Let $f_a = \xi_a f$, $f_b = \xi_b f$, and let $\xi = \xi_a + \xi_b$. Then $a_1 + b_1 = f_a + f_b = \xi f$. On the other hand it follows that $\sqrt{ab} = \sqrt{a_0 b_0} = \sqrt{(1 - \xi_a)(1 - \xi_b)}f \leq \frac{(1-\xi_1)+(1-\xi_b)}{2}f = (1 - \frac{\xi}{2})f \leq (1 - \frac{\xi}{2})p^t$. Now let η be given. If $\xi < \eta$, then (ii) holds. If $\xi \geq \eta$, then (i) holds by taking γ slightly smaller than $1/2$. This completes the whole proof of Proposition 3.1. $\quad\square$

Proof of Theorem 1.3. This follows from Proposition 3.1 if $\mathcal{A}$ and $\mathcal{B}$ are shifted. (Recall that if $0 < p \leq \frac{1}{t+1}$ then $\mu_p(\mathcal{F}_0^t) \geq \mu_p(\mathcal{F}_1^t)$ with equality holding iff $p = \frac{1}{t+1}$.) If they are not shifted, then we use Lemma 2.3(iii) to get shifted families $\mathcal{A}'$ and $\mathcal{B}'$ starting from $\mathcal{A}$ and $\mathcal{B}$. By Lemma 2.3(i) they have the same p-weights as $\mathcal{A}$ and $\mathcal{B}$, and so by Proposition 3.1,

$$\sqrt{\mu_p(\mathcal{A}')\mu_p(\mathcal{B}')} = \sqrt{\mu_p(\mathcal{A})\mu_p(\mathcal{B})} \leq \mu_p(\mathcal{F}_s^t) \leq p^t \quad (s = 0, 1).$$

Moreover if both equalities hold then either $\mathcal{A}' = \mathcal{B}' = \mathcal{F}_0^t$, or $p = \frac{1}{t+1}$ and $\mathcal{A}' = \mathcal{B}' = \mathcal{F}_1^t$, and Lemma 2.3(iv) (or Lemma 2.7) gives us that either $\mathcal{A} = \mathcal{B} \cong \mathcal{F}_0^t$, or $p = \frac{1}{t+1}$ and $\mathcal{A} = \mathcal{B} \cong \mathcal{F}_1^t$. $\quad\square$

Proof of Theorem 1.4. This directly follows from Proposition 3.1 unless (ii) of Proposition 3.1 happens with $s = 1$. In this last case, we notice that

$$g(p) := \mu_p(\mathcal{F}_1^t)/\mu_p(\mathcal{F}_0^t) = (t+2)p(1-p) + p^2$$

is an increasing function of p on $(0, \frac{1}{t+1}(1 + \frac{t}{2})]$, and $g(\frac{1}{t+1}) = 1$. Thus we have

$$\sqrt{\mu_p(\mathcal{A})\mu_p(\mathcal{B})} \le \mu_p(\mathcal{F}_1^t(n)) < g\left(\frac{1}{t+1} - \epsilon\right)p^t.$$

This gives (i) of Theorem 1.4 by choosing γ so that $g(\frac{1}{t+1} - \epsilon) = 1 - \gamma\eta$. $\quad\square$

4. Results about uniform families

In this section, we prove Proposition 4.1 about k-uniform cross t-intersecting families, from which Theorems 1.1 and 1.2 will follow.

Proposition 4.1. *For every $k \ge t \ge 14$, $n \ge (t+1)k$ and $\eta \in (0, 1]$ we have the following. If $\mathcal{A}$ and $\mathcal{B}$ are shifted cross t-intersecting families in $\binom{[n]}{k}$, then one of the following holds.*

(i) $\sqrt{|\mathcal{A}||\mathcal{B}|} < (1 - \gamma^*\eta)\binom{n-t}{k-t}$, *where $\gamma^* \in (0, 1]$ depends only on t.*
(ii) $|\mathcal{A} \triangle \mathcal{F}_s^t(n, k)| + |\mathcal{B} \triangle \mathcal{F}_s^t(n, k)| < \eta|\mathcal{F}_s^t(n, k)|$, *where $s = 0$ or 1.*

If (ii) *happens then $\sqrt{|\mathcal{A}||\mathcal{B}|} \le |\mathcal{F}_s^t(n, k)|$ with equality holding iff $\mathcal{A} = \mathcal{B} = \mathcal{F}_s^t(n, k)$.*

Our proof of Proposition 4.1 closely follows the proof of Proposition 3.1. We will use k-uniform versions of the concepts of that proof, but instead of introducing another index k, we redefine our notation. In particular we let $\mathcal{F}^u$ be the family of walks from $(0, 0)$ to $(n - k, k)$ that hit the line $y = x + u$, or equivalently,

$$\mathcal{F}^u = \left\{F \in \binom{[n]}{k} : |F \cap [j]| \ge (j + u)/2 \text{ for some } j\right\}.$$

Let $\tilde{\mathcal{F}}^u, \dot{\mathcal{F}}^u$, and $\ddot{\mathcal{F}}^u$ be defined as before, but with respect to this new $\mathcal{F}^u$. Similarly, we now use $\mathcal{F}_i^t$ to mean $\mathcal{F}_i^t(n, k)$. (One can think of this redefinition as applying the function first_k to everything in the previous definitions of the families.)

4.1. Proof of Proposition 4.1: setup

Let $k \ge t \ge 14$ and $n \ge (t+1)k$ be given ($\eta \in (0, 1]$ will be given later). The following basic inequalities will be used frequently without referring to explicitly.

$$\frac{k}{n} \leq \frac{1}{t+1}, \qquad \frac{n-k}{n} \geq \frac{t}{t+1}, \qquad \frac{n}{n-k} \leq \frac{t+1}{t},$$

$$\left(\frac{n}{n-k}\right)^t \leq \left(1+\frac{1}{t}\right)^t < e, \qquad \frac{k}{n-k} \leq \frac{1}{t}, \qquad \frac{k(n-k)}{n^2} \leq \frac{t}{(t+1)^2}.$$

Let $\mathcal{A}$ and $\mathcal{B}$ be non-empty shifted cross t-intersecting families in $\binom{[n]}{k}$. Let $u = \lambda(\mathcal{A})$ and $v = \lambda(\mathcal{B})$. By Lemma 2.17 we have $u + v \geq 2t$. If $u + v \geq 2t+1$, then (i) of Lemma 2.13 gives

$$|\mathcal{A}||\mathcal{B}| \leq \binom{n}{k-u}\binom{n}{k-v} \leq \binom{n}{k-t}\binom{n}{k-t-1} <_t \binom{n-t}{k-t}^2,$$

which gives (i) of the proposition. In fact, the last inequality can be shown as follows:

$$\binom{n}{k-t}\binom{n}{k-t-1}\binom{n-t}{k-t}^{-2}$$

$$= \frac{n\cdots(n-t+1)}{(n-k+t)\cdots(n-k+1)} \frac{n\cdots(n-t+1)(k-t)}{(n-k+t+1)\cdots(n-k+1)}$$

$$= \left(\frac{n\cdots(n-t+1)}{(n-k+t)\cdots(n-k+1)}\right)^2 \frac{n}{n-k+t+1} \frac{k-t}{n}$$

$$< \left(\frac{n-t+1}{n-k+1}\right)^{2t+1} \frac{1}{t+1} < \left(1+\frac{1}{t}\right)^{2t+1} \frac{1}{t+1} < \frac{e^{2+1/t}}{t+1} <_t 1,$$

where the last inequality holds for $t \geq 8$.

So we may assume that $u + v = 2t$, and by symmetry that $u \leq v$. For later use, we also notice that $\frac{e^{2+1/t}}{t+1} <_t \frac{1}{2}$ for $t \geq 15$, while $(1+\frac{1}{t})^{2t+1}\frac{1}{t+1} < \frac{1}{2}$ is true even when $t = 14$. Thus we have

$$\binom{n}{k-t}\binom{n}{k-t-1}\binom{n-t}{k-t}^{-2} <_t \frac{1}{2} \tag{24}$$

for $t \geq 14$.

We partition $\mathcal{A}$ and $\mathcal{B}$ into families $\dot{\mathcal{A}}$, $\ddot{\mathcal{A}}$, $\tilde{\mathcal{A}}$ and $\dot{\mathcal{B}}$, $\ddot{\mathcal{B}}$, $\tilde{\mathcal{B}}$, as we do near the beginning of Section 3.1 (but relative to the k-uniform versions of $\tilde{\mathcal{F}}^u, \dot{\mathcal{F}}^u$, and $\ddot{\mathcal{F}}^u$.)

If $\dot{\mathcal{A}} = \emptyset$, then $\mathcal{A} = \ddot{\mathcal{A}} \cup \tilde{\mathcal{A}}$. Using (iii) and (i) of Lemma 2.13 we have $|\ddot{\mathcal{A}}| \leq \binom{n}{k-u-1}$, $|\tilde{\mathcal{A}}| \leq \binom{n}{k-u-1}$ and $|\mathcal{B}| \leq \binom{n}{k-v}$. Thus we get

$$|\mathcal{A}||\mathcal{B}| \leq 2\binom{n}{k-u-1}\binom{n}{k-v} \leq 2\binom{n}{k-t}\binom{n}{k-t-1} <_t \binom{n-t}{k-t}^2,$$

where the last inequality follows from (24), and this is one of the points we really need $t \geq 14$. The same holds for the case when $\dot{\mathcal{B}} = \emptyset$. Thus if $\dot{\mathcal{A}} = \emptyset$ or $\dot{\mathcal{B}} = \emptyset$ then (i) of the proposition holds.

From now on we assume that $\dot{\mathcal{A}} \neq \emptyset$ and $\dot{\mathcal{B}} \neq \emptyset$. Then Lemma 3.2 holds in our k-uniform setting as well, namely, there exist unique nonnegative integers s and s' such that $s - s' = (v - u)/2$, $\mathcal{A}_s := \dot{\mathcal{A}} \cup \ddot{\mathcal{A}} \subset \mathcal{F}_s^u$, and $\mathcal{B}_{s'} := \dot{\mathcal{B}} \cup \ddot{\mathcal{B}} \subset \mathcal{F}_{s'}^v$. It then follows from $\emptyset \neq \dot{\mathcal{B}} \subset \mathcal{F}_{s'}^v$ that

$$k \geq v + s' = v + \left(s - \frac{v - u}{2}\right) = \frac{u + v}{2} + s = t + s.$$

In summary, we may assume the following **setup**.

- $t \geq 14$, $s \geq s' \geq 0$, $k \geq t + s$, and $n \geq (t + 1)k$.
- $u + v = 2t$, $0 \leq u \leq t \leq v \leq 2t$, and $s - s' = (v - u)/2$.
- $u = t - (s - s')$ and $v = t + (s - s')$.
- $\mathcal{A} = \dot{\mathcal{A}} \cup \ddot{\mathcal{A}} \cup \tilde{\mathcal{A}} \subset \mathcal{F}^u$, $\mathcal{B} = \dot{\mathcal{B}} \cup \ddot{\mathcal{B}} \cup \tilde{\mathcal{B}} \subset \mathcal{F}^v$, $\dot{\mathcal{A}} \neq \emptyset$ and $\dot{\mathcal{B}} \neq \emptyset$.
- $\mathcal{A}_s := \dot{\mathcal{A}} \cup \ddot{\mathcal{A}} \subset \mathcal{F}_s^u$ and $\mathcal{B}_{s'} := \dot{\mathcal{B}} \cup \ddot{\mathcal{B}} \subset \mathcal{F}_{s'}^v$.

From here, our division into cases is the same as in the p-weight version.

4.2. Proof of Proposition 4.1: easy cases

In this subsection, we prove the following.

Lemma 4.2. *If $s \geq 2$ then $\sqrt{|\mathcal{A}||\mathcal{B}|} < 0.89 \binom{n-t}{k-t}$.*

Proof. Let $\bar{\mathcal{F}}_s^u := (\dot{\mathcal{F}}^u \cup \ddot{\mathcal{F}}^u) \cap \mathcal{F}_s^u$. Since $\mathcal{A} = \tilde{\mathcal{A}} \cup \mathcal{A}_s$, $\tilde{\mathcal{A}} \subset \tilde{\mathcal{F}}^u$, $\mathcal{A}_s \subset \bar{\mathcal{F}}_s^u$, we have $\mathcal{A} \subset \tilde{\mathcal{F}}^u \cup \bar{\mathcal{F}}_s^u$ and

$$|\mathcal{A}| \leq \left|\tilde{\mathcal{F}}^u\right| + \left|\bar{\mathcal{F}}_s^u\right|.$$

By (i) of Lemma 2.13 we have

$$\left|\tilde{\mathcal{F}}^u\right| = \binom{n}{k - u - 1}.$$

Since all walks in $\bar{\mathcal{F}}_s^u$ hit $(s, u + s)$, by counting the number of walks from $(0, 0)$ to $(s, u + s)$, and from $(s, u + s)$ to $(n - k, k)$, we get

$$\left|\bar{\mathcal{F}}_s^u\right| \leq \binom{u + 2s}{s}\binom{n - u - 2s}{k - u - s}.$$

Thus we have

$$|\mathcal{A}| \leq (a_1 + a_2)\binom{n - u}{k - u},$$

where

$$a_1 := f(n, k, u, s) := \binom{n}{k-u-1}\binom{n-u}{k-u}^{-1},$$

$$a_2 := g(n, k, u, s) := \binom{u+2s}{s}\binom{n-u-2s}{k-u-s}\binom{n-u}{k-u}^{-1}.$$

This rather generous estimation is enough for our purpose (if $s \geq 2$ and $t \geq 14$) as we will see. In the same way we have

$$|\mathcal{B}| \leq (b_1 + b_2)\binom{n-v}{k-v},$$

where $b_1 = f(n, k, v, s')$ and $b_2 = g(n, k, v, s')$. Notice that

$$\binom{n-u}{k-u}\binom{n-v}{k-v} \leq \cdots \leq \binom{n-(t-1)}{k-(t-1)}\binom{n-(t+1)}{k-(t+1)} \leq \binom{n-t}{k-t}^2.$$

Thus, to prove the lemma, it is enough to show that

$$(a_1 + a_2)(b_1 + b_2) < 0.89.$$

First we find bounds on the individual components.

Claim 4.3. *For $s \geq 2$, we have $a_1 < 0.195$, $b_1 < 0.528$, and $a_1 b_1 < 0.038$. For $s = 2$, we have $b_1 < 0.224$.*

Proof. Using $n \geq (t+1)k$ we have

$$a_1 = \frac{n \cdots (n-u+1)(k-u)}{(n-k+u+1)(n-k+u) \cdots (n-k+1)}$$

$$\leq \left(\frac{n-u+1}{n-k+1}\right)^u \frac{k-u}{n-k+u+1}$$

$$< \left(\frac{(t+1)k+1}{tk+1}\right)^u \frac{k}{kt} < \left(\frac{t+1}{t}\right)^u \frac{1}{t} < \frac{e^{u/t}}{t}.$$

The RHS is decreasing in t and increasing in u. So $e^{u/t}/t$ is maximized when $u = t$ (recall that $u \leq t$). Using also $t \geq 14$ we have

$$a_1 < \frac{e}{t} \leq \frac{e}{14} < 0.195.$$

In the same way we have

$$b_1 = f(n,k,v,s') < \frac{e^{v/t}}{t}.$$

Since $v \le 2t$, the RHS is maximized when $v = 2t$, and we get

$$b_1 < \frac{e^{v/t}}{t} \le \frac{e^2}{t} \le \frac{e^2}{14} < 0.528$$

in general. Further, when $s = 2$, we have $v = t + s - s' \le t + 2$, and

$$b_1 < \frac{e^{v/t}}{t} \le \frac{e^{1+\frac{2}{t}}}{t} \le \frac{e^{\frac{16}{14}}}{14} < 0.224.$$

Also we have

$$a_1 b_1 < \frac{e^{u/t}}{t}\frac{e^{v/t}}{t} = \frac{e^{(u+v)/t}}{t^2} = \frac{e^2}{t^2} \le \frac{e^2}{14^2} < 0.038. \qquad \square$$

Claim 4.4. *For $s \ge 3$, $a_2 < 0.34$, $b_2 < 2.21$ and $a_2 b_2 < 0.12$. For $s = 2$, $a_2 < 0.68$, $b_2 < 1.14$, and $a_2 b_2 < 0.47$.*

Proof. Using that $n \ge (t+1)k$ and that $k \ge t + s \ge u + s$ we have

$$a_2 \binom{u+2s}{s}^{-1} = \frac{(k-u)\cdots(k-u-s+1)(n-k)\cdots(n-k-s+1)}{(n-u)\cdots(n-u-2s+1)}$$

$$< \left(\frac{k-u}{n-u}\right)^s \left(\frac{n-k}{n-u-s}\right)^s \le \left(\frac{k}{n}\right)^s \le \left(\frac{1}{t+1}\right)^s,$$

and

$$a_2 = g(n,k,u,s) < \binom{u+2s}{s}\left(\frac{1}{t+1}\right)^s =: h(t,u,s).$$

Similarly, noting that $k \ge v + s'$, we have

$$b_2 = g(n,k,v,s') < h(t,v,s').$$

We check that $h(t,u,s)$ is decreasing in s for $s \ge 2$. In fact

$$h(t,u,s) > h(t,u,s+1)$$

is (after some computation) equivalent to

$$s^2(t-3) + s(tu + 2t - 3u - 4) + (tu - u^2 + t - 2u - 1) > 0.$$

Considering the LHS as a quadric of s, it is minimized at $s = -(\frac{u+2}{2} + \frac{1}{t-3}) < 0$. So the LHS is increasing in s for $s \geq 2$, and it suffices to check the above inequality at $s = 2$, that is,

$$3tu + 9t - u^2 - 8u - 21 > 0.$$

This is certainly true for $u = 0$. If $u \geq 1$, then, using $t \geq u$, the LHS satisfies

$$u(t - u) + 8(t - u) + (2u + 1)t - 21 \geq 3t - 21 > 0,$$

which verifies that $h(t, u, s)$ is decreasing in s.

Thus if $s \geq 3$, then, noting that $h(t, u, 3)$ is increasing in u, we have

$$h(t, u, s) \leq h(t, u, 3) \leq h(t, t, 3) = \binom{t+6}{3} \left(\frac{1}{t+1} \right)^3.$$

The derivative of the RHS is $-\frac{(2t+11)(3t+13)}{3(t+1)^4} < 0$, and $h(t, t, 3)$ is decreasing in t. Consequently, if $s \geq 3$ and $t \geq 14$, then

$$a_2 < h(14, 14, 3) < 0.34.$$

Similarly, if $s = 2$ and $t \geq 14$, then

$$a_2 < h(14, 14, 2) = 0.68.$$

Since $b_2 = h(t, v, s')$ is increasing in v and $v \leq 2t$, we have $b_2 < h(t, 2t, s')$. For $s' = 0, 1$, we have $h(t, 2t, 0) = 1$ and $h(t, 2t, 1) = 2$. Now let $s' \geq 2$. Then $h(t, 2t, s')$ is decreasing in t. In fact, we have

$$
\begin{aligned}
\frac{h(t, 2t, s')}{h(t+1, 2(t+1), s')} &= \frac{(2t + s' + 2)(2t + s' + 1)}{(2t + 2s' + 2)(2t + 2s' + 1)} \left(1 + \frac{1}{t+1} \right)^{s'} \\
&> \frac{(2t + s' + 2)(2t + s' + 1)}{(2t + 2s' + 2)(2t + 2s' + 1)} \left(1 + \frac{s'}{t+1} \right) \\
&= 1 + \frac{s'(s' - 1)}{2(t+1)(2s + 2s' + 1)} > 1.
\end{aligned}
$$

Thus, for $s' \geq 2$ and $t \geq 14$, we have

$$h(t, 2t, s') \leq h(14, 28, s') = \binom{28 + 2s'}{s'} \frac{1}{15^{s'}},$$

where the RHS is decreasing in s', and $h(14, 28, s') \leq h(14, 28, 2) < 2.21$. Consequently, for $s' \geq 0$ and $t \geq 14$, we get

$$b_2 = h(h, v, s') \leq h(14, 28, 2) < 2.21.$$

If $s = 2$, then we replace $v \leq 2t$ with $v = t + s - s' \leq t + 2$, and we get $b_2 = h(t, v, s') \leq h(t, t + 2, s')$. Since $0 \leq s' \leq s$, by computing $h(t, t + 2, s')$ for $s' = 0, 1, 2$, it turns out that the maximum is taken when $s' = 1$. Namely, for $s = 2$ and $t \geq 14$, we have

$$b_2 \leq h(t, t + 2, 1) = \frac{t + 4}{t + 1} \leq h(14, 16, 1) = 1.2.$$

Finally, using that $u + 2s = v + 2s' = t + s + s'$, we have

$$a_2 b_2 = h(t, u, s) h(t, v, s') = \binom{t + s + s'}{s} \binom{t + s + s'}{s'} \left(\frac{1}{t + 1} \right)^{s + s'}$$

$$\leq \binom{t + s + s'}{(s + s')/2}^2 \left(\frac{1}{t + 1} \right)^{s + s'} = h\left(t, t, \frac{s + s'}{2}\right)^2 \leq h(t, t, s)^2.$$

Computing this for $s = 2$ and $s = 3$ gives the required bounds on $a_2 b_2$. $\quad\square$

Now, in the case that $s \geq 3$ we have

$$(a_1 + a_2)(b_1 + b_2) = a_1 b_1 + a_1 b_2 + a_2 b_1 + a_2 b_2$$
$$< 0.038 + 0.195 \cdot 2.21 + 0.34 \cdot 0.528 + 0.12 < 0.77,$$

and when $s = 2$ we have

$$a_1 b_1 + a_1 b_2 + a_2 b_1 + a_2 b_2 < 0.038 + 0.195 \cdot 1.14 + 0.68 \cdot 0.224 + 0.47 < 0.89,$$

which completes the proof of Lemma 4.2. $\quad\square$

4.3. Proof of Proposition 4.1: a harder case

Lemma 4.5. *For $(s, s') = (1, 0)$, we have $\sqrt{|\mathcal{A}||\mathcal{B}|} <_t \binom{n-t}{k-t}$.*

Proof. Setting $(s, s') = (1, 0)$ yields that $u = t - 1$ and $v = t + 1$. We again follow the proof of Lemma 3.5, redefining the constructions of that proof by applying the first_k operation to them.

That is, let us define $D_i^{\mathcal{A}} \in \dot{\mathcal{F}}^{t-1} \cap \mathcal{F}_1^{t-1}$ $(1 \leq i \leq n - 2k + t - 1)$ and $D_j^{\mathcal{B}} \in \dot{\mathcal{F}}^{t+1} \cap \mathcal{F}_0^{t+1}$ $(1 \leq j \leq n - 2k + t + 1)$ by

$$D_i^{\mathcal{A}} := \text{first}_k\big([t - 2] \cup \{t, t + 1\} \cup \{t + 1 + i + 2\ell \in [n]: \ell = 1, 2, \ldots\}\big),$$
$$D_j^{\mathcal{B}} := \text{first}_k\big([t + 1] \cup \{t + 1 + j + 2\ell \in [n]: \ell = 1, 2, \ldots\}\big).$$

Since $\emptyset \neq \dot{\mathcal{A}} \subset \mathcal{A}_1$ and $A \to D_1^{\mathcal{A}}$ for any $A \in \dot{\mathcal{A}}$, we have $D_1^{\mathcal{A}} \in \mathcal{A}$ and $\{i: D_i^{\mathcal{A}} \in \mathcal{A}\} \neq \emptyset$. Similarly, $\{j: D_j^{\mathcal{B}} \in \mathcal{B}\} \neq \emptyset$. So the following values are well defined:

$$I := \max\{i: D_i^{\mathcal{A}} \in \mathcal{A}\}, \qquad J := \max\{j: D_j^{\mathcal{B}} \in \mathcal{B}\}.$$

The argument below is almost the same as in Section 3.3. The only difference is that all walks considered here are from $(0,0)$ to $(n-k,k)$. So we use Lemma 2.13 instead of Lemma 2.2. In each case we will show that

$$|\mathcal{A}||\mathcal{B}| = \big(|\tilde{\mathcal{A}}| + |\mathcal{A}_1|\big)\big(|\tilde{\mathcal{B}}| + |\mathcal{B}_0|\big) <_t \binom{n-t}{k-t}^2.$$

Case 1. When $I \geq 2$ and $J \geq 2$.

We divide walks in $\tilde{\mathcal{A}}$ into three types (1a), (1b), and (1c) as in Case 1 of Section 3.3. The number of walks of (1a) is $\binom{n}{k-t-J+1}$ by (i) of Lemma 2.13. For (1b) use (ii) of Lemma 2.13 and we get $\binom{n-t}{k-t} - \binom{n-t}{k-t-J+1}$. Similarly, with aid of Example 2.1, we get $t\big(\binom{n-t-1}{k-t-1} - \binom{n-t-1}{k-t-J}\big)$ for (1c). Thus we have

$$\begin{aligned}
|\tilde{\mathcal{A}}| &\leq \binom{n}{k-t-J+1} + \left(\binom{n-t}{k-t} - \binom{n-t}{k-t-J+1}\right) \\
&\quad + t\left(\binom{n-t-1}{k-t-1} - \binom{n-t-1}{k-t-J}\right) \\
&\leq \binom{n}{k-t-1} + \binom{n-t}{k-t} + t\binom{n-t-1}{k-t-1}.
\end{aligned}$$

As for $\mathcal{A}_1 \subset \bar{\mathcal{F}}_1^{t-1}$ we notice that all walks in $\bar{\mathcal{F}}_1^{t-1}$ hit $(1,t)$ without hitting $(0,t)$, and then from $(1,t)$ they never hit $y = x+1$. This gives

$$|\mathcal{A}_1| \leq t\left(\binom{n-t-1}{k-t} - \binom{n-t-1}{k-t-1}\right).$$

Any walk in $\tilde{\mathcal{B}}$ hits the line $y = x + (t+1+I)$, or hits both $(0,t+1)$ and $y = x+t+2$ (see Section 3.3 for details). Thus we get

$$|\tilde{\mathcal{B}}| \leq \binom{n}{k-t-1-I} + \binom{n-t-1}{k-t-2} \leq \binom{n}{k-t-3} + \binom{n-t-1}{k-t-2}.$$

Any walk in $\mathcal{B}_0 \subset \bar{\mathcal{F}}_0^{t+1}$ hits $(0,t+1)$ but does not hit the line $y = x+t+2$. This gives

$$|\mathcal{B}_0| \leq \binom{n-t-1}{k-t-1} - \binom{n-t-1}{k-t-2}.$$

In summary, we get

$$|\mathcal{A}| \leq \binom{n}{k-t-1} + \binom{n-t}{k-t} + t\binom{n-t-1}{k-t},$$
$$|\mathcal{B}| \leq \binom{n}{k-t-3} + \binom{n-t-1}{k-t-1}.$$

Then

$$|\mathcal{A}|\binom{n-t}{k-t}^{-1} \le \frac{n\cdots(n-t+1)(k-t)}{(n-k+t+1)\cdots(n-k+2)(n-k+1)} + 1 + t\frac{n-k}{n-t}$$

$$< \left(\frac{n}{n-k}\right)^t \frac{k}{n-k} + 1 + t\frac{n-k}{n-t},$$

$$|\mathcal{B}|\binom{n-t}{k-t}^{-1} \le \frac{n\cdots(n-t+1)(k-t)(k-t-1)(k-t-2)}{(n-k+t+3)\cdots(n-k+1)} + \frac{k-t}{n-t}$$

$$< \left(\frac{n}{n-k}\right)^t \left(\frac{k}{n-k}\right)^2 \frac{k-t-2}{n-k+1} + \frac{k-t}{n-t}.$$

We also use

$$\frac{k-t-2}{n-k+1} < \frac{k}{n-k}, \qquad \frac{k-t}{n-t} < \frac{k}{n},$$

$$\frac{n-k}{n-t}\frac{k-t-2}{n-k+1} < \frac{n-k}{n}\frac{k}{n-k}, \qquad \frac{n-k}{n-t}\frac{k-t}{n-t} < \frac{n-k}{n}\frac{k}{n}.$$

Then

$$|\mathcal{A}||\mathcal{B}|\binom{n-t}{k-t}^{-2}$$

$$\le \left(\left(\frac{n}{n-k}\right)^t \frac{k}{n-k} + 1 + t\frac{n-k}{n}\right)\left(\left(\frac{n}{n-k}\right)^t \left(\frac{k}{n-k}\right)^3 + \frac{k}{n}\right)$$

$$= \left(\left(\frac{n}{n-k}\right)^t \frac{n}{n-k}\left(\frac{k}{n}\right)^2 + \frac{k}{n} + t\frac{k(n-k)}{n^2}\right)\left(\left(\frac{n}{n-k}\right)^t \left(\frac{n}{n-k}\right)^3\left(\frac{k}{n}\right)^2 + 1\right)$$

$$< \left(\frac{e}{t(t+1)} + \frac{1}{t+1} + \frac{t^2}{(t+1)^2}\right)\left(e\frac{t+1}{t^3} + 1\right).$$

(For the first inequality, we remark that we did not estimate $|\mathcal{A}|$ and $|\mathcal{B}|$ separately. Instead, we estimated each term appeared in the expansion of $|\mathcal{A}||\mathcal{B}|$ first, then we factorized the sum of the terms afterwards.) The RHS is equal to the $g(t)$ from (16), and thus $<_t 1$ for $t \ge 7$.

Case 2. When $I = 1$.

Since walks in $\tilde{\mathcal{A}}$ hit $y = x + t$, and walks in $\mathcal{B}$ hit $y = x + t + 1$, we get

$$|\tilde{\mathcal{A}}| \le \binom{n}{k-t} \quad \text{and} \quad |\mathcal{B}| \le \binom{n}{k-t-1}.$$

As for $\mathcal{A}_1$ we look at the walks that hit $(1, t)$ without hitting $y = x + t$. Among them, we delete the walks that hit all of $(1, t-2)$, $(1, t)$, $(3, t)$, and do not hit $y = x + t - 2$ after hitting $(3, t)$. (Here we use the fact that $D_2^{\mathcal{A}} \notin \mathcal{A}$.) Thus we get

$$|\mathcal{A}_1| \leq t\left(\binom{n-t-1}{k-t} - \binom{n-t-1}{k-t-1}\right) - (t-1)\left(\binom{n-t-3}{k-t} - \binom{n-t-3}{k-t-1}\right).$$

For $\tilde{\mathcal{A}}$ and $\mathcal{B}$ we simply use the following estimation.

$$|\tilde{\mathcal{A}}|\binom{n-t}{k-t}^{-1} \leq \frac{n\cdots(n-t+1)}{(n-k+t)\cdots(n-k+1)} \leq \left(\frac{n-t+1}{n-k+1}\right)^t < \left(1+\frac{1}{t}\right)^t,$$

$$|\mathcal{B}|\binom{n-t}{k-t}^{-1} \leq \frac{n\cdots(n-t+1)}{(n-k+t)\cdots(n-k+1)}\frac{k-t}{n-k+t+1} < \left(1+\frac{1}{t}\right)^t\frac{1}{t}.$$

For $\mathcal{A}_1$ we need to estimate

$$|\mathcal{A}_1|\binom{n-t}{k-t}^{-1} \leq t\left(\frac{n-k}{n-t}\left(1-\frac{k-t}{n-k}\right)\right)$$

$$- (t-1)\left(\frac{(n-k)(n-k-1)(n-k-2)}{(n-t)(n-t-1)}\left(1-\frac{k-t}{n-k-2}\right)\right)$$

$$= t\frac{n-2k+t}{n-t} - (t-1)\frac{(n-k)(n-k-1)(n-2k+t+2)}{(n-t)(n-t-1)(n-t-2)}$$

$$< \left(t\frac{n-2k}{n} - (t-1)\frac{(n-k)^2(n-2k)}{n^3}\right) + t\left(\frac{n-2k+t}{n-t} - \frac{n-2k}{n}\right).$$

Let $p = \frac{k}{n} \leq \frac{1}{t+1}$. Then the first term of the RHS is

$$t(1-2p) - (t-1)(1-p)^2(1-2p) = (1-\alpha)\left(tq - (t-1)q^3\right),$$

where $q = 1-p$ and $\alpha = \frac{p}{q}$, and thus we can reuse (17). For the second term we note that

$$\frac{n-2k+t}{n-t} - \frac{n-2k}{n} \leq \frac{n-2k+2t}{n} - \frac{n-2k}{n} = \frac{2t}{n}.$$

Consequently we get

$$|\mathcal{A}_1|\binom{n-t}{k-t}^{-1} < \frac{t(t-1)(3t+1)}{(t+1)^3} + \frac{2t^2}{n}.$$

Finally we use $g(t)$ from (19), $g_2(t)$ from (20), and note that $n \geq (t+1)k \geq (t+1)t$ to obtain

$$|\mathcal{A}||\mathcal{B}|\binom{n-t}{k-t}^{-2} < g(t) + \frac{2t^2}{n}\left(1+\frac{1}{t}\right)^t\frac{1}{t} < g(t) + \frac{2e}{t+1} \tag{25}$$

$$< g_2(t) + \frac{2e}{t+1}. \tag{26}$$

The RHS (26) is decreasing in t for $t > 0$ and is less than 1 when $t = 20$, and using $g(19)$ instead of $g_2(19)$ we get that $g(19) + \frac{2e}{19+1} < 1$. This means that $|\mathcal{A}||\mathcal{B}| <_t \binom{n-t}{k-t}^2$ for $t \geq 19$. For the remaining cases $14 \leq t \leq 18$ we check $|\mathcal{A}||\mathcal{B}| <_t \binom{n-t}{k-t}^2$ by brute force as follows. If $n > n_0(t) := \frac{2t}{1-g(t)}(1 + \frac{1}{t})^t$ then the RHS of (25) is still < 1. For smaller $n \leq n_0(t)$ we use the trivial bounds for $|\tilde{\mathcal{A}}|, |\mathcal{A}_1|$ and $|\mathcal{B}|$ given above to get:

$$|\mathcal{A}||\mathcal{B}| < \left\{ \binom{n}{k-t} + t\left(\binom{n-t-1}{k-t} - \binom{n-t-1}{k-t-1} \right) \right.$$
$$\left. - (t-1)\left(\binom{n-t-3}{k-t} - \binom{n-t-3}{k-t-1} \right) \right\} \binom{n}{k-t-1},$$

and check that the RHS is less than 1 for all $t \leq k, (t+1)k \leq n \leq n_0(t)$ with the aid of computer. For example, in the case $t = 14$, we have $\lfloor n_0(14) \rfloor = 1023$, and we compute the RHS of the above inequality for all k and n with $14 \leq k, 15k \leq n \leq 1023$. The cases $15 \leq t \leq 18$ are similar and easier. In the end, it turns out that $|\mathcal{A}||\mathcal{B}| <_t \binom{n-t}{k-t}^2$ for all $k \geq t \geq 14$, $n \geq (t+1)k$ in Case 2.

Case 3. When $J = 1$.

Using the same reasoning as in Section 3.3, we get

$$|\mathcal{A}| \leq \binom{n}{k-t+1},$$
$$|\tilde{\mathcal{B}}| \leq \binom{n}{k-t-2},$$
$$|\mathcal{B}_0| \leq \binom{n-t-1}{k-t-1} - \binom{n-t-1}{k-t-2} - \left(\binom{n-t-3}{k-t-1} - \binom{n-t-3}{k-t-2} \right).$$

We continue to bound as follows:

$$|\mathcal{A}| \binom{n-t}{k-t}^{-1} \leq \frac{n \cdots (n-t+2)(n-t+1)}{(n-k+t-1) \cdots (n-k+1)(k-t+1)}$$
$$\leq \left(\frac{n-t+2}{n-k+1} \right)^{t-1} \frac{n-t+1}{k-t+1} < \left(1 + \frac{1}{t} \right)^{t-1} \frac{n-t}{k-t},$$
$$|\tilde{\mathcal{B}}| \binom{n-t}{k-t}^{-1} \leq \frac{n \cdots (n-t+1)}{(n-k+t+2) \cdots (n-k+3)} \frac{(k-t)(k-t-1)}{(n-k+2)(n-k+1)}$$
$$\leq \left(\frac{n-t+1}{n-k+3} \right)^t \frac{(k-t)(k-t-1)}{(n-k+2)(n-k+1)} < \left(1 + \frac{1}{t} \right)^t \left(\frac{k-t}{n-k} \right)^2,$$
$$|\mathcal{B}_0| \binom{n-t}{k-t}^{-1} \leq \frac{k-t}{n-t} \left(\left(1 - \frac{k-t-1}{n-k+1} \right) - \frac{(n-k)(n-k-1)}{(n-t-1)(n-t-2)} \left(1 - \frac{k-t-1}{n-k-1} \right) \right)$$
$$< \frac{k-t}{n-t} \left(1 - \left(\frac{n-k}{n} \right)^2 + \frac{k(n-k)}{n^2} \right).$$

(For simplicity we just threw away the first $-\frac{k-t-1}{n-k+1}$ from the last inequality, while this term was used in the p-weight version.) Finally we get

$$
\begin{aligned}
|\mathcal{A}||\mathcal{B}|\binom{n-t}{k-t}^{-2} &< \left(1+\frac{1}{t}\right)^{t-1}\left(\left(1+\frac{1}{t}\right)^t \frac{nk}{(n-k)^2}+1-\left(\frac{n-k}{n}\right)^2+\frac{k(n-k)}{n^2}\right) \\
&\leq \left(1+\frac{1}{t}\right)^{t-1}\left(\left(1+\frac{1}{t}\right)^t \frac{1+t}{t^2}+1-\left(\frac{t}{t+1}\right)^2+\frac{t}{(t+1)^2}\right) \\
&< e\left(\frac{e(t+1)}{t^2}+\frac{3t+1}{(t+1)^2}\right).
\end{aligned}
\tag{27}
$$

The RHS is decreasing in t, and less than 1 when $t = 16$. Further, (27) is less than 1 when $t = 14, 15$. Thus $|\mathcal{A}||\mathcal{B}| <_t \binom{n-t}{k-t}^2$ for $t \geq 14$. This completes the proof of Case 3, and so of Lemma 4.5. □

4.4. Proof of Proposition 4.1: extremal cases

This is the case that $s = s' \in \{0, 1\}$. Let $s \in \{0, 1\}$ and let

$$
D_i' = \text{first}_k(D_i) = \text{first}_k\big([1, t-1] \cup \{t+s, t+2s\} \cup \{t+2s+i+2j \colon j \geq 1\}\big)
$$

for $1 \leq i \leq k-t-s =: i_{\text{kmax}}$. Notice that

$$
D_{i_{\text{kmax}}}' = \text{first}_k\big([1, t-1] \cup \{t+s, t+2s\} \cup \{k+s+2j \colon j \geq 1\}\big)
$$

and

$$
D_{i_{\text{kmax}}-1}' = \text{first}_k\big([1, t-1] \cup \{t+s, t+2s\} \cup \{k+s+2j-1 \colon j \geq 1\}\big).
$$

For any $A \in \dot{\mathcal{A}} \neq \emptyset$ it is easy to check that $A \to D_1'$, and hence $D_1' \in \mathcal{A}$. Similarly $D_1' \in \mathcal{B}$. Let $I' := \max\{i \colon D_i' \in \mathcal{A}\}$ and $J' := \max\{j \colon D_j' \in \mathcal{B}\}$.

Claim 4.6. If $I' \neq i_{\text{kmax}}$, then there is $\beta = \beta(t) > 0$ such that

$$
\big|\mathcal{B} \setminus \mathcal{F}_s^t(n, k)\big| \leq (1-\beta)\big|\mathcal{F}_s^t(n, k) \setminus \mathcal{A}\big|.
$$

Proof. First we show that

$$
\big|\mathcal{F}_s^t(n, k) \setminus \mathcal{A}\big| \geq \binom{t}{s}\binom{n-2s-t-I'}{k-s-t}\frac{n-2k+t-I'}{n-2s-t-I'}
\tag{28}
$$

and

$$
\big|\mathcal{B} \setminus \mathcal{F}_s^t(n, k)\big| \leq \binom{n}{k-t-I'}.
\tag{29}
$$

Consider a walk W that hits $(s, s + t)$ and satisfies $W \to D_{I'+1}$. Since $D'_{I'+1} \notin \mathcal{A}$ we have $W \in \mathcal{F}^t_s(n, k) \setminus \mathcal{A}$. Also W must hit $Q_1 = (s, t - s)$ and $Q_2 = (s + I' + 1, s + t)$. There are $\binom{t}{s}$ ways for W to go from $(0, 0)$ to Q_1, then the next $2s + I' + 1$ steps to Q_2 are unique. From Q_2 the walk must not hit $y = x + (t - I')$. The number of such walks is equal to the number of walks from $(0, 0)$ to (x_0, y_0) that hit $y = x + c$ where $x_0 = (n - k) - (s + I' + 1)$, $y_0 = k - (s + t)$, and $c = 1$. So we can count this number using (ii) of Lemma 2.13 as follows:

$$\binom{n - 2s - t - I' - 1}{k - s - t} - \binom{n - 2s - t - I' - 1}{k - s - t - 1} = \binom{n - 2s - t - I'}{k - s - t} \frac{n - 2k + t - I'}{n - 2s - t - I'}.$$

Thus the number of walks in $\mathcal{F}^t_s(n, k) \setminus \mathcal{A}$ is at least the RHS of (28).

Next we show (29). Since $\mathrm{first}_k(\mathrm{dual}_t(D'_{I'})) \notin \mathcal{B}$, each walk in $\mathcal{B}$ hits at least one of $(0, t + s)$, $(s, t + s)$, and $y = x + (t + I')$. Since each walk that hits $(0, t + s)$ or $(s, t + s)$ is in $\mathcal{F}^t_s(n, k)$, each walk in $\mathcal{B} \setminus \mathcal{F}^t_s(n, k)$ hits $y = x + (t + I')$. This yields (29).

Now we consider a lower bound for $|\mathcal{F}^t_s(n, k) \setminus \mathcal{A}|$ based on (28). We have

$$\frac{n - 2k + t - I'}{n - t - 2s - I'} > \frac{n - 3k - s + 1}{n - k - s + 1} > \frac{n - 3k}{n - k} > \frac{(t + 1)k - 3k}{(t + 1)k - k} = \frac{t - 2}{t}.$$

We also have

$$\binom{n - t - 2s - I'}{k - t - s} \Big/ \binom{n}{k - t - s}$$

$$> \left(\frac{n - k - s - I'}{n - k + t + s} \right)^{k - t - s} = \left(1 + \frac{t + 2s + I'}{n - k - s - I'} \right)^{-(k - t - s)}$$

$$> \left(1 + \frac{t + 2 + I'}{(t - 1)(k - t - s)} \right)^{-(k - t - s)} > e^{-\frac{t + 2 + I'}{t - 1}}.$$

Thus we infer

$$(\text{RHS of (28)}) \Big/ \binom{n}{k - t - s} > t^s \frac{t - 2}{t} e^{-\frac{t + 2 + I'}{t - 1}}.$$

Finally we consider an upper bound of $|\mathcal{B} \setminus \mathcal{F}^t_s(n, k)|$ based on (29). We have

$$\binom{n}{k - t - I'} \Big/ \binom{n}{k - t - s} \leq \left(\frac{k - t - s}{n - k + t + I'} \right)^{I' - s} \leq \left(\frac{1}{t} \right)^{I' - s}.$$

Therefore, it suffices to show that

$$t^s \frac{t - 2}{t} e^{-\frac{t + 2 + I'}{t - 1}} >_t \left(\frac{1}{t} \right)^{I' - s}, \quad \text{or} \quad f(t, i) := \frac{t - 2}{t} e^{-\frac{t + 2 + i}{t - 1}} t^i >_t 1.$$

By direct computation, we have $\frac{\partial f}{\partial t} > 0$ for $t \geq 2$. Further we have $\frac{\partial f(8,i)}{\partial i} > 0$ for $i \geq 1$, and $f(8,1) > 1.2$. Hence, $f(t,i) >_t 1$ for every $t \geq 8$ and $i \geq 1$. $\square$

Let $f = |\mathcal{F}_s^t(n,k)|$, $a = |\mathcal{A}|$, $a_0 = |\mathcal{A} \cap \mathcal{F}_s^t(n,k)|$, $a_1 = |\mathcal{A} \triangle \mathcal{F}_s^t(n,k)|$, $a_f = |\mathcal{A} \setminus \mathcal{F}_s^t(n,k)|$, and $f_a = |\mathcal{F}_s^t(n,k) \setminus \mathcal{A}|$. Define b, b_0, b_1, b_f, f_b similarly. The proof of the next lemma is identical to that of Lemma 3.8 (use Claim 4.6 in place of Claim 3.7).

Lemma 4.7. *Let $\eta > 0$ be given. If $I' \neq i_{\mathrm{kmax}}$, then one of the following holds.*

(i) $\sqrt{ab} < (1 - \frac{\beta\eta}{4})f$, *where $\beta \in (0,1]$ depends only on t.*
(ii) $a_1 + b_1 < \eta f$ *and* $\sqrt{ab} < f$.

Finally we finish the proof of Proposition 4.1. If $I \neq i_{\mathrm{kmax}}$, then one of (i) or (ii) of Proposition 4.1 holds by Lemma 4.7. (In this case we always have $\sqrt{ab} < f$.) The same holds for the case $J \neq i_{\mathrm{kmax}}$.

Consequently we may assume that $I' = J' = i_{\mathrm{kmax}}$. Since $I' = i_{\mathrm{kmax}}$ we have $D'_{i_{\mathrm{kmax}}} \in \mathcal{A}$. Then $C := [k+s+1] \setminus \{t+s, t+2s\} \notin \mathcal{B}$ because $|D'_{i_{\mathrm{kmax}}} \cap C| = t-1$. Thus all walks B in $\mathcal{B}$ satisfy $B \nrightarrow C$, and $\mathcal{B} \subset \mathcal{F}_s^t(n,k)$ follows. Similarly, $J' = i_{\mathrm{kmax}}$ yields $\mathcal{A} \subset \mathcal{F}_s^t(n,k)$. Thus we have $ab \leq f^2$ with equality holding iff $\mathcal{A} = \mathcal{B} = \mathcal{F}_s^t(n,k)$. Now we show that one of (i) or (ii) of Proposition 4.1 holds. Let $f_a = \xi_a f$, $f_b = \xi_b f$, and let $\xi = \xi_a + \xi_b$. Then $a_1 + b_1 = f_a + f_b = \xi f$. On the other hand it follows that $\sqrt{ab} = \sqrt{a_0 b_0} = \sqrt{(1-\xi_a)(1-\xi_b)} f \leq \frac{(1-\xi_1)+(1-\xi_b)}{2} f = (1 - \frac{\xi}{2})f \leq (1 - \frac{\xi}{2})\binom{n-t}{k-t}$. Let η be given. If $\xi < \eta$, then (ii) holds. If $\xi \geq \eta$, then (i) holds by taking γ^* slightly smaller than $1/2$. This completes the proof of Proposition 4.1. $\square$

Proof of Theorem 1.1. This follows from Proposition 3.1 if $\mathcal{A}$ and $\mathcal{B}$ are shifted. (Recall that if $n > (t+1)k$ then $|\mathcal{F}_0^t(n,k)| > |\mathcal{F}_1^t(n,k)|$.) If they are not shifted, then let $\mathcal{A}'$ and $\mathcal{B}'$ be shifted families we get from shifting $\mathcal{A}$ and $\mathcal{B}$. Then the result holds for $\mathcal{A}'$ and $\mathcal{B}'$. By Lemma 2.6 the same is true for $\mathcal{A}$ and $\mathcal{B}$, yielding the theorem. $\square$

Proof of Theorem 1.2. This follows from Proposition 4.1 unless (ii) of Proposition 4.1 happens with $s = 1$. In this last case, we have $\sqrt{|\mathcal{A}||\mathcal{B}|} \leq |\mathcal{F}_1^t(n,k)|$. Let $p := k/n$. We will show that

$$|\mathcal{F}_1^t(n,k)|/|\mathcal{F}_0^t(n,k)| < (t+2)p(1-p) + p^2 =: g(p). \tag{30}$$

Then, as in the proof of Theorem 1.4, we get (i) of Theorem 1.2 by choosing γ so that $g(\frac{1}{t+1+\delta}) = 1 - \gamma\eta$, and this completes the proof.

Now noting that the LHS of (30) is

$$\frac{k-t}{(n-t)(n-t-1)}\big((t+2)(n-k) - (k-t-1)\big),$$

we can rearrange (30) as follows:

$$f(p) := \big((t+2) - p(t+1)\big)n^2 - (t+1)(t+2)n + t(t+1)^2 > 0.$$

Since $p \le \frac{1}{t+1+\delta}$ we have $f(p) > f(\frac{1}{t+1})$. Then $f(\frac{1}{t+1}) > 0$ is equivalent to $n^2 - (t+2)n + t(t+1) > 0$, which is certainly true for $n \ge (t+1)k \ge t(t+1)$. □

We proved Proposition 3.1 for $p \le \frac{1}{t+1}$, but our proof works for $p \le \frac{1}{t+1-\epsilon}$ as well, where $\epsilon > 0$ is a sufficiently small constant depending on t only. To see this we just notice that the functions used to bound the p-weights of families are continuous as functions of p. (This is not surprising. In fact it seems very likely that Proposition 3.1 holds for $p \le \frac{2}{t+3+\delta}$, where $\delta > 0$ is any given constant.) In the same way, one can verify that Proposition 4.1 is true for $n \ge (t+1-\epsilon)k$ as well, where $\epsilon > 0$ is a sufficiently small constant depending on t only. Thus the upper bound for $|\mathcal{A}||\mathcal{B}|$ in Theorem 1.1 is also true even if we replace the condition $n \ge (t+1)k$ with $n \ge (t+1-\epsilon)k$. If k is sufficiently large for fixed t, then $(t+1)(k-t+1) > (t+1-\epsilon)k$. Namely we have the following.

Theorem 4.8. *For every $t \ge 14$ there is some k_0 such that for every $k > k_0$ and $n \ge (t+1)(k-t+1)$ we have the following. If $\mathcal{A} \subset \binom{[n]}{k}$ and $\mathcal{B} \subset \binom{[n]}{k}$ are cross t-intersecting, then*

$$|\mathcal{A}||\mathcal{B}| \le \binom{n-t}{k-t}^2$$

with equality holding iff $\mathcal{A} = \mathcal{B} \cong \mathcal{F}_0^t(n,k)$, or $n = (t+1)(k-t+1)$ and $\mathcal{A} = \mathcal{B} \cong \mathcal{F}_1^t(n,k)$.

5. An application to integer sequences

As an application of Theorem 1.3 we consider families of t-intersecting integer sequences, see e.g., [8]. Let n, m, t be positive integers with $m \ge 2$ and $n \ge t$. Then $\mathcal{H} \subset [m]^n$ is considered to be a family of integer sequences $(a_1, \ldots, a_n)$, $1 \le a_i \le m$. We say that $\mathcal{H}$ is t-intersecting if any two sequences intersect in at least t positions, more precisely, $\#\{i : a_i = b_i\} \ge t$ holds for all $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathcal{H}$. To relate a family of sequences with a family of subsets, let us define an obvious surjection $\sigma : [m]^n \to 2^{[n]}$ by $\sigma((a_1, \ldots, a_n)) = \{i : a_i = 1\}$. Then

$$\mathcal{H}_i^t(n) := \big\{ a \in [m]^n : \sigma(a) \in \mathcal{F}_i^t(n) \big\}$$

is a t-intersecting family of integer sequences of size

$$\big|\mathcal{H}_i^t(n)\big| = m^n \mu_{\frac{1}{m}}\big(\mathcal{F}_i^t(n)\big).$$

It is known from [2,10,3] that if $r = \lfloor \frac{t-1}{m-2} \rfloor$, $n \ge t + 2r$, and $\mathcal{H} \subset [m]^n$ is a family of t-intersecting integer sequences, then

$$|\mathcal{H}| \le \big|\mathcal{H}_r^t(n)\big|. \tag{31}$$

Observe that $|\mathcal{H}_0^t(n)| = m^{n-t}$. We extend (31) in the case of $r = 0$ to cross t-intersecting families of integer sequences. We say that $\mathcal{A}, \mathcal{B} \subset [m]^n$ are cross t-intersecting if $\#\{i\colon a_i = b_i\} \geq t$ for all $(a_1, \ldots, a_n) \in \mathcal{A}$ and $(b_1, \ldots, b_n) \in \mathcal{B}$. Two such families are called isomorphic, denoted $\mathcal{A} \cong \mathcal{B}$, if there are permutations $f_1, \ldots, f_n$ of $[m]$ and a permutation g of $[n]$ such that

$$\{(f_1(a_1), \ldots, f_n(a_n))\colon (a_1, \ldots, a_n) \in \mathcal{A}\} = \{(b_{g(1)}, \ldots, b_{g(n)})\colon (b_1, \ldots, b_n) \in \mathcal{B}\}.$$

Using Theorem 1.3 we prove a conjecture posed in [26] as follows.

Theorem 5.1. *Let $t \geq 14$, $m \geq t + 1$ and $n \geq t$. If $\mathcal{A}$ and $\mathcal{B}$ are cross t-intersecting families of integer sequences in $[m]^n$, then $|\mathcal{A}||\mathcal{B}| \leq (m^{n-t})^2$. Equality holds iff either $\mathcal{A} = \mathcal{B} \cong \mathcal{H}_0^t(n)$, or $m = t + 1$ and $\mathcal{A} = \mathcal{B} \cong \mathcal{H}_1^t(n)$.*

To prove Theorem 5.1 we need some more preparation. For $\mathcal{H} \subset [m]^n$, $j \in [n]$ and $c \in [m]$, define another shifting operation $S_j^c(\mathcal{H}) = \{S_j^c(a)\colon a \in \mathcal{H}\} \subset [m]^n$ as follows. For $a = (a_1, \ldots, a_n)$ let $S_j(a_1, \ldots, a_n) := (b_1, \ldots, b_n)$ where $b_\ell = a_\ell$ for $\ell \in [n] \setminus \{j\}$ and $b_j = 1$. Then let $S_j^c(a) = S_j(a)$ if $a_j = c$ and $S_j(a) \notin \mathcal{H}$, otherwise let $S_j^c(a) = a$. Namely, by $S_j^c(a)$, we replace a_j with 1 if $a_j = c$, but we do this replacement only if the resulting sequence is not in the original family $\mathcal{H}$. We say that $\mathcal{H}$ is shifted if $S_j^c(\mathcal{H}) = \mathcal{H}$ for all $j \in [n]$ and $c \in [m]$.

Lemma 5.2. *For $\mathcal{A}, \mathcal{B} \subset [m]^n$, $j, t \in [n]$, and $c \in [m]$, we have the following.*

(i) *$|S_j^c(\mathcal{A})| = |\mathcal{A}|$.*
(ii) *If $\mathcal{A}$ and $\mathcal{B}$ are cross t-intersecting families, then $S_j^c(\mathcal{A})$ and $S_j^c(\mathcal{B})$ are cross t-intersecting families as well.*
(iii) *Starting from $\mathcal{A}$ and $\mathcal{B}$ we obtain shifted families of sequences by repeatedly shifting two families simultaneously finitely many times.*
(iv) *Let $m \geq 3$, and let ℓ be chosen so that $\max_i |\mathcal{H}_i^t(n)| = |\mathcal{H}_\ell^t(n)|$. If $\mathcal{A}$ and $\mathcal{B}$ are cross t-intersecting families with $S_j^c(\mathcal{A}) = S_j^c(\mathcal{B}) = \mathcal{H}_\ell^t(n)$, then $\mathcal{A} = \mathcal{B} \cong \mathcal{H}_\ell^t(n)$.*
(v) *If $\mathcal{A}$ and $\mathcal{B}$ are shifted cross t-intersecting, then $\sigma(\mathcal{A})$ and $\sigma(\mathcal{B})$ are cross t-intersecting families of subsets in $2^{[n]}$.*

One can prove the above (i)–(iv) similarly as the proof of Lemmas 2.3 and 2.6. See [26] for the proof of (v). We mention that (ii) is due to Kleitman [17], and (v) is observed by Frankl and Füredi [8].

Proof of Theorem 5.1. Let $\mathcal{A}$ and $\mathcal{B}$ be cross t-intersecting families in $[m]^n$, and let $\mathcal{A}'$ and $\mathcal{B}'$ be corresponding shifted families guaranteed by Lemma 5.2. By letting $\mathcal{F} := \sigma(\mathcal{A}) \subset 2^{[n]}$ we have

$$|\mathcal{A}| = |\mathcal{A}'| \leq \sum_{x \in \mathcal{F}} (m-1)^{n-|x|} = m^n \mu_{\frac{1}{m}}(\mathcal{F}). \tag{32}$$

Similarly $|\mathcal{B}| = m^n \mu_{\frac{1}{m}}(\mathcal{G})$, where $\mathcal{G} := \sigma(\mathcal{B})$. Since $\mathcal{F}$ and $\mathcal{G}$ are cross t-intersecting families it follows from Theorem 1.3 that

$$\mu_{\frac{1}{m}}(\mathcal{F})\mu_{\frac{1}{m}}(\mathcal{G}) \leq (1/m)^{2t}. \tag{33}$$

By (32) and (33) we have

$$|\mathcal{A}||\mathcal{B}| \leq \left(m^n\right)^2 \mu_{\frac{1}{m}}(\mathcal{F})\mu_{\frac{1}{m}}(\mathcal{G}) \leq \left(m^n\right)^2 (1/m)^{2t} = \left(m^{n-t}\right)^2.$$

Now suppose that $|\mathcal{A}||\mathcal{B}| = (m^{n-t})^2$. Then we need equality in (33). By Theorem 1.3 we have $\mathcal{F} = \mathcal{G} \cong \mathcal{F}_0^t(n)$, or $m = t+1$ and $\mathcal{F} = \mathcal{G} \cong \mathcal{F}_1^t(n)$. We also need equality in (32). By the definition of $\mathcal{F}$ and $\mathcal{G}$ we have $\mathcal{A}_1' = \mathcal{B}_2' \cong \mathcal{H}_0^t(n)$, or $m = t+1$ and $\mathcal{A}_1' = \mathcal{B}_2' \cong \mathcal{H}_1^t(n)$. By this together with Lemma 5.2(v) we can conclude that $\mathcal{A} = \mathcal{B} \cong \mathcal{H}_0^t(n)$, or $m = t+1$ and $\mathcal{A} = \mathcal{B} \cong \mathcal{H}_1^t(n)$. This completes the proof of Theorem 5.1. □

Acknowledgments

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