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On cross-intersecting families

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Dedicated to the memory of Professor Frolík.

Abstract

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Let $n \geq t \geq 1$ be integers. Let $\mathcal{F}$, $\mathcal{G}$ be families of subsets of the n-element set X. They are called cross t-intersecting if $|F \cap G| \geq t$ holds for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$. If $\mathcal{F} = \mathcal{G}$ then $\mathcal{F}$ is called t-intersecting. Let $m(n, t)$ denote the maximum possible cardinality of a t-intersecting family. Our main result says that if $\mathcal{F}$, $\mathcal{G}$ are cross s-intersecting with $|\mathcal{G}| \leq |\mathcal{F}| \leq m(n, t)$, $t \leq s$, then $|\mathcal{F}| + |\mathcal{G}| \leq m(n, t) + m(n, 2s - t)$ holds and this is best possible.

1. Introduction

Let X be an n-element set. A family $\mathcal{F}$ of subsets of X, i.e., $\mathcal{F} \subset 2^X$ is called t-intersecting if $|F \cap F^1| \geq t$ holds for all $F, F^1 \in \mathcal{F}$.

For $n + t$ even define

$$\mathcal{F}(n, t) = \{F \subset X: |F| \geq (n + t)/2\}.$$

Fix an element $x \in X$ and for $n + t$ odd define

$$\mathcal{F}(n, t) = \{F \subset X: |F \cap (X - \{x\})| \geq (n - 1 + t)/2\}.$$

Clearly, $\mathcal{F}(n, t)$ is t-intersecting. Recall the following classical result.

Katona Theorem [1]. *If $\mathcal{F} \subset 2^X$ is t-intersecting, then $|\mathcal{F}| \leq |\mathcal{F}(n, t)|$ holds. Moreover, for $t \geq 2$ equality holds only if $\mathcal{F} = \mathcal{F}(n, t)$.*

For a family $\mathcal{F} \subset 2^X$ and an integer $s \geq 1$ define $\partial_s(\mathcal{F}) = \{G \subset X: \exists F \in \mathcal{F}, |F \Delta G| \leq s\}$, where $F \Delta G = (F - G) \cup (G - F)$ is the symmetric difference of F and G.

Another classical result is the isoperimetric theorem of Harper which we state here in a slightly weaker, but more convenient form. For a short proof see [2].

Isoperimetric Theorem (Harper [4]). *Let* $\mathcal{F} \subset 2^X$ *satisfy* $|\mathcal{F}| = \binom{x}{b} + \sum_{i=b+1}^{n} \binom{n}{i}$ *where* $1 \leq b \leq n$ *is an integer and* $b \leq x \leq n$ *is a real number. Then*

$$|\partial_s(\mathcal{F})| \geq \binom{x}{b-s} + \sum_{i=b-s+1}^{n} \binom{n}{i}. \tag{1}$$

The families $\mathcal{F}$, $\mathcal{G} \subset 2^X$ are called cross t-intersecting if $|F \cap G| \geq t$ holds for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$. Such families often occur in inductive proofs and it is important to have good bounds on their sizes. The first such result was obtained by Ahlswede and Katona, who used the isoperimetric theorem to prove the following.

Theorem 1. *Suppose that* $\mathcal{F}$, $\mathcal{G} \subset 2^X$ *are cross t-intersecting. Then* $\min\{|\mathcal{F}|, |\mathcal{G}|\} \leq |\mathcal{F}(n, t)|$. *Moreover for* $t \geq 2$ *equality holds if and only if* $\mathcal{F} = \mathcal{G} = \mathcal{F}(n, t)$.

Improving earlier results of Rődl and the author [3], Matsumoto and Tokushige [6] proved the following stronger result.

Theorem 2. *Suppose that* $\mathcal{F}$, $\mathcal{G} \subset 2^X$ *are cross t-intersecting. Then* (i) *or* (ii) *holds*:

(i) $|\mathcal{F}||\mathcal{G}| \leq |\mathcal{F}(n, t)|^2$,

(ii) $n + t$ *is odd and* $|\mathcal{F}||\mathcal{G}| \leq |\mathcal{F}(n, t-1)||\mathcal{F}(n, t+1)|$.

Estimating $|\mathcal{F}| + |\mathcal{G}|$ turns out to be almost trivial. Namely, if $\mathcal{F}$ and $\mathcal{G}$ are cross t-intersecting, then $\{X - F: F \in \mathcal{F}\} \cap \mathcal{G} = \emptyset$ must hold, implying $|\mathcal{F}| + |\mathcal{G}| \leq 2^n$. On the other hand taking $|\mathcal{F}| = 2^X$ and $\mathcal{G} = \emptyset$ shows that this bound is best possible.

However, recently Sali [7] found the following interesting result.

Theorem 3. *Suppose that* $\mathcal{F}$, $\mathcal{G} \subset 2^X$ *are cross s-intersecting. Moreover, both* $\mathcal{F}$ *and* $\mathcal{G}$ *are t-interesting,* $1 \leq t \leq s \leq n$. *Then*

$$|\mathcal{F}| + |\mathcal{G}| \leq |\mathcal{F}(n, t)| + |\mathcal{F}(n, 2s - t)|. \tag{2}$$

In view of the Katona Theorem the following result, which is the main result of the present paper, is stronger.

Theorem 4. *Suppose that* $1 \leq t \leq s \leq n$ *and* $\mathcal{F}$, $\mathcal{G} \subset 2^X$ *are cross s-intersecting families satisfying* $|\mathcal{G}| \leq |\mathcal{F}| \leq |\mathcal{F}(n, t)|$. *Then*

$$|\mathcal{F}| + |\mathcal{G}| \leq |\mathcal{F}(n, t)| + |\mathcal{F}(n, 2s - t)|. \tag{3}$$

Our proof, which is based on the isoperimetric theorem, is different and shorter than Sali's argument.

2. Proof of Theorem 4 Let us start with an easy inequality.

Lemma 1. *Suppose that* $1 \leq t < s \leq n$. *Then*

$$|\mathcal{F}(n, t)| + |\mathcal{F}(n, 2s - t)| \geq |\mathcal{F}(n, n + 1)| + |\mathcal{F}(n, 2s - t - 1)| \tag{4}$$

holds, with equality if and only if $s = t + 1$ *and* $n + t$ *is odd.*

Proof. Suppose first that $n + t$ is odd, $n + t = 2a + 1$. From the definition it is clear that the RHS of (4) is

$$\sum_{a+1 \leq i \leq n} \binom{n}{i} + \sum_{a+s-t \leq j \leq n} \binom{n}{j}.$$

Similarly, the LHS can be written as

$$\binom{n-1}{a} + \sum_{a+1 \leq i \leq n} \binom{n}{i} + \binom{n-1}{a+s-t} + \sum_{a+s-t+1 \leq j \leq n} \binom{n}{j}.$$

Thus (4) is equivalent to

$$\binom{n-1}{a} \geq \binom{n}{a+s-t} - \binom{n-1}{a+s-t} = \binom{n-1}{a+s-t-1},$$

which holds because of $s \geq t + 1$ and $a \geq n/2$. Moreover, the inequality is strict unless $s = t + 1$.

Suppose next that $n + t = 2a$. By very similar computation (4) turns out to be equivalent to

$$\binom{n-1}{a-1} > \binom{n-1}{a+s-t-1}$$

which is true because of $a > n/2$ and $s \geq t + 1$. □

In view of Lemma 1, when proving Theorem 4 we may assume that $|\mathcal{F}(n, t + 1) < |\mathcal{F}| \leq |\mathcal{F}(n, t)|$ holds.

Proof of Theorem 4. Set $a = \lfloor (n + t)/2 \rfloor$. Then we may assume that

$$|\mathcal{F}| = \binom{x}{a} + \sum_{a+1 \leq i \leq n} \binom{n}{i}$$

holds for some real number $a \leq x \leq n$.

Let us observe that $\partial_{s-1}\mathcal{F} \cap \{X - G: G \in \mathcal{G}\} = \emptyset$. Indeed, otherwise for some $F \in \mathcal{F}$ one has $|F \,\Delta\, (X - G)| < s$, yielding $|F \cap G| < |(X - G) \cap G| + s = s$, a contradiction.

We distinguish two cases.

Case (a): $n+t=2a$.

From (1) we infer

$$|\partial_{s-1}\mathcal{F}| \geq \binom{x}{a-s+1} + \sum_{a-s+2\leq i\leq n} \binom{n}{i},$$

consequently

$$|\mathcal{G}| \leq \sum_{0\leq i\leq a-s+1} \binom{n}{i} - \binom{x}{a-s+1} = \sum_{(n+2s-t-2)/2\leq j\leq n} \binom{n}{j} - \binom{x}{a-s+1}.$$

Equivalently,

$$|\mathcal{G}| \leq |\mathcal{F}(n, 2s-t)| + \binom{n}{a-s+1} - \binom{x}{a-s+1}.$$

Since $|\mathcal{F}| = |\mathcal{F}(n, t)| - \binom{n-1}{a} + \binom{x}{a}$, we have to prove that

$$\binom{n-1}{a} - \binom{x}{a} \geq \binom{n}{a-s+1} - \binom{x}{a-s+1}$$

or equivalently

$$\binom{n-1}{a} - \binom{n}{a-s+1} \geq \binom{x}{a} - \binom{x}{a-s+1}. \tag{5}$$

The RHS is negative unless $x \geq 2a-s+1$, so we may assume that this inequality holds. Then, however

$$\binom{x}{a} - \binom{x}{a-s+1}$$
$$= \binom{x}{a-s+1}\left(\frac{(x-a+1)(x-a+2)\cdots(x-a+s-1)}{(a-s+2)\cdots a} - 1\right)$$

which is an increasing function of x, proving (5). It follows also, that the inequality is strict unless $x=n$, i.e., $\mathcal{F} = \mathcal{F}(n, t)$ holds.

Case (b): $n+t=2a+1$.

In this case $x \leq n-1$ holds by assumption and the same argument gives

$$|\mathcal{G}| \leq \sum_{0\leq j\leq a-s+1} \binom{n}{j} - \binom{x}{a-s+1}$$
$$= |\mathcal{F}(n, 2s-t)| + \binom{n}{a-s+1} - \binom{x}{a-s+1}.$$

Now the desired inequality becomes

$$\binom{n-1}{a} - \binom{n-1}{a-s+1} \geq \binom{x}{a} - \binom{x}{a-s+1},$$

which holds again by monotonicity, in view of $x \leq n-1$. The inequality is strict again unless $x = n-1$. This concludes the proof. □

Remark. One can prove that equality holds in (3) only if $\mathscr{F} = \mathscr{F}(n, t)$, $\mathscr{G} = \mathscr{G}(n, 2s - t)$ or if $\mathscr{F} = \mathscr{F}(n, t+1)$, $\mathscr{G} = \mathscr{F}(n, 2s - t - 1)$, $s = t + 1$ and $n + t$ is odd.

References

[1] R. Ahlswede and G.O.H. Katona, On the geometry of Hamming spaces, Discrete Math. 17 (1977) 1–22.

[2] P. Frankl, A lower bound on the size of a complex generated by an antichain, Discrete Math. 76 (1989) 51–56.

[3] P. Frankl and V. Rődl, Forbidden intersections, Trans AMS 300 (1987) 259–286.

[4] L. H. Harper, Optimal numberings and isoperimetric problems on graphs, J. Combin. Theory 1 (1966) 385–394.

[5] G.O.H. Katona, Intersection theorems for finite sets, Acta Math. Hungr. 15 (1964) 329–337.

[6] M. Matsumoto and N. Tokushige, A generalization of the Katona theorem for cross t-intersecting families, Graphs Combin. 5 (1989) 159–171.

[7] A. Sali, An Erdős–Ko–Rado theorem for posets, preprint.