JOURNAL OF COMBINATORIAL THEORY, Series B 53, 195–234 (1991)

Multiply-Intersecting Families

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Communicated by the Editors

Received February 20, 1989

Intersection problems occupy an important place in the theory of finite sets. One of the central notions is that of a r-wise t-intersecting family, that is, a collection $F_1, ..., F_m$ of distinct subsets of the n-element set X such that $|F_{i_1} \cap \cdots \cap F_{i_r}| \geqslant t$ holds for all choices of $1 \leqslant i_1 < \cdots < i_r \leqslant m$. What is the maximal size $m = m(n, r, t)$ of a r-wise t-intersecting family? Taking all subsets containing a fixed t-element set shows that $m(n, r, t) \geqslant 2^{n-t}$ holds for all $n \geqslant t \geqslant 0$. One of the main results of the paper is that $m(n, r, t) = 2^{n-t}$ holds if and only if $n < r + t$ or $t \leqslant 2^r - r - 1$ with the possible (but unlikely) exception of the case $(r, t) = (3, 4)$. Many more best possible results are obtained. Another one is the following. Suppose that $\mathscr{G}_1, ..., \mathscr{G}_r$ are cross t-intersecting (see definition in the paper) and $t \leqslant 2^r - r - 2$, then $|\mathscr{G}_1| \, |\mathscr{G}_2| \cdot \ldots \cdot |\mathscr{G}_r| \leqslant 2^{r(n-t)}$.

1. INTRODUCTION

A family $\mathscr{F}$ of subsets of $[n] = \{1, 2, ..., n\}$ is called *r-wise t-intersecting* if $|F_1 \cap \cdots \cap F_r| \geqslant t$ holds for all $F_1, ..., F_r \in \mathscr{F}$. Such families were widely investigated we refer the interested reader to the surveys [F1, Fü].

Let $m(n, r, t)$ denote the maximum of $|\mathscr{F}|$ over all $\mathscr{F} \subset 2^{[n]}$, $\mathscr{F}$ r-wise t-intersecting. If $\mathscr{F}$ is maximal then necessarily $F \subset G \subset [n]$ and $F \in \mathscr{F}$ imply $G \in \mathscr{F}$, a family with this property is called a *co-complex* or *filter*. Recall that $\mathscr{G}$ is a *complex* or *ideal* if $H \subset G \in \mathscr{G}$ implies $H \in \mathscr{G}$.

Note that $m(n, r, t) = 0$ for $t > n$ and we usually assume $n \geqslant t$, $r \geqslant 2$. Even for $t \leqslant n < t + r$ trivially $m(n, r, t) = 2^{n-t}$ holds.

For $0 \leqslant i \leqslant (n-t)/r$ define the families

$$\mathscr{A}_i = \mathscr{A}_i(n, r, t) = \{A \subset [n] : |A \cap [t + ri]| \geqslant t + (r-1)\, i\}.$$

It is easy to see that $\mathscr{A}_i$ is r-wise t-intersecting, $|\mathscr{A}_0| = 2^{n-t}$. The basic open problem is the following.

Conjecture 1.1 [F2].

$$m(n, r, t) = \max\{|\mathscr{A}_i| : 0 \leqslant i \leqslant (n-t)/r\}. \tag{1.1}$$

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0095-8956/91 $3.00

In [F2], (1.1) was proved for $t \leqslant 2^r \cdot r/150$. For $r=2$ it follows from a classical result of Katona [K]. Actually, $m(n, 2, t) = \mathscr{A}_{\lfloor (n-t)/2 \rfloor}$ holds.

In [F1] it was shown that

$$m(n, r, t) = 2^{n-t} \qquad \text{holds for} \quad r \geqslant t. \tag{1.2}$$

This result is applied there to give a very short proof of the following important result of Brace and Daykin, which was also discovered by Kleitman [P].

THEOREM 1.2 [BD]. *Suppose that $\mathscr{F} \subset 2^{[n]}$ is r-wise 1-intersecting, $r \geqslant 3$, and satisfies $\cap \mathscr{F} = \varnothing$. Then $|\mathscr{F}| \leqslant |\mathscr{A}_1(n, r, 1)|$ with equality holding if and only if $\mathscr{F}$ is isomorphic to $\mathscr{A}_1(n, r, 1)$.*

Let us note that $|\mathscr{A}_0(n, r, t)| \gtreqqless |\mathscr{A}_1(n, r, t)|$ holds according as

$$2^r - r - 1 \gtreqqless t.$$

We have the following:

Conjecture 1.3. Suppose that $\mathscr{F} \subset 2^{[n]}$ is r-wise t-intersecting with $|\cap \mathscr{F}| < t$. Suppose further that $t \leqslant 2^r - r - 1$. Then $|\mathscr{F}| \leqslant |\mathscr{A}_1(n, r, t)|$ and equality holds if and only if $\mathscr{F}$ is isomorphic to $\mathscr{A}_1(n, r, t)$.

In this paper we prove this conjecture for all but six choices of (r, t).

Let us mention the trivial inequalities

$$m(n, r, t) \geqslant 2m(n-1, r, t), \tag{1.3}$$

$$m(n, r, t) \geqslant m(n-1, r, t-1). \tag{1.4}$$

Let $\alpha(r)$ be the unique positive root of $(x^r - 2x + 1)/(x-1)$.

It is easy to see that

$$\frac{1}{2} + \frac{1}{2^{r+1}} < \alpha(r) < \frac{1}{2} + \frac{1}{2^r} \qquad \text{holds for} \quad r \geqslant 3. \tag{1.5}$$

THEOREM 1.4.

$$m(n, r, t) \leqslant 2m(n-1, r, t-1)\, \alpha(r) \qquad \textit{holds for} \quad n \geqslant 1, \qquad t \geqslant 1. \tag{1.6}$$

Inequality (1.6) complements (1.4) in a certain way.

COROLLARY 1.5.

$$m(n+s, r, t+s) \leqslant 2^s \alpha(r)^s\, m(n, r, t). \tag{1.7}$$

Looking at the *dual family* $\mathscr{F}^c = \{[n] - F: F \in \mathscr{F}\}$ of a r-wise t-intersecting family $\mathscr{F}$, we see that any r of its members have union of size at most $n-t$. We call this property *dually r-wise t-intersecting*. For $t=1$, 1 is omitted. For $i \in [n]$ define

$$\mathscr{F}(i) = \{F - \{i\} : i \in F \in \mathscr{F}\}.$$

Similarly, $\mathscr{F}(\bar{i}) = \{F \in \mathscr{F} : i \notin F\}$. We use also the notation $[i, j]$ for $\{i, i+1, ..., j\}$. The *minimum degree* $\delta(\mathscr{F})$ is defined by $\delta(\mathscr{F}) = \min_i |\mathscr{F}(i)|$.

Note that the dual family of $\mathscr{A}_1(n, r, 1)$ is dually r-wise intersecting and has minimum degree 2^{n-r-1}.

Conjecture 1.6 (Daykin [D]). If $\mathscr{G}$ is dually r-wise intersecting and $\bigcup \mathscr{G} = [n]$ then $\delta(\mathscr{G}) \leqslant 2^{n-r-1}$ holds for $r \geqslant 3$.

In [DF] this conjecture was proved for $r \geqslant 25$ (for some partial results see [D, BSW]).

THEOREM 1.7. *Conjecture* 1.6 *holds for all* $r \geqslant 5$.

This leaves two cases, $r=3$ and 4 open. Especially the case $r=3$ seems to need new methods.

Making new conjectures is easier.

Conjecture 1.8. If $\mathscr{G}$ is dually r-wise t-intersecting then $\delta(\mathscr{G}) \leqslant 2^{n-r-t}$ holds for $t \leqslant 2^r - 2r$.

Note that—if true—Conjecture 1.8 is best possible, namely $\mathscr{A}_1(n, r, t)^c = \{[n] - A: A \in \mathscr{A}_1(n, r, t)\}$ has minimum degree 2^{n-r-t} while $\delta(\mathscr{A}_2(n, r, t)^c) = (t+2r)\, 2^{n-t-2r}$.

For convenience set $\mathscr{B}_i(n, r, t) = \mathscr{A}_i(n, r, t)^c$.

The paper is organized as follows.

Section 2 introduces shifting, the most useful operation on intersecting families. Except for some simple results in that section the paper is self-contained.

Apart from the very short proof of Theorem 1.4, Section 3 contains a short proof of the Brace–Daykin Theorem and of Theorem 3.4, which shows, how the function $m(n, r, t)$ is related to Conjecture 1.3.

Theorem 3.1 is included here because part of it is needed for the short proof of Theorem 1.2.

Section 4 develops the necessary tools and gives the somewhat lengthy proof of Theorem 1.7 for $r \geqslant 7$. The cases $r=5$ and 6 rely on some stronger results and are proved only in Section 8.

The main result of Section 5 is Theorem 5.5 which together with Theorem 5.8 establishes

$$m(n, r, t) = 2^{n-t} \quad \text{if and only if} \quad t \leqslant 2^r - r - 1 \quad \text{or} \quad n < t + r$$

(with the possible exception of the case $(t, r) = (4, 3)$).

In Section 6 the validity of Conjecture 1.1 is established for $t \leqslant 2^{r-2}(2^{r-2} - 2)/(r-1)$ which is considerable improvement on earlier results. Theorem 6.4 establishes Conjecture 1.3 for $r \geqslant 5$.

Section 7 is probably the highlight of the paper. Several best possible results are obtained in a unified way for cross-intersecting families $\mathscr{F}_1, ..., \mathscr{F}_r$. We should stress that the idea of estimating $|\mathscr{F}_1| \cdots |\mathscr{F}_r|$ instead of $\min_i |\mathscr{F}_i|$ goes back to Moon [M].

Since $\min_i |\mathscr{F}_i| \leqslant (|\mathscr{F}_1| \cdots |\mathscr{F}_r|)^{1/r}$, the results provide a full proof of some conjectures of [DF], which the author believed to be beyond reach. Proposition 7.7, which gives very good bounds on the value of the positive roots of the polynomials $x^r - 2x + 1$, plays a surprisingly important role in the proofs. Theorems 1.2 and 3.1 and parts of Theorem 5.5 are consequences of the results of this section. Results for cross-intersecting families are not simply interesting in themselves but they are also very useful. This was already demonstrated in the proof (for $r \geqslant 7$) of Theorem 1.7. The results of Section 7 appear to be indispensable for $r = 5, 6$. These cases are presented in Section 8.

Section 9 contains extensions of Theorem 1.7 for cross-intersecting families and proves Conjecture 1.8 in a wide range. We could further extend this range but preferred to have a proof of the present result only, because it is much shorter.

In Section 10 some possible extensions of Theorem 1.2 are discussed.

2. Preliminaries

Let us call the families $\mathscr{F}_1, ..., \mathscr{F}_r \subset 2^{[n]}$ *r-cross t-intersecting* if $|F_1 \cap \cdots \cap F_r| \geqslant t$ holds for all $F_1 \in \mathscr{F}_1, ..., F_r \in \mathscr{F}_r$.

The operation S_{ij}—called (i, j)-shift—was essentially defined by Erdős, Ko, and Rado [EKR].

$$S_{ij}(\mathscr{F}) = \{S_{ij}(F) : F \in \mathscr{F}\},$$

where

$$S_{ij}(F) = \begin{cases} F' = (F - \{j\}) \cup \{i\} & \text{if} \quad i \notin F, \quad j \in F, \quad F' \notin \mathscr{F} \\ F & \text{otherwise.} \end{cases}$$

LEMMA 2.1 [F1]. *If $\mathscr{F}_1, \ldots, \mathscr{F}_r$ are r-cross t-intersecting then so are $S_{ij}(\mathscr{F}_1), \ldots, S_{ij}(\mathscr{F}_r)$ as well.*

Iterating S_{ij} for all $1 \leqslant i < j \leqslant n$ will provide us eventually with families $\mathscr{G}_1, \ldots, \mathscr{G}_r$ which are r-cross t-intersecting, satisfy $|\mathscr{G}_l| = |\mathscr{F}_l|$ for $1 \leqslant l \leqslant r$, moreover

$$S_{ij}(\mathscr{G}_l) = \mathscr{G}_l \quad \text{for all} \quad 1 \leqslant i < j \leqslant n \quad \text{and} \quad 1 \leqslant l \leqslant r. \tag{2.1}$$

Families satisfying (2.1) are called *shifted.* The next proposition is very easy to prove (cf. [F1]).

PROPOSITION 2.2. *$\mathscr{G} \subset 2^{[n]}$ is shifted if and only if for all $G \in \mathscr{G}$, $i \notin G$, $j \in G$, and $1 \leqslant i < j \leqslant n$ one has $((G - \{j\}) \cup \{i\}) \in \mathscr{G}$.*

The following proposition exhibits an important property of shifted r-cross t-intersecting families.

PROPOSITION 2.3 [F1]. *Suppose that $\mathscr{G}_1, \ldots, \mathscr{G}_r \subset 2^{[n]}$ are shifted and r-cross t-intersecting. Let $G_j \in \mathscr{G}_j$, $1 \leqslant j \leqslant r$. Then there exists $0 \leqslant i \leqslant (n-t)/r$ such that*

$$|G_1 \cap [t + ir]| + \cdots + |G_r \cap [t + ir]| \geqslant r(t + i(r-1)). \tag{2.2}$$

Inequality (2.2) was used to prove the following—recall the definition of $\alpha(r)$ from Section 1.

THEOREM 2.4 [F1]. *If $\mathscr{F}_1, \ldots, \mathscr{F}_r \subset 2^{[n]}$ are r-cross t-intersecting then*

$$|\mathscr{F}_1|\,|\mathscr{F}_2| \cdots |\mathscr{F}_r| \leqslant (2^n \alpha(r)^t)^r.$$

The following is an easy consequence of shiftedness.

PROPOSITION 2.5 [F1]. *If $\mathscr{F}_1, \ldots, \mathscr{F}_r$ are r-cross t-intersecting and shifted then $\mathscr{F}_1(\bar{1}), \ldots, \mathscr{F}_r(\bar{1})$ are r-cross $(t + r - 1)$-intersecting.*

COROLLARY 2.6.

$$m(n, r, t) \leqslant m(n-1, r, t-1) + m(n-1, r, t+r-1). \tag{2.3}$$

Proof. Let $\mathscr{G} \subset 2^{[n]}$ be shifted, r-wise t-intersecting with $|\mathscr{G}| = m(n, r, t)$. Note $|\mathscr{G}| = |\mathscr{G}(1)| + |\mathscr{G}(\bar{1})|$.

Since $\mathscr{G}(1)$ is r-wise $(t-1)$-intersecting and by Proposition 2.5 (applied with $\mathscr{F}_1 = \cdots = \mathscr{F}_r = \mathscr{G}$) $\mathscr{G}(\bar{1})$ is r-wise $(t+r-1)$-intersecting we have $m(n, r, t) = |\mathscr{G}(1)| + |\mathscr{G}(\bar{1})| \leqslant m(n-1, r, t-1) + m(n-1, r, r+t-1)$. ∎

We will often use the following easy result.

PROPOSITION 2.7 [F2]. *If $\mathscr{F}$ is r-wise t-intersecting and for some $1 \leqslant i < j \leqslant n$ one has $S_{ij}(\mathscr{F}) \cong \mathscr{A}_l(r, t)$ then $\mathscr{F} \cong \mathscr{A}_l(r, t)$ holds too.*

Note the fact, that if $\mathscr{F}$ is shifted then the dual family $\mathscr{F}^c = \{[n] - F: F \in \mathscr{F}\}$ is shifted in the opposite direction, i.e., $S_{ji}(\mathscr{F}^c) = \mathscr{F}^c$ for all $1 \leqslant i < j \leqslant n$.

We will need the following inequality concerning the numbers $\alpha(r)$, the roots in $(\frac{1}{2}, 1)$ of $x^r - 2x + 1$.

PROPOSITION 2.8.

$$\alpha(r)^{2^{r-1}+1} < 2^{-2^{r-1}} \qquad \textit{holds for} \quad r \geqslant 5. \tag{2.4}$$

Proof. Define $b = 1/(2^{r-1} + 1)$. We can rewrite (2.4) as

$$2\alpha(r) < 2^b.$$

Since $2\alpha(r)$ is the only root of $f(y) = (y/2)^r - y + 1$ between 1 and 2, it will be sufficient to show that $f(2^b) < 0$ holds, because $f(1) = 2^{-r} > 0$. Equivalently, we have to show that $(2^b) - 1 > 2^{br-r}$. By $2^b = e^{(\ln 2)b} > 1 + b \ln 2 > 1 + 0.69b$ and $2^{br} < 2^{5/17} < 1.23$, it is enough to show that $0.69/(2^{r-1} + 1) > 1.23/2^r$, which is true for $r \geqslant 5$. ∎

3. THE PROOF OF THEOREM 1.4 AND SOME APPLICATIONS

Proof of Theorem 1.4. Apply induction on n. The case $n = 1$ is trivial. Suppose the statement has been proved for $n - 1$ and use (2.3) and (1.3)

$$\begin{aligned} m(n, r, t) &\leqslant m(n-1, r, t-1) + m(n-1, r, r+t-1) \\ &\leqslant m(n-1, r, t-1) + m(n-1-r, r, t-1)\, 2^r \alpha(r)^r \\ &\leqslant m(n-1, r, t-1)(1 + \alpha(r)^r) = 2m(n-1, r, t-1)\, \alpha(r). \quad ∎ \end{aligned}$$

THEOREM 3.1. *Suppose that $\mathscr{F} \subset 2^{[n]}$ is 3-wise t-intersecting, then*

$$|\mathscr{F}| \leqslant 2^{n-t} \qquad \textit{holds for} \quad t = 1, 2, 3. \tag{3.1}$$

Moreover, if $|\bigcap \mathscr{F}| < t$ then (3.2), (3.3), *and* (3.4) *hold. Finally, for $t = 4$ we have* (3.5).

$$|\mathscr{F}| \leqslant 5 \cdot 2^{n-4} \qquad \textit{if} \quad t = 1 \tag{3.2}$$

$$|\mathscr{F}| < 5(\sqrt{5} - 1)\, 2^{n-5} < 0.773 \cdot 2^{n-2} \qquad \textit{if} \quad t = 2 \tag{3.3}$$

$$|\mathscr{F}| < 10(3 - \sqrt{5})\, 2^{n-6} < 0.955 \cdot 2^{n-3} \qquad \textit{if} \quad t = 3 \tag{3.4}$$

$$|\mathscr{F}| < 5(\sqrt{5} - 2)\, 2^{n-4} < 1.181 \cdot 2^{n-4} \qquad \textit{if} \quad t = 4. \tag{3.5}$$

Remark. Note that (3.2) is the important special case $r=3$ of Theorem 1.2. Also, (3.3) and (3.4) improve earlier bounds of [F1, F3].

Proof. First note that $\alpha(3)=(\sqrt{5}-1)/2$. We apply induction on n and prove all statements simultaneously. Let first $t=1$. If $\bigcap \mathscr{F} \neq \emptyset$ we have nothing to prove. Otherwise $([n]-\{i\}) \in \mathscr{F}$ for all i, can be supposed. This property is unaltered by shifting, thus (by Lemma 2.1) we may assume that $\mathscr{F}$ is shifted. Consider $\mathscr{F}(1)$ and $\mathscr{F}(\bar{1})$ (on $[2, n]$). Since $[2, n] \in \mathscr{F}$, $\mathscr{F}(1)$ is 2-wise 1-intersecting and thus $|\mathscr{F}(1)| \leqslant (1/2)\, 2^{n-1} = 2^{n-2}$.

Also, $\mathscr{F}(\bar{1})$ is 3-wise 3-intersecting (cf. Proposition 2.5). Thus, by (3.1)—using the induction hypothesis—$|\mathscr{F}(\bar{1})| \leqslant 2^{n-4}$. Consequently,

$$|\mathscr{F}| \leqslant 2^{n-2} + 2^{n-4} = 5 \cdot 2^{n-4}$$

as desired.

In case of equality, $\mathscr{F}(\bar{1}) = \{G \subset [2, n] : [2, 4] \subset G\}$ follows from (3.4). Since $\mathscr{F}(\bar{1}) \subset \mathscr{F}$ and $\mathscr{F}$ is shifted, $\mathscr{F} = \mathscr{A}_1(n, 3, 1)$ must hold.

For the case $t>1$ and for later use we need a lemma.

PROPOSITION 3.2. *Suppose that $\mathscr{F} \subset 2^{[n]}$ is a co-complex. Then for all $1 \leqslant i < j \leqslant n$ we have $|\bigcap \mathscr{F}| = |\bigcap S_{ij}(\mathscr{F})|$. If $\mathscr{F}$ is shifted then $|\bigcap \mathscr{F}(1)| \leqslant$* $\max\{0, |\bigcap \mathscr{F}| - 1\}$.

Proof. If $\mathscr{G}$ is a co-complex, then $|\bigcap \mathscr{G}|$ is just n minus the number of $(n-1)$-element sets in $\mathscr{G}$. This quantity obviously does not change by shifting. To prove the second assertion we may suppose that $A = \bigcap \mathscr{F}(1)$ is not empty. We claim that $1 \in F$ for all $F \in \mathscr{F}$. Suppose the contrary. Since $\mathscr{F}$ is a co-complex, $[2, n] \in \mathscr{F}$ follows. Choose an element $a \in A$. Then $[n] - \{a\}$ must be in $\mathscr{F}$ by Proposition 2.2. However, this contradicts $a \in F$ for all $1 \in F \in \mathscr{F}$. Thus $|\mathscr{F}| = |\mathscr{F}(1)|$ and $\bigcap \mathscr{F} = \{1\} \cup A$ follow. ∎

Suppose next that $t=2$. If $|\bigcap \mathscr{F}| = 2$, we have nothing to prove.

Thus let $|\bigcap \mathscr{F}| < 2$. By Proposition 3.2, we may assume that $\mathscr{F}$ is shifted and consider $\mathscr{F}(1)$ and $\mathscr{F}(\bar{1})$.

By Proposition 3.2, $\mathscr{F}(1)$ is 3-wise 1-intersecting with $\bigcap \mathscr{F}(1) = \emptyset$. Thus $|\mathscr{F}(1)| \leqslant 5 \cdot 2^{n-5}$.

Also $\mathscr{F}(\bar{1})$ is 3-wise 4-intersecting (by Proposition 2.5). Using (3.5) gives

$$|\mathscr{F}(\bar{1})| \leqslant 5(\sqrt{5}-2)\, 2^{n-5}.$$

Now (3.3) follows from $|\mathscr{F}| = |\mathscr{F}(1)| + |\mathscr{F}(\bar{1})|$.

The case $t=3$ is very similar. Therefore we shall be somewhat sketchy. If $|\bigcap \mathscr{F}| \geqslant 3$ then we have nothing to prove.

If $|\bigcap \mathscr{F}| \leqslant 2$, then we may assume that $\mathscr{F}$ is shifted. By Proposition 3.2

we may apply (3.3) to the 3-wise 2-intersecting family $\mathscr{F}(1)$ and (3.5) with Corollary 1.5 to the 3-wise 5-intersecting family $\mathscr{F}(\bar{1})$. This yields

$$|\mathscr{F}| \leqslant 5(\sqrt{5}-1)\,2^{n-6} + 5(\sqrt{5}-2)(\sqrt{5}-1)\,2^{n-6} = 10(3-\sqrt{5})\,2^{n-6}.$$

Finally, let $t=4$. If $\bigcap \mathscr{F} \neq \varnothing$, then $|\mathscr{F}| \leqslant 2^{n-4}$ follows from the preceding cases. Thus we may assume that $\bigcap \mathscr{F} = \varnothing$, $\mathscr{F}$ is shifted. We can apply (3.4) to the 3-wise 3-intersecting family $\mathscr{F}(1)$ and (3.5) together with Corollary 1.5 to the 3-wise 6-intersecting family $\mathscr{F}(\bar{1})$. This gives

$$|\mathscr{F}| \leqslant 10(3-\sqrt{5})\,2^{n-7} + 5(\sqrt{5}-2)(\sqrt{5}-1)^2\,2^{n-7} = 5(\sqrt{5}-2)\,2^{n-4}$$

as desired. ∎

Now we want to give a short proof of Theorem 1.2.

LEMMA 3.3. *If* $m(n, r, t) = 2^{n-t}$ *then* $m(n, r+1, t+1) = 2^{n-t-1}$ *holds for* $n \geqslant t+1$, $r \geqslant 3$ *with* $\mathscr{A}_0(n, r+1, t+1)$ *the unique optimal family.*

Proof. Let $\mathscr{F} \subset 2^{[n]}$ be $(r+1)$-wise $(t+1)$-intersecting. If $\mathscr{F}$ is r-wise $(t+2)$-intersecting then by the assumption and Corollary 1.5 we have

$$|\mathscr{F}| \leqslant 2^{n-t}\alpha(r)^2 \leqslant 2^{n-t}\left(\frac{1}{2}+\frac{1}{2^r}\right)^2 < 2^{n-t-1}.$$

Otherwise there exist $F_1, ..., F_r \in \mathscr{F}$ with $|F_1 \cap \cdots \cap F_r| = t+1$. Thus this $(t+1)$-element set is contained in all members of $\mathscr{F}$. This yields $|\mathscr{F}| \leqslant 2^{n-t-1}$ with equality holding if and only if $\mathscr{F} \cong \mathscr{A}_0(n, r+1, t+1)$. ∎

Proof of Theorem 1.2. Apply induction on r. The case $r=3$ is (3.2). By Proposition 3.2 we may suppose that $\mathscr{F}$ is shifted. Now $\mathscr{F}(1)$ is $(r-1)$-wise intersecting and $\bigcap \mathscr{F}(1) = \varnothing$. Thus—by induction—$|\mathscr{F}(1)| \leqslant (r+1)\,2^{n-1-r}$.

As $\mathscr{F}(\bar{1})$ is r-wise r-intersecting by Proposition 2.5, (3.1) and repeated applications of Lemma 3.3 imply $|\mathscr{F}(\bar{1})| \leqslant 2^{n-r-1}$.

Thus $|\mathscr{F}| = |\mathscr{F}(1)| + |\mathscr{F}(\bar{1})| \leqslant (r+2)\,2^{n-r-1}$ as desired. In case of equality we have $|\mathscr{F}(\bar{1})| = 2^{n-r-1}$, consequently,

$$\mathscr{F}(\bar{1}) = \{G \subset [2, n] : [2, r+1] \subset G\}.$$

Using shiftedness, $\mathscr{F} = \mathscr{A}_1(n, r, 1)$ follows. ∎

THEOREM 3.4. *Suppose that* $m(n, r, t+r-1) = 2^{n-t-r+1}$ *with* $\mathscr{A}_0$ *as the only optimal family. Then Conjecture* 1.3 *holds for* r, t.

Proof. Apply induction on t. The case $t=1$ is Theorem 1.2. Consider $\bigcap \mathscr{F} = A$. Suppose first $1 \leqslant |A| < t$. Choose $i \in A$ and consider $\mathscr{F}(i)$ which satisfies the assumptions with t replaced by $t-1$. We infer

$$|\mathscr{F}| = |\mathscr{F}(i)| \leqslant (r+t)\, 2^{n-1-r-t+1} < |\mathscr{A}_1(n, r, t)|.$$

If $\bigcap \mathscr{F} = \varnothing$, then we may suppose that $\mathscr{F}$ is shifted.

Now $\mathscr{F}(1)$ satisfies the assumptions with t replaced by $t-1$. Thus

$$|\mathscr{F}(1)| \leqslant |\mathscr{A}_1(n-1, r, t-1)| \leqslant (r+2)\, 2^{n-r-t}.$$

By Proposition 2.5, $\mathscr{F}(\bar{1})$ is $(r+t-1)$-intersecting. Thus $|\mathscr{F}(\bar{1})| \leqslant 2^{n-r-t}$, yielding

$$|\mathscr{F}| = |\mathscr{F}(1)| + |\mathscr{F}(\bar{1})| \leqslant (r+t+1)\, 2^{n-r-t} = |\mathscr{A}_1(n, r, t)|.$$

In case of equality, $\mathscr{F}(\bar{1}) = \{G \subset [2, n] : [2, r+t] \subset G\}$. By shiftedness $\mathscr{F} = \mathscr{A}_1(n, r, t)$ follows. ∎

4. The Minimum Degree of Dually r-Wise Intersecting Families

In this section we shall use Theorem 5.2, which is proved in the next section. Let $\mathscr{G} \subset 2^{[n]}$ be a dually r-wise intersecting family with $\bigcup \mathscr{G} = [n]$ throughout this whole section. For $2 \leqslant s \leqslant r$ define

$$t(s) = \min\{t : \exists G_1, ..., G_s \in \mathscr{G}, |G_1 \cup \cdots \cup G_s| = n - t\}.$$

Thus $t(s)$ is the maximal integer t for which $\mathscr{G}$ is dually s-wise t-intersecting. We can assume without loss of generality that $t(r) = 1$ and that $F \subset G \in \mathscr{G}$ implies $F \in \mathscr{G}$, i.e., $\mathscr{G}$ is a *complex*.

Call a subset $A \subset [n]$ a *hole* if $|A \cap G| \leqslant 1$ for all $G \in \mathscr{G}$.

Proposition 4.1. *If A, B are holes, $i \in A \cap B$ then $|\mathscr{G}(i)| \leqslant 2^{n-|A \cup B|}$.*

Proof. If $i \in G \in \mathscr{G}$, then $G \cap (A - \{i\}) = \varnothing$ and $G \cap (B - \{i\}) = \varnothing$. Thus $\mathscr{G}(i) \subset 2^{[n]-(A \cup B)}$. ∎

Proposition 4.2. *For $2 \leqslant s < r$ one has $t(s) \geqslant r - s + 1$. Moreover, if equality holds for some s, then there exists a hole of size $r - s + 1$.*

Proof. Choose $G_1, ..., G_s \in \mathscr{G}$ to satisfy $|G_1 \cup \cdots \cup G_s| = n - t(s)$ and consider $A = [n] - (G_1 \cup \cdots \cup G_s)$. Then $A \not\subset H_1 \cup \cdots \cup H_{r-s}$ must hold for all $H_1, ..., H_{r-s} \in \mathscr{G}$. Since $\bigcup \mathscr{G} = [n]$, $|A| \geqslant r - s + 1$, and in case of equality $|H \cap A| \leqslant 1$ for all $H \in \mathscr{G}$, i.e., A is a hole. ∎

PROPOSITION 4.3. *Suppose that* $\alpha(k)^{r-k+1}2^{r-k}<1$ *and there is a hole of size* $k+1$. *Then the minimum degree* $\delta(\mathscr{G})$ *satisfies*

$$\delta(\mathscr{G})\leqslant 2^{n-r-1}.$$

Proof. By symmetry let $[k+1]$ be a hole. Consider the families $\mathscr{A}(1),...,\mathscr{A}(k)\subset 2^{[n]-[k+1]}$. If these families are dually k-cross $(r-k+1)$-intersecting then by Theorem 2.4 and the assumption on $\alpha(k)$

$$\min_i |\mathscr{A}(i)|\leqslant (|\mathscr{A}(1)|\cdots|\mathscr{A}(k)|)^{1/k}\leqslant 2^{n-k-1}\alpha(k)^{r-k+1}<2^{n-r-1},$$

as desired. Otherwise we can choose $H_i\in\mathscr{G}(i)$, $i=1,...,k$ such that $|H_1\cup\cdots\cup H_k| = n-r-1$, i.e., $|(H_1\cup\{1\})\cup\cdots\cup(H_k\cup\{k\})| = n-(r-k+1)$. Let B be the complement of this last set. By Proposition 4.2, B is a hole $B\cap[k+1]=\{k+1\}$. From Proposition 4.1, $|\mathscr{G}(k+1)|\leqslant 2^{n-r-1}$ follows. ∎

PROPOSITION 4.4. $\alpha(k)^{l+1}2^l<1$ *holds in each of the following cases.*

(i) $k=3$, $l=2$

(ii) $k=4$, $l\leqslant 7$

(iii) $k\geqslant 5$, $l\leqslant 2k$.

Proof. Parts (i) and (ii) can be checked by direct computation. To prove (iii) consider the following sequence of equalities

$$(2\alpha(k))^{l+1}\leqslant\left(1+\frac{1}{2^{k-1}}\right)^{l+1}\leqslant e^{(l+1)2^{1-k}}<e^{11/16}<2. \quad ∎$$

PROPOSITION 4.5. *If there is no hole of size* $t(s)+1$ *for some* $3\leqslant s\leqslant r$ *then* $t(s-d)\geqslant t(s)+2d$ *holds for* $1\leqslant d\leqslant s-2$.

Proof. Since any subset of a hole is a hole itself, it is sufficient to consider the $d=1$ case. Let B be a set of size $t(s-1)$ whose complement is the union of $s-1$ sets in $\mathscr{G}$. Since $\bigcup\mathscr{G}=[n]$, $|B|>t(s)$. If $|B|=t(s)+1$ then by definition of $t(s)$, $|G\cap B|\leqslant 1$ must hold for all $G\in\mathscr{G}$; i.e., B is a hole. This would contradict our assumptions. Thus $t(s-1)=|B|\geqslant t(s)+2$. ∎

Proof of Theorem 1.7 *for* $r\geqslant 10$. If $\mathscr{G}$ has a hole of size $\lfloor r/2\rfloor$ then $\delta(\mathscr{G})\leqslant 2^{n-r-1}$ follows from Propositions 4.3 and 4.4.

Let b be the size of the largest hole. We may assume that $1\leqslant b<\lfloor r/2\rfloor$ holds.

Since $t(r)\geqslant 1$, $t(r-b+1)\geqslant b$ must hold. Thus we may apply Proposition 4.5 with $s=r-b+1$. Consequently,

$$t(4)\geqslant b+2(r-b-3)=2r-b-6. \tag{4.1}$$

For $r=10$ and 11 using Theorem 5.2 we obtain

$$|\mathscr{G}| \leqslant m(n, 4, 10) = 2^{n-10} \tag{4.2}$$

$$|\mathscr{G}| \leqslant m(n, 4, 12) = 2^{n-10}\alpha(4)^2 < 2^{n-11}. \tag{4.3}$$

In general, going from r to $r+2$ the minimum value of $t(4)$ goes up by 3: from $2r-\lfloor r/2 \rfloor - 5$ to $2(r+2)-\lfloor (r+2)/2 \rfloor - 5$. This results in a change in the upper bound of $|\mathscr{G}|$ by a factor of $\alpha(4)^3 < 1/4$. This proves in view of (4.2) and (4.3),

$$|\mathscr{G}| \leqslant m(n, 4, 2r-\lfloor r/2 \rfloor - 5) < 2^{n-r}.$$

Since $\mathscr{G}$ is a complex, $|\mathscr{G}(i)| \leqslant (1/2)\,|\mathscr{G}| \leqslant 2^{n-r-1}$ follows. ∎

Note that the above proof goes through unchanged for $r=9$ if $\mathscr{G}$ has no hole of size 4. This will be our starting point in the following.

Proof of Theorem 1.7 *for* $r=9$, 8, 7 *in the Case of Existence of a Hole of Size* 4. We may assume that $[4]$ is a hole in $\mathscr{G}$. If $\mathscr{G}(1)$, $\mathscr{G}(2)$, $\mathscr{G}(3)$ are dually 3-cross r-intersecting on $[5, n]$ then by Theorem 2.4 we obtain

$$\delta(\mathscr{G}) \leqslant (|\mathscr{G}(1)|\;|\mathscr{G}(2)|\;|\mathscr{G}(3)|)^{1/3} \leqslant 2^{n-4}\alpha(3)^r < 2^{n-r-1}.$$

Since $\alpha(3)^6 < 2^{-4}$, for $r=7$ we obtain $\delta(\mathscr{G}) < 2^{n-r-1}$ even if $\mathscr{G}(1)$, $\mathscr{G}(2)$, $\mathscr{G}(3)$ are dually 3-cross $(r-1)$-intersecting.

Let $H_i \in \mathscr{G}(i)$, $i=1,2,3$, be sets such set $T = \bigcup_{1 \leqslant i \leqslant 3} H_i \cup \{i\}$ has maximal size.

By the above considerations and the fact that $\mathscr{G}$ is dually r-wise intersecting, $n-r+2 \geqslant |T| \geqslant n-r$, where the second inequality is strict for $r=7$.

Note that by Proposition 4.3 we may assume

$$\text{there is no hole of size 5.} \tag{4.4}$$

If $|T| = n-r+2$ then $[n]-T$ is a hole, contradicting (4.4).

Suppose next $|T| = n-r+1$ and consider $B=[n]-T$. By (4.4), B is not a hole. Consequently, there exists some $G_4 \in \mathscr{G}$ with $|B \cap G_4| = 2$. Since $\mathscr{G}$ is dually r-wise intersecting, the $(r-3)$-element set $B_0 = B - G_4$ is a hole. For $r=8, 9$ this contradicts (4.4). Thus let $r=7$.

If possible, we choose G_4 in a way that is does not contain $\{4\}$. If it is not possible then $B-\{4\}$ is a hole of size $r-2=5$, contradicting (4.4). Thus we may assume $4 \in B_0$.

Consider $\mathscr{G}(4) \subset 2^{n-([4] \cup B_0)}$. If $\mathscr{G}(4)$ is 2-wise 1-intersecting, then $|\mathscr{G}(4)| \leqslant (1/2)\,2^{n-7} = 2^{n-8}$ and we are done. Otherwise there exist

$H_1, H_2 \in \mathscr{G}(4)$ such that $H_1 \cup H_2 = [n] - ([4] \cup B_0)$. Thus $t(2) \leqslant 6$. By Proposition 4.2 there is a hole of size 6, contradicting (4.4).

The final case is $|B| = r$, $r = 8$ or 9. Again, we choose $G_4 \in \mathscr{G}$ with $4 \notin G_4$, $|G_4 \cap B| = 2$ and define $B_0 = B - G_4$. If this was impossible, then $B - \{4\}$ is a hole of size $r-1$, contradicting (4.4). Defining B_1 in a similar way and using (4.4) we obtain a hole B_2 of size $r-4$, containing $\{4\}$. For $r = 9$, this contradicts (4.4).

Thus let $r = 8$. By the maximal choice of T and $|T| = r$, we have $t(3) = 8$. This implies that $\mathscr{G}(4)$ is a dually 3-wise 2-intersecting family on $[n] - ([4] \cup B_2)$. By (3.1) we conclude $\mathscr{G}(4) \leqslant 2^{n-7-2} = 2^{n-9}$. ∎

Proof of Theorem 1.7 *for* $r = 8$, *If There Is No Hole of Size* 4. Let b be the size of the largest hole, $1 \leqslant b \leqslant 3$. Using Proposition 4.5, we infer that $t(4) \geqslant b + 2(8 - b - 3) = 10 - b$.

For $1 \leqslant b \leqslant 2$ using Theorem 5.2 gives $|\mathscr{G}| \leqslant 2^{n-8}$ and thus $\delta(\mathscr{G}) \leqslant 2^{n-9}$.

The only remaining case is $b = 3$. Let $[3]$ be a hole. If $\mathscr{G}(1)$, $\mathscr{G}(2)$, $\mathscr{G}(3)$ are dually 3-cross 9-intersecting then Theorem 2.4 gives $\delta(\mathscr{G}) \leqslant 2^{n-3}\alpha(3)^9 < 2^{n-9}$.

Otherwise we can find $H_i \in \mathscr{G}(i)$, $1 \leqslant i \leqslant 3$, with union of size $n - 11$. Consequently $H_i \cup \{i\}$, $i = 1, 2, 3$, which are in $\mathscr{G}$, have union of size $n - 8$. However, using Proposition 4.5, $t(3) \geqslant 3 + 2 \cdot 3 = 9$, a contradiction. ∎

Finally we come to the technically most difficult case, $r = 7$. In view of the above cases we may assume that there is no hole of size 4.

Proof of Theorem 1.7 *for* $r = 7$, *No Hole of Size* 4. We shall concentrate on $t(4)$. If $t(4) \geqslant 7$, then by Theorem 5.2 we have $\delta(\mathscr{G}) \leqslant (1/2)\,|\mathscr{G}| \leqslant 2^{n-7-1} = 2^{n-8}$, as desired.

On the other hand, by Proposition 4.2, $t(4) \geqslant 5$. This leaves 2 cases.

(a) $t(4) = 5$. Suppose that $G_1, ..., G_4 \in \mathscr{G}$ with $G_1 \cup \cdots \cup G_4 = [6, n]$. Consider the family $\mathscr{H} = \{G \cap [5] : G \in \mathscr{G}\}$. Since $\mathscr{H}$ must be dually 3-intersecting, $|H| \leqslant 2$ for all $H \in \mathscr{H}$. Moreover, $\mathscr{H}$ contains no two disjoint 2-element sets.

If some $i \in [5]$ is not contained in any 2-element member of $\mathscr{H}$, then $\mathscr{G}(i) \subset 2^{[6,n]}$ and $\mathscr{G}(i)$ must be dually 3-wise 3-intersecting (because Proposition 4.5 implies $t(3) \geqslant 7$). Thus $|\mathscr{G}(i)| \leqslant 2^{n-5-3} = 2^{n-8}$ by (3.1).

Thus all $i \in [5]$ are contained in some 2-element members of $\mathscr{H}$. It follows that $\{H \in \mathscr{H} : |H| = 2\}$ is a star, i.e., for some $j \in [5]$, $\{\{j, i\} : i \in [5], i \neq j\}$ are the 2-element members. Now $[5] - \{j\}$ is a hole of size 4, a contradiction.

(b) $t(4) = 6$. Choose $G_1, ..., G_4 \in \mathscr{G}$ with $G_1 \cup \cdots \cup G_4 = T$, $|T| = n - 6$. Say $T = [7, n]$.

Again by Proposition 4.5 we have $t(3) \geqslant 8$.

We claim that $|G \cap [6]| \leqslant 2$ for all $G \in \mathscr{G}$.

Indeed, the contrary implies that, $[6]-G$, is a 3-element set, say $[3]$, which is a hole. By $t(3)=8$ the families $\mathscr{G}(1)$, $\mathscr{G}(2)$, $\mathscr{G}(3)$ are dually 3-cross 8-intersecting on $[4, n]$. Using Theorem 2.4

$$\delta(\mathscr{G}) \leqslant (|\mathscr{G}_1|\ |\mathscr{G}_2|\ |\mathscr{G}_3|)^{1/3} \leqslant 2^{n-3}\left(\frac{\sqrt{5}-1}{2}\right)^8 < 2^{n-8}$$

follows.

Consequently, $\mathscr{H} = \{G \cap [6] : G \in \mathscr{G}\}$ is a family consisting of the empty set, $\{i\}$, $1 \leqslant i \leqslant 6$, and some 2-element sets. Define

$$\mathscr{K} = \{H \in \mathscr{H} : |H| = 2\}.$$

By the above argument, there is no hole of size 3. Consequently, every $B \in \binom{[6]}{3}$ contains some member of $\mathscr{K}$.

On the other hand $\mathscr{K}$ contains no 3 pairwise disjoint sets ($\mathscr{G}$ is dually 7-wise intersecting).

Viewing $\mathscr{K}$ as a graph on $[6]$, the first condition and the simplest case of Ramsey's theorem imply that $\mathscr{K}$ contains a triangle. Applying the condition once more gives an edge, disjoint to the triangle. The remaining vertex cannot be joined by the second condition to any of the vertices of the triangle. Using the first condition shows that it must be joined to both endpoints of the edge which was disjoint to the triangle. Consequently, $\mathscr{K}$ is the disjoint union of two triangles, say $\mathscr{K} = \binom{[3]}{2} \cup \binom{[4,6]}{2}$.

Now we are close to the final contradiction, only we have to make some calculations using Theorem 2.4.

Set $\mathscr{B}(A) = \{G - A : G \in \mathscr{G}, G \cap [6] = A\} \subset 2^{[7,n]}$.

For $A_1, A_2, A_3, A_4 \subset [6]$ the families $\mathscr{B}(A_i)$, $i = 1, 2, 3, 4$, are dually 4-cross $|A_1 \cup \cdots \cup A_4|$-intersecting because of $t(4) = 6$. By Theorem 2.4 we have

$$|\mathscr{B}(A_1)| \cdots |\mathscr{B}(A_4)| \leqslant (2^{n-6}\alpha(4)^{|A_1 \cup \cdots \cup A_4|})^4. \tag{4.5}$$

If for some 1-element set $\{i\} \subset [6]$ one has $|\mathscr{B}(\{i\})| \leqslant (1/3)\, 2^{n-8}$ then using $|\mathscr{B}(A)| \leqslant |\mathscr{B}(A')|$ for $A \subset A'$, $|\mathscr{G}(i)| \leqslant 2^{n-8}$ follows. Thus we may assume that

$$|\mathscr{B}(\{i\})| \geqslant \tfrac{1}{3} 2^{n-8} \qquad \text{for all} \qquad i \in [6]. \tag{4.6}$$

Using (4.5) with three 1-element and one disjoint 2-element sets and taking into consideration (4.6) gives

$$\begin{aligned} |\mathscr{B}(A)| &< (2^{n-6}\alpha(4)^5)^4 \Big/ \left(\frac{1}{3}\, 2^{n-8}\right)^3 \\ &= 2^n \alpha(4)^{20} \cdot 27 < 2^{n-8}/28 \qquad \text{for all} \qquad A \in \binom{[6]}{2}. \end{aligned} \tag{4.7}$$

Using (4.5) with $A_i = \{i\}$, we may assume by symmetry that

$$|\mathscr{B}(\{1\})| \leqslant 2^{n-6}\alpha(4)^4 < 0.35 \cdot 2^{n-8}. \tag{4.8}$$

Combining (4.7) and (4.8) gives finally

$$|\mathscr{G}(1)| = |\mathscr{B}(\{1\})| + |\mathscr{B}(\{1, 2\})| + |\mathscr{B}(\{1, 3\})| < 0.43 \cdot 2^{n-8}. \quad \blacksquare$$

5. The Erdős–Ko–Rado Case

As we mentioned before, among the families $\mathscr{A}_i(n, r, t)$, $\mathscr{A}_0(n, r, t)$ is largest for $t < 2^r - r - 1$ while for $t = 2^r - r - 1$, $|\mathscr{A}_0(n, r, t)| = |\mathscr{A}_1(n, r, t)|$ holds (for $n \geqslant r + t$) and these are the largest.

Consider the following statement which, except for the uniqueness, would follow from Conjecture 1.1.

> $m(n, r, t) = 2^{n-t}$ for all $t \leqslant 2^r - r - 1$ with $\mathscr{A}_0(n, r, t)$ as the only optimal family except for $t = 2^r - r - 1$, where $\mathscr{A}_1(n, r, t)$ is the only other optimal family. (5.1)

Proposition 5.1. *Suppose that* (5.1) *holds for some* $r \geqslant 5$. *Then it holds for* $r + 1$ *as well.*

Proof. Suppose first that $t = 2^{r+1} - r - 2$. Let $\mathscr{F} \subset 2^{[n]}$ be $(r+1)$-wise t-intersecting, $|\mathscr{F}|$ maximal. Consequently, $F \in \mathscr{F}$, $F \subset H \subset [n]$ imply $H \in \mathscr{F}$. We may suppose that $\mathscr{F}$ is shifted.

If $\mathscr{F}$ is r-wise $(t+2)$-intersecting, then using the validity of (5.1) for r, Corollary 1.5, and Proposition 2.8, we infer

$$|\mathscr{F}| \leqslant 2^{n-(2^r - r - 1)}\alpha(r)^{2^r + 1} < 2^{n-(2^{r+1} - r - 2)}.$$

Thus we may assume that $\mathscr{F}$ is not r-wise $(t+2)$-intersecting. Looking at the dual family $\mathscr{G} = \{[n] - F \colon F \in \mathscr{F}\}$, it is dually $(r+1)$-wise t-intersecting but not dually r-wise $(t+2)$-intersecting. Therefore there exist $G_1, ..., G_r \in \mathscr{G}$ with $G_1 \cup \cdots \cup G_r = [t+2, n]$.

Consequently,

$$|G \cap [t+1]| \leqslant 1 \qquad \text{holds for all} \quad G \in \mathscr{G}. \tag{5.2}$$

If $G \cap [t] = \varnothing$ for all $G \in \mathscr{G}$, then $\mathscr{F} \subset \mathscr{A}_0(r+1, t)$. Suppose that this is not the case.

Using shiftedness, $\{t\} \in \mathscr{G}$ follows. By shiftedness, $\{j\} \in \mathscr{G}$ for all $t < j \leqslant n$. (This implies $n \geqslant t + r + 1$.)

Define the families $\mathscr{G}_0, \mathscr{G}_1, ..., \mathscr{G}_t$ by

$$\mathscr{G}_0 = \{G \in \mathscr{G} : G \cap [t] = \varnothing\} \subset 2^{[t+1, n]},$$
$$\mathscr{G}_i = \{G - \{i\} : G \cap [t] = \{i\}; G \in \mathscr{G}\} \subset 2^{[t+2, n]}, \qquad 1 \leqslant i \leqslant t.$$

Obviously,

$$|\mathscr{G}| = |\mathscr{G}_0| + \cdots + |\mathscr{G}_t| \qquad \text{holds.} \tag{5.3}$$

If $|G \cap [t+r+1]| \leqslant 1$ for all $G \in \mathscr{G}$, then $\mathscr{F} \subset \mathscr{A}_1(n, r+1, t)$ follows and we are done. Thus we may assume that $G_{r+1} = \{t+r, t+r+1\} \in \mathscr{G}$.

We prove two inequalities.

$$|\mathscr{G}_0| \leqslant \frac{r+2}{2^{r+1}} 2^{n-t} \tag{5.4}$$

$$|\mathscr{G}_i| \leqslant 2^{n-t-r-2}, \qquad 1 \leqslant i \leqslant t. \tag{5.5}$$

To prove (5.4) just note that $\mathscr{G}_0$ is dually r-wise intersecting and $\bigcup \mathscr{G}_0 = [t+1, n]$. Thus (5.4) follows from Theorem 1.2.

To prove (5.5) it is enough to show that $\mathscr{G}_i$ is dually r-wise $(r+1)$-intersecting (and apply (5.1)).

Otherwise by shiftedness there exist $H_1, ..., H_r \in \mathscr{G}_i$ with $H_1 \cup \cdots \cup H_r = [t+r+2, n]$.

Again by shiftedness, $G_i = H_i \cup \{t+i-1\}$ is in $\mathscr{G}$. Together with $G_{r+1} = \{t+r, t+r+1\}$ this gives $G_1 \cup \cdots \cup G_{r+1} = [t, n]$, a contradiction.

Now (5.4) and (5.5) give in view of (5.3),

$$|\mathscr{G}| < (t+r+2)\, 2^{n-t-r-1} = 2^{n-t},$$

as desired.

If $s = 2^{r+1} - r - 2 - t$ is positive, then define

$$\tilde{\mathscr{F}} = \{[n+1, n+s] \cup F : F \in \mathscr{F}\} \subset 2^{[n+s]}.$$

Then $\tilde{\mathscr{F}}$ is $(r+1)$-wise $t+s = 2^{r+1} - r - 2$-intersecting. Thus either $|\mathscr{F}| = |\tilde{\mathscr{F}}| < 2^{n-t}$ or $\mathscr{F} \cong \mathscr{A}_0(n, r+1, t)$ follows (the case $\tilde{\mathscr{F}} \cong \mathscr{A}_1(n, r+1, 2^{r+1} - r - 2)$ is impossible because of $\bigcap \tilde{\mathscr{F}} \neq \varnothing$). ∎

In view of Proposition 5.1 it would be sufficient to show (5.1) for $r = 3$, 4, and 5. However, we did not succeed in doing so. We will prove it for $r = 5$, using the following partial result for $r = 4$.

THEOREM 5.2. *If $\mathscr{F}$ is 4-wise t-intersecting with $t \leqslant 10$, then $|\mathscr{F}| \leqslant 2^{n-t}$ with equality holding if and only if $\mathscr{F} \cong \mathscr{A}_0(n, 4, t)$.*

Proof. As in the case of Proposition 5.1, it is sufficient to deal with the case $t=10$. Let $\mathscr{F}\subset 2^{[n]}$ be a shifted 4-wise 10-intersecting family of maximal size.

In view of (1.6) and Theorem 3.1 we have

$$m(n, 3, 13) < 2^{n-4}5(\sqrt{5}-2)\left(\frac{\sqrt{5}-1}{2}\right)^9 < 2^{n-10}. \tag{5.6}$$

Let $\mathscr{G}=\{[n]-F: F\in\mathscr{F}\}$ be the dual family. Then by (5.6) we may suppose that there exist $G', G'', G'''\in\mathscr{G}$ with $G'\cup G''\cup G'''=[13, n]$. Consequently,

$$|G\cap[1, 12]|\leqslant 2 \qquad \text{for all} \quad G\in\mathscr{G}. \tag{5.7}$$

We may suppose that $\mathscr{G}\not\subset 2^{[11,n]}$ and thus $\{10\}\in\mathscr{G}$ by shiftedness. For $1\leqslant i\leqslant n$ and $A\subset[i]$ define

$$\mathscr{G}(i, A)=\{G-A: G\in\mathscr{G}, G\cap[i]=A\}.$$

Then $\mathscr{G}(10, \varnothing)$ is dually 3-wise intersecting because of $\{10\}\in\mathscr{G}$. By shiftedness $\{j\}\in\mathscr{G}$ for all $10<j\leqslant n$. Thus by Theorem 1.2 we have

$$|\mathscr{G}(10, \varnothing)|\leqslant 5\cdot 2^{n-14}. \tag{5.8}$$

Since $|\mathscr{A}_2(n, 4, 10)| < |\mathscr{A}_1(n, 4, 10)| < |\mathscr{A}_0(n, 4, 10)|$, we may assume that $\mathscr{F}\not\subset\mathscr{A}_i(n, 4, 10)$ for $i=1, 2$. Consequently, $\{13, 14\}\in\mathscr{G}$ and $\{16, 17, 18\}\in\mathscr{G}$.

CLAIM 5.3. *$\mathscr{G}(12, \{i\})$ is dually 3-wise 3-intersecting on $[13, n]$ for $1\leqslant i\leqslant 10$.*

Proof of Claim. Otherwise there exist $H_1, H_2, H_3\in\mathscr{G}(12, \{i\})$ with $H_1\cup H_2\cup H_3=[15, n]$. By shiftedness, $G_1=H_1\cup\{10\}$, $G_2=H_2\cup\{11\}$, and $G_3=H_3\cup\{12\}$ are in $\mathscr{G}$ and their union together with $\{13, 14\}$ is $[10, n]$, a contradiction. ∎

By Theorem 3.1 we infer

$$|\mathscr{G}(12, \{i\})|\leqslant 2^{n-15}. \tag{5.9}$$

CLAIM 5.4. *If $A\in\binom{[12]}{2}$, $A\cap[10]\neq\varnothing$ then $\mathscr{G}(12, A)$ is dually 3-wise 7-intersecting on $[13, n]$.*

Proof of Claim. Otherwise we find $H_1, H_2, H_3\in\mathscr{G}(12, A)$ with $H_1\cup H_2\cup H_3=[19, n]$. By shiftedness $H_1\cup\{10, 13\}$, $H_2\cup\{11, 14\}$, and $H_3\cup\{12, 15\}$ are in $\mathscr{G}$. However, the union of these sets with $\{16, 17, 18\}$ is $[10, n]$, a contradiction. ∎

Using Theorem 3.1 together with Corollary 1.5 we infer

$$|\mathscr{G}(12, A)| \leqslant 2^{n-19} 5(\sqrt{5}-2)(\sqrt{5}-1)^3$$
$$\text{for} \quad A \in \binom{[12]}{2}, \qquad A \cap [10] \neq \varnothing. \tag{5.10}$$

From (5.8), (5.9), and (5.10) we obtain

$$\begin{aligned} |\mathscr{G}| &= |\mathscr{G}(10, \varnothing)| + \sum_{1 \leqslant i \leqslant 10} |\mathscr{G}(12, \{i\})| + \sum_{A \in \binom{[12]}{2}, A \cap [10] \neq \varnothing} |\mathscr{G}(12, A)| \\ &\leqslant 160 \cdot 2^{n-19} + 160 \cdot 2^{n-19} + 325(\sqrt{5}-2)(\sqrt{5}-1)^3 2^{n-19} \\ &< \frac{465}{512} 2^{n-10}. \quad \blacksquare \end{aligned}$$

Now we are ready to prove the main result of this section.

THEOREM 5.5. *Statement* (5.1) *holds for all* $r \geqslant 5$.

Proof. In view of Proposition 5.1 it is sufficient to prove (5.1) for $r = 5$. Also, as in the preceding proofs, it is sufficient to consider the case $t = 2^5 - 5 - 1 = 26$.

Let $\mathscr{F}$ be a 5-wise 26-intersecting family of maximal size. If $\mathscr{F}$ is 4-wise 29-intersecting, then by Theorem 5.2 and Corollary 1.5 we have

$$|\mathscr{F}| \leqslant 2^{n-10} \alpha(4)^{19} < 2^{n-26}.$$

Thus we may assume that the dual family $\mathscr{G}$ is not dually 4-wise 29-intersecting , $\mathscr{G}$ is shifted and therefore contains 4 sets whose union is $[29, n]$. Thus

$$|G \cap [28]| \leqslant 2 \qquad \text{for all} \quad G \in \mathscr{G}. \tag{5.11}$$

We may assume that $\mathscr{F} \not\subset \mathscr{A}_i(5, 26)$ for $i = 0, 1, 2$ and consequently $\{26\}$, $\{30, 31\}$, and $\{34, 35, 36\}$ are in $\mathscr{G}$.

As in the proof of Theorem 5.2, we infer

CLAIM 5.6. (i) *$\mathscr{G}(26, \varnothing)$ is dually* 4-*wise intersecting with* $\bigcup \mathscr{G}(26, \varnothing) = [27, n]$

(ii) *$\mathscr{G}(28, i)$ is dually* 4-*wise* 4-*intersecting on* $[29, n]$ *for* $1 \leqslant i \leqslant 26$

(iii) *$\mathscr{G}(28, A)$ is dually* 4-*wise* 9-*intersecting on* $[29, n]$ *for all* $A \subset \binom{[28]}{2}$ *with* $A \cap [26] \neq \varnothing$.

From (5.11) and Claim 5.6 we obtain applying Theorem 1.2 and Theorem 5.2,

$$|\mathscr{G}| \leqslant 6 \cdot 2^{n-31} + 26 \cdot 2^{n-32} + \left(\binom{28}{2} - 1\right) 2^{n-37} = \frac{1593}{2^{11}} 2^{n-26} < 2^{n-26}.$$

By very much the same proof one can show the following:

THEOREM 5.7. *Suppose that* $27 \leqslant t \leqslant 31$. *Then* $m(n, 5, t) = |\mathscr{A}_1(n, 5, t)|$ *with* $\mathscr{A}_1(n, 5, t)$ *as the unique optimal family.*

Remark. For $t = 32$ one has already $2^{n-10}\alpha(4)^{25} > 2^{n-32}$, thus the proof breaks down in the first step.

In view of Theorems 3.1, 5.2, and 5.6 the only unsolved cases in the Erdős–Ko–Rado case are $(r, t) = (3, 4)$ and $(r, t) = (4, 11)$. We succeeded in solving the latter.

THEOREM 5.8. *If* $\mathscr{F} \subset 2^{[n]}$ *is* 4-*wise* 11-*intersecting then* $|\mathscr{F}| \leqslant 2^{n-11}$ *with equality holding if and only if* $\mathscr{F} \cong \mathscr{A}_i(n, 4, 11)$ *for* $i = 0$ *or* 1 *holds.*

Proof. Since this proof is similar to the other proofs in this section, we will be somewhat sketchy. We may suppose that $|\mathscr{F}| \geqslant 2^{n-11}$, $\mathscr{F}$ is shifted, and $\mathscr{F} \not\subseteq \mathscr{A}_i(n, 4, 11)$ for $i = 0, 1, 2, 3$. For the dual family, $\mathscr{G}$ this implies that $K_0 = \{11\}$, $K_1 = \{14, 15\}$, $K_2 = \{17, 18, 19\}$, and $K_3 = \{20, 21, 22, 23\}$ are all in $\mathscr{G}$.

If $\mathscr{F}$ is 3-wise 15-intersecting, then we obtain from Theorem 3.1 and Corollary 1.5 that

$$|\mathscr{F}| < 2^{n-3}\alpha(3)^{12} < 2^{n-11}.$$

Thus the existence of 3 sets with union $[15, n]$ follows in $\mathscr{G}$, implying

$$|G \cap [14]| \leqslant 3 \qquad \text{for all} \quad G \in \mathscr{G}. \tag{5.12}$$

We distinguish two cases.

(a) $\{11, 13, 14\} \notin \mathscr{G}$. Note that by shiftedness, $G_0 = \{12, 13, 14\}$ is the only possible 3-element set satisfying $G_0 = G \cap [14]$ for some $G \in \mathscr{G}$.

Set again $\mathscr{G}_0 = \{G \in \mathscr{G} : G \subset [12, n]\} \subset 2^{[12, n]}$. Then

$$|\mathscr{G}_0| \leqslant \frac{5}{16} 2^{n-11} \qquad \text{by Theorem 1.2.} \tag{5.13}$$

Also for $A \subset [14]$, $A \cap [11] \neq \varnothing$ define

$$\mathscr{G}(14, A) = \{G - A : G \cap [14] = A, G \in \mathscr{G}\} \subset 2^{[15, n]}.$$

CLAIM 5.9. *$\mathscr{G}(14, A)$ is dually 3-wise $(4|A|-2)$-intersecting for $|A|=1, 2$.*

Proof. Let $|A|=2$. Choose $H_1, H_2, H_3 \in \mathscr{G}(14, A)$ with maximal union. By shiftedness we may assume that $H_1 \cup H_2 \cup H_3 = [m, n]$ for some m. Again by shiftedness, $H_1 \cup \{11, 14\}$, $H_2 \cup \{12, 15\}$, and $H_3 \cup \{13, 16\}$ are in $\mathscr{G}$. Together with $K_2 = \{17, 18, 19\}$ their union is $[11, 19] \cup [m, n]$. Since $\mathscr{G}$ is dually 4-wise 11-intersecting, $m \geqslant 21$ follows, i.e., $\mathscr{G}(14, A)$ is dually 3-wise 6-intersecting.

The case $|A|=1$ is almost the same, except that we use $H_1 \cup \{11\}$, $H_2 \cup \{12\}$, $H_3 \cup \{13\}$ along with $K_1 = \{14, 15\}$. ∎

COROLLARY 5.10. *Let $A \subset [14]$ satisfy $A \cap [11] \neq \varnothing$. Then*

$$|\mathscr{G}(14, A)| \leqslant 2^{n-16} \qquad \textit{for} \quad |A|=1 \tag{5.14}$$

and

$$|\mathscr{G}(14, A)| \leqslant 2^{n-20} 5(\sqrt{5}-2)(\sqrt{5}-1)^2 \qquad \textit{for} \quad |A|=2. \tag{5.15}$$

Proof. Use Claim 5.9, Theorem 3.1, and Corollary 1.5 and $\alpha(3) = (\sqrt{5}-1)/2$. ∎

Since $|\mathscr{G}| = |\mathscr{G}_0| + \sum \{|\mathscr{G}(14, A)| : A \subset [14], A \cap [11] \neq \varnothing\}$ and $\{11, 13, 14\} \notin \mathscr{G}$, using shiftedness we infer from (5.13), (5.14), and (5.15),

$$\begin{aligned} |\mathscr{G}| &\leqslant 5 \cdot 2^{n-15} + 11 \cdot 2^{n-16} + 88 \cdot 2^{n-20} \cdot 5(\sqrt{5}-2)(\sqrt{5}-1)^2 \\ &< 31 \cdot 2^{n-16} < 2^{n-11}, \end{aligned}$$

as desired.

(b) $\{11, 13, 14\} \in \mathscr{G}$. In this case we can slightly improve Claim 5.9 for $|A|=1$.

CLAIM 5.11. *Let $A \subset [14]$ satisfy $A \cap [11] \neq \varnothing$. Then $\mathscr{G}(14, A)$ is dually 3-wise 3-intersecting for $|A|=1$ and 3-wise 10-intersecting for $|A|=3$.*

Proof. As in the proof of Claim 5.9 choose $H_1, H_2, H_3 \in \mathscr{G}(14, A)$ with $H_1 \cup H_2 \cup H_3 = [m, n]$, m as small as possible.

If $|A|=1$, then by shiftedness $H_1 \cup \{12\}$, $H_2 \cup \{15\}$, $H_3 \cup \{16\}$ are in $\mathscr{G}$. Their union along with $\{11, 13, 14\}$ is $[11, 16] \cup [m, n]$. This yields $m \geqslant 18$, as desired.

The proof for $|A|=3$ is the same as that of Claim 5.9, using $H_1 \cup \{11, 14, 17\}$, $H_2 \cup \{12, 15, 18\}$, $H_3 \cup \{13, 16, 19\}$, and $K_3 = \{20, 21, 22, 23\}$. ∎

Using Claim 5.11 together with Theorem 3.1 and Corollary 1.5 we infer

$$|\mathscr{G}(14, \{i\})| \leqslant 2^{n-17}, \qquad 1 \leqslant i \leqslant 11, \tag{5.16}$$

$$|\mathscr{G}(14, A)| \leqslant 2^{n-24}5(\sqrt{5}-2)(\sqrt{5}-1)^6$$
$$\text{for} \quad A \subset [14], \qquad A \cap [11] \neq \varnothing, \qquad |A| = 3. \tag{5.17}$$

From (5.13), (5.15), (5.16), and (5.17) we obtain

$$|\mathscr{G}| \leqslant 5 \cdot 2^{n-15} + 11 \cdot 2^{n-17} + 88 \cdot 2^{n-20}5(\sqrt{5}-2)(\sqrt{5}-1)^2$$
$$+ 363 \cdot 2^{n-24}5(\sqrt{5}-2)(\sqrt{5}-1)^6 < 31.4 \cdot 2^{n-16} < 2^{n-11},$$

as desired. ∎

6. An Extended Range for Conjectures 1.1 and 1.3

Throughout this section $\mathscr{F} \subset 2^{[n]}$ is a shifted, dually r-wise t-intersecting complex of maximal size $m(n, r, t)$. Recall from the Introduction the definition of the dually r-wise t-intersecting families $\mathscr{B}_i(n, r, t) = \{B \subset [n] : |B \cap [t+ir]| \leqslant i\}$.

Usually we consider n, r, and t as fixed and write simply $\mathscr{B}_i$ for $\mathscr{B}_i(n, r, t)$.

Theorem 6.1. *Conjecture* 1.1 *is true for* $t \leqslant 2^{r-2}(2^{r-2}-2)/(r-1)$.

Proof. In view of Theorem 5.5 we may suppose that $t \geqslant 2^r - r$ and thus $r \geqslant 7$.

Let s be the largest integer such that $\mathscr{F}$ is dually $(r-1)$-wise s-intersecting and suppose that $G_1, ..., G_{r-1} \in \mathscr{F}$ with $G_1 \cup \cdots \cup G_{r-1} = [s+1, n]$. Thus

$$|F \cap [s]| \leqslant s - t \qquad \text{for all} \quad F \in \mathscr{F}. \tag{6.1}$$

Since $|\mathscr{F}| = m(n, r, t) \geqslant 2^{n-t}$, Theorems 1.4, 5.5, and Proposition 2.8 imply

$$s/t < 1 + 1/2^{r-2}, \qquad \text{i.e.,} \qquad s - t < t/2^{r-2}. \tag{6.2}$$

Combining with (6.1) we obtain

$$|F \cap [t+b]| \leqslant b, \qquad \text{where} \quad b = \lfloor t/2^{r-2} \rfloor. \tag{6.3}$$

For convenience, set also,

$$m = \max_i |\mathscr{B}_i|. \tag{6.4}$$

Supposing that $\mathscr{F} \not\subset \mathscr{B}_i$, we must show that $|\mathscr{F}| < m$.

Let h be the smallest non-negative integer such that

$$|F \cap [t+h]| \leqslant h \qquad \text{holds for all} \quad F \in \mathscr{F}. \tag{6.5}$$

By $\mathscr{F} \not\subseteq \mathscr{B}_0$ and by (6.3) we have

$$1 \leqslant h < t/2^{r-2}. \tag{6.6}$$

The minimality of h implies $F_0 = [t, t+h-1] \in \mathscr{F}$.

For $A \subset [t+h-1]$ recall the definition $\mathscr{F}(t+h-1, A) = \{F - A : F \in \mathscr{F}, F \cap [t+h-1] = A\}$.

CLAIM 6.2. *For $A \subset [t+h-1]$ the family $\mathscr{F}(t+h-1, A)$ is dually $(r-1)$-wise $((r-1)|A|+1)$-intersecting for $|A| < h$ and $(r-1)h+2$-intersecting for $|A| = h$.*

Proof. (i) $|A| < h$. Suppose the contrary and choose by shiftedness $G_1, G_2, ..., G_{r-1} \in \mathscr{F}(t+h-1, A)$ satisfying $[t+h+(r-1)|A|, n] = G_1 \cup \cdots \cup G_{r-1}$. Using shiftedness $F_j = G_j \cup [t+h+(j-1)|A|, t+h+j|A|-1] \in \mathscr{F}$ hold for $1 \leqslant j < r$. Consequently, $F_0 \cup \cdots \cup F_{r-1} = [t, n]$, a contradiction.

(ii) $|A| = h$. Since $\mathscr{F} \not\subseteq \mathscr{B}_h$, $H_0 = [t+(r-1)h, t+rh] \in \mathscr{F}$ holds. Suppose again for contradiction that there exist $G_1, ..., G_{r-1} \in \mathscr{F}(t+h-1, A)$ with $G_1 \cup \cdots \cup G_{r-1} = [t+rh+1, n]$. Using shiftedness $H_j = G_j \cup [t+(j-1)h, t+jh-1] \in \mathscr{F}$ follows. Now $H_0 \cup \cdots \cup H_{r-1} = [t, n]$ gives the desired contradiction. ∎

Now we can easily prove $|\mathscr{F}| < 2m$. Namely, by Claim 6.2 and Theorem 5.5 we have $|\mathscr{F}(t+h-1, A)| \leqslant 2^{n-t-h-(r-1)|A|}$. Consequently,

$$|\mathscr{F}| \leqslant \sum_{0 \leqslant i \leqslant h} \binom{t+h}{i} 2^{n-t-ri-(h-i)} < \sum_{0 \leqslant i \leqslant h} |\mathscr{B}_i| \, 2^{-(h-i)} < 2m. \tag{6.7}$$

To remove the 2 we need the following.

CLAIM 6.3.

$$|\mathscr{F}(t+h-1, A)| \leqslant 2^{n-t-h-(r-1)|A|-1}$$

$$\textit{holds for} \quad A \subset [t+h-1], \ |A| \leqslant h. \tag{6.8}$$

Proof of (6.8). Note that for $0 \leqslant i \leqslant h-1$ the inequality $h \leqslant t/2^{r-2}$ implies $(r-1)i + r + 1 \leqslant 2^{r-2}$. Thus (6.8) follows from Claim 6.2, Theorem 5.5, and Theorem 3.4 unless $\mathscr{F}(t+h-1, A) \subset 2^{[t+h+(r-1)|A|+1, n]}$. To obtain (6.8) it is sufficient to show that $\mathscr{F}(t+h-1, A)$ is dually 2-wise intersecting on $[t+h+(r-1)|A|+1, n]$.

Actually it is even dually $(r-2)$-wise $(h-|A|)$-intersecting, because the contrary would mean the existence of $G_1, ..., G_{r-2} \in \mathscr{F}(t+h-1, A)$ satisfying $G_1 \cup \cdots \cup G_{r-2} = [t+2h+(r-2)|A|, n]$. Arguing as in the proof of Claim 6.2, it follows that the following sets are in $\mathscr{F}$.

$$F_j = [t+2h+(j-1)|A|, t+2h+j|A|-1] \cup G_j, \qquad 1 \leqslant j \leqslant r-2,$$

$$F_{r-1} = [t, t+h-1], \qquad F_r = [t+h, t+2h-1].$$

However, $F_1 \cup \cdots \cup F_r = [t, n]$, a contradiction.

For $|A| = h$ the inequality (6.8) follows directly from Claim 6.2 and Theorem 5.5 using $(r-1)h+2 \leqslant 2^{r-1}-r$, i.e., $h \leqslant (2^{r-1}-r-2)/(r-1)$ which is true for $t \leqslant 2^{r-2}(2^{r-1}-r-2)/(r-1)$. ∎

Now using (6.8) we infer

$$\begin{aligned} |\mathscr{F}| &= \sum_{A \subset [t+h-1]} |\mathscr{F}(t+h-1, A)| \\ &\leqslant \sum_{0 \leqslant i \leqslant h} \binom{t+h-1}{i} 2^{n-t-ri-(h-i+1)} \\ &< \sum_{0 \leqslant i \leqslant h} |\mathscr{B}_i|/2^{h-i+1} \leqslant (1-2^{-h-1})\, m. \end{aligned}$$ ∎

THEOREM 6.4. *Conjecture* 1.3 *is true for* $r \geqslant 5$.

Proof. Let $\mathscr{F} \subset 2^{[n]}$ be a dually r-wise t-intersecting shifted complex with $|\bigcup \mathscr{F}| > n-t$, $\mathscr{F} \not\cong \mathscr{B}_1(n, r, t)$. Supposing that $|\mathscr{F}| \geqslant |\mathscr{B}_1(n, r, t)|$ holds, we have to derive a contradiction. In view of Theorems 3.4 and 5.5 we may suppose that

$$2^r - 2r \leqslant t \leqslant 2^r - r - 2. \tag{6.9}$$

Thus

$$|\mathscr{F}|/2^{n-t} \geqslant (2^r - 2r + 1)/2^r \tag{6.10}$$

holds.

The proof will go along the lines of that of Theorem 6.1 and therefore we will be somewhat sketchy.

We claim that $\mathscr{F}$ is not dually $(r-1)$-wise $(t+3)$-intersecting. Indeed, the contrary would imply by Theorems 1.4, 5.5, and 5.8

$$|\mathscr{F}| \leqslant 2^{n-t-3}\alpha(r-1)^{t+3-2^{r-1}+r}.$$

Using (6.9) this contradicts (6.10) by direct computation if $r = 5$ and by Proposition 2.8 if $r \geqslant 6$.

By shiftedness $|F \cap [t+2]| \leqslant 2$ follows for all $F \in \mathscr{F}$. Let again h be the minimal integer such that

$$|F \cap [t+h]| \leqslant h \qquad \text{for all} \quad F \in \mathscr{F}. \tag{6.11}$$

Since $h=0$ would imply $\mathscr{F} \subset 2^{[t+1,n]}$, we have $h=1$ or 2.

Claim 6.3 is still valid giving

$$|\mathscr{F}| \leqslant 2^{n-t-2} + t \cdot 2^{n-t-r-1} \qquad \text{for} \quad h=1 \qquad \text{and} \tag{6.12}$$

$$|\mathscr{F}| \leqslant 2^{n-t-3} + (t+1) \cdot 2^{n-t-r-2} + \binom{t+1}{2} 2^{n-t-2r-1}$$

$$\text{for} \quad h=2. \tag{6.13}$$

Note that $2^r + 2(t+1) < 4(t+r+1)$ holds by (6.9). Thus the RHS of (6.12) is less than $|\mathscr{B}_1(n, r, t)|$ and the sum of the first two terms on the RHS of (6.13) is less than $|\mathscr{B}_1(n, r, t)|/2$. Since the third term is less than $|\mathscr{B}_2(n, r, t)|/2 < |\mathscr{B}_1(n, r, t)|/2$, both for $h=1$ and 2 the contradiction $|\mathscr{F}| < |\mathscr{B}_1(n, r, t)|$ is obtained. ∎

Remark 6.5. In view of Theorems 3.4, 3.1, 5.2, and 5.8 the only unproved cases of Conjecture 1.3 are $r=3$, $2 \leqslant t \leqslant 4$, and $r=4$, $8 \leqslant t \leqslant 10$.

7. Improved Bounds for Cross Intersecting Families

First we shall give an improvement of Theorem 2.4 along the lines of Theorem 1.4.

Let us define the quantity $b(n, r, t)$ by $b(n, r, t)^r = \max\{|\mathscr{F}_1| \cdots |\mathscr{F}_r| : \mathscr{F}_1, ..., \mathscr{F}_r \subset 2^{[n]}$ are r-cross t-intersecting$\}$. We need a simple lemma.

Proposition 7.1. *Let $\mathscr{G}_1, ..., \mathscr{G}_r \subset 2^{[n]}$ be shifted and r-cross t-intersecting. Let $0 \leqslant i \leqslant r$ be arbitrary. Then $\mathscr{G}_1(\bar{1}), ..., \mathscr{G}_i(\bar{1})$, $\mathscr{G}_{i+1}(1), ..., \mathscr{G}_r(1)$ are r-cross $(t+i-1)$-intersecting on $[2, n]$.*

Proof. Let $H_1, ..., H_r$ be arbitrary sets satisfying $H_j \in \mathscr{G}_j(\bar{1})$ for $1 \leqslant j \leqslant i$ and $H_j \in \mathscr{G}_j(1)$ for $i < j \leqslant r$. Set $H_1 \cap \cdots \cap H_r = \{a_1, ..., a_s\}$.

Define $K_j = (H_j - \{a_j\}) \cup \{1\}$ for $1 \leqslant j \leqslant \min\{i, s\}$ and $K_j = H_j \cup \{1\}$ for the remaining values of j. Then $K_j \in \mathscr{G}_j$ implies $|K_1 \cap \cdots \cap K_r| \geqslant t$. If $s < i$, then by construction $K_1 \cap \cdots \cap K_r = \varnothing$, a contradiction. Thus $s \geqslant i$ and consequently,

$$|H_1 \cap \cdots \cap H_r| = |K_1 \cap \cdots \cap K_r| + i - 1 \geqslant t + i - 1$$

follows. ∎

One more definition is needed.

DEFINITION 7.2. For $1 \leqslant i \leqslant r$ let $b^{(i)}(n, r, t)$ be defined by $b^{(i)}(n, r, t)^r = \max\{|\mathscr{F}_1| \cdots |\mathscr{F}_r| : \mathscr{F}_1, ..., \mathscr{F}_r \subset 2^{[n]}$ are r-cross t-intersecting and $|\bigcap \mathscr{F}_j| < t$ for $1 \leqslant j \leqslant i\}$.

Note that $b^{(i)}(n, r, t) = 0$ for $t + i > n$. Therefore we shall tacitly assume that $n \geqslant t + i$.

EXAMPLE 7.3. For $1 \leqslant i \leqslant r$ and $t \geqslant 1$ fixed define

$$\mathscr{F}_1 = \cdots = \mathscr{F}_i = \{F \subset [n] : |F \cap [t+i]| \geqslant t + i - 1\}$$

and

$$\mathscr{F}_{i+1} = \cdots = \mathscr{F}_r = \{F \subset [n] : [t+i] \subset F\}.$$

These families are r-cross t-intersecting with $\bigcap \{F : F \in \mathscr{F}_j\} = \varnothing$ for $1 \leqslant j \leqslant i$.

The following definition is slightly complicated but it is central for the proof of the main results of this section.

DEFINITION 7.4. Let M be an $r+1$ by infinity array with general entry $m(i, t)$, $0 \leqslant i \leqslant r$, $t \geqslant 1$. Then M is called an *admissible array of bounds* if

$m(i, t)\, 2^{rs} \geqslant b^{(i)}(i + t + s, r, t)^r$ holds for some pairs (i, t) and, in particular for *all* pairs with $t = 1$ ($s \geqslant 0$, arbitrary). (7.1)

Moreover,

$$m(i, t) \geqslant \sum_{0 \leqslant g \leqslant i} \binom{i}{g} m(i - g, t + g - 1)$$

holds for each of the remaining pairs (7.2)

and finally

$m(i, t)\, 2^{-ir}$ is monotone non-increasing for fixed t as a function of i. (7.3)

Hopefully, the next theorem will convince the reader that he did not lose his time by struggling through this definition.

THEOREM 7.5. *Let M be an admissible array of bounds. Then* (7.1), *that is,*

$$b^{(i)}(i + t + s, r, t)^r \leqslant m(i, t)\, 2^{rs} \qquad \textit{holds for all pairs } (i, t) \textit{ and all } s \geqslant 0. \tag{7.4}$$

Proof. Let $\mathscr{F}_1, ..., \mathscr{F}_r \subset 2^{[i+t+s]}$ be r-cross t-intersecting shifted filters (co-complexes) satisfying $|\bigcap \mathscr{F}_j| < t$ for $1 \leqslant j \leqslant i$ and $|\mathscr{F}_1| \cdots |\mathscr{F}_r| = b^{(i)}(i+t+s, r, t)^r$.

By (7.3) we have $m(i, t)\, 2^{rs} \leqslant m(i-1, t)\, 2^{r(s+1)} \leqslant \cdots \leqslant m(0, t)\, 2^{r(s+i)}$, that is, for t and $i+t+s$ fixed the RHS of (7.4) is a monotone non-increasing function of i. Therefore, in proving (7.4) we may assume that $|\bigcap \mathscr{F}_j| \geqslant t$ for $i < j \leqslant r$. Consequently, shiftedness implies $|\mathscr{F}_j(1)| = |\mathscr{F}_j|$ for $i \leqslant j \leqslant r$.

We apply induction on $i+t$. The case $i+t=1$ means $i=0$, $t=1$, and it is covered by (7.1). For the same reason we may assume that (i, t) is not covered by (7.1) and, in particular, $t \geqslant 2$.

For $A \subset [i]$ define

$$m(A) = \Big(\prod_{j \in A} |\mathscr{F}_j(\bar{1})|\Big)\Big(\prod_{j \in [r]-A} |\mathscr{F}_j(1)|\Big).$$

Then

$$|\mathscr{F}_1| \cdots |\mathscr{F}_r| = \sum_{A \subset [i]} m(A) \qquad \text{holds.} \tag{7.5}$$

On the other hand Proposition 7.1 and the induction hypothesis imply

$$m(A) \leqslant m(i-|A|, t+|A|-1)\, 2^{rs}. \tag{7.6}$$

Combining (7.5) and (7.6) and using (7.2) gives $|\mathscr{F}_1| \cdots |\mathscr{F}_r| \leqslant \sum_{0 \leqslant g \leqslant i} \binom{i}{g} m(i-g, t+g-1)\, 2^{rs} \leqslant m(i, t)\, 2^{rs}$, as desired. ∎

Now we have to exhibit some admissible arrays of bounds.

DEFINITION 7.6. For a fixed $r \geqslant 3$ define the array M by

$$m(i, t) = \begin{cases} (i+t+1)^i & \text{for } \; i+t \leqslant 2^r - r - 2 \\ (2^r - r)^r\, (2\alpha(r))^{r(t-2^r+2r+1)}\, 2^{-r(r-i)} & \text{for } \; i+t \geqslant 2^r - r - 1. \end{cases}$$

Our next task will be to show that M is admissible. We need some preparation.

PROPOSITION 7.7.

$$\lfloor 1/\alpha(r)^r \rfloor = 2^r - r - 1 \qquad \textit{holds for} \quad r \geqslant 3. \tag{7.7}$$

Proof. Equation (7.7) can be checked directly for $r=3$. Suppose $r \geqslant 4$ and recall that $\alpha(r)$ is the only root in $(1/2, 1)$ of $x^r - 2x + 1$. That is, $1/\alpha(r)$

is the only root in $(1, 2)$ of $f(y) = 1 - 2y^{r-1} + y^r$. Since $f(1) = 0, f(2) = 1$ it will be sufficient to show that

$$f((2^r - r - 1)^{1/r}) < 0 < f((2^r - r)^{1/r}) \qquad \text{holds.}$$

This inequality is equivalent to the inequalities

$$2^r(2^r - r - 1)^{r-1} > (2^r - r)^r$$

and

$$2^r(2^r - r)^{r-1} < (2^r - r + 1)^r.$$

The second one is a consequence of the inequality between arithmetic and geometric means.

The first can be rewritten as

$$\left(1 - \frac{1}{2^r - r}\right)^{r-1} > \left(1 - \frac{r}{2^r}\right).$$

However, using Bernoulli's inequality and $r \geqslant 4$ we deduce

$$\left(1 - \frac{1}{2^r - r}\right)^{r-1} > 1 - \frac{r-1}{2^r - r} \geqslant 1 - \frac{r(1 - 1/r)}{2^r(1 - 1/r)} = 1 - \frac{r}{2^r}$$

as desired. ▮

PROPOSITION 7.8. $(i+t)^{i-1} 2^r > (i+t+1)^i$ holds for $1 \leqslant i \leqslant r$ if $i + t \leqslant 2^r - r - 1$.

Proof. The desired inequality can be rewritten as

$$\left(1 - \frac{1}{i+t+1}\right)^{i-1} > \frac{i+t+1}{2^r}.$$

For $i+t$ fixed the LHS is a decreasing function of i, therefore we may assume that $i = r$. Taking rth roots and setting $y = (i+t)^{1/r}$ we can rewrite the original inequality as

$2y^{r-1} > y^r + 1,$ that is,

$y^r - 2y^{r-1} + 1 < 0,$ which holds for

$1 < y < 1/\alpha(r),$ thus the statement follows from the previous proposition. ▮

CLAIM 7.9. *The array M satisfies the monotonicity condition* (7.3).

Proof. We have to show that $2^r m(i-1, t) \geqslant m(i, t)$.

For $i+t \leqslant 2^r - r - 2$ this is immediate from Proposition 7.8. For $i+t = 2^r - r - 1$ by Propositions 7.7 and 7.8 we have the chain of inequalities

$$\begin{aligned}(2^r - r - 1)^{i-1}\, 2^r &= (2^r - r - 1)^{r-1}\, 2^r/(2^r - r - 1)^{r-i} \\ &> (2^r - r)^r\, \alpha(r)^{r(r-i)} \\ &= m(i, 2^r - r - 1 - t),\end{aligned}$$

as desired.

Finally, for $i+t \geqslant 2^r - r$ clearly $m(i-1, t)\, 2^r = m(i, t)$ holds. ∎

We also need the following simple fact.

PROPOSITION 7.10. *Let $\mathscr{G}_1, ..., \mathscr{G}_h \subset 2^Y$ be h-cross 1-intersecting. Then $|\mathscr{G}_1| \cdots |\mathscr{G}_h| \leqslant 2^{h(|Y|-1)}$ holds.*

Proof. For $1 \leqslant i < j \leqslant h$ the families $\mathscr{G}_i$ and $\mathscr{G}_j$ are 2-cross 1-intersecting. Thus $\mathscr{G}_i$ and $\mathscr{G}_j^c = \{Y - G : G \in \mathscr{G}_j\}$ are disjoint families. This yields

$$|\mathscr{G}_i| + |\mathscr{G}_j| \leqslant 2^{|Y|}, \qquad 1 \leqslant i < j \leqslant h. \tag{7.8}$$

Since $i \neq j$ were arbitrary, it follows that the arithmetic mean of $|\mathscr{G}_i|$, $1 \leqslant i \leqslant h$, is at most $2^{|Y|-1}$. Their geometric mean cannot be larger. ∎

Now we are ready to prove the main result.

THEOREM 7.11. *The array M is admissible for all $r \geqslant 3$.*

Proof. We apply induction on n. The case $n = 1$ is trivially true. For $(i, t) = (0, 1)$ the validity of (7.1) follows from Proposition 7.10 for all r. Now we prove by induction on $i+t$ that (7.1) holds for $t = 1$ and all $r \geqslant 3$ and (7.2) is satisfied for all $t \geqslant 2$, $r \geqslant 3$.

Let first $t = 1$ and let $\mathscr{F}_1, ..., \mathscr{F}_r \subset 2^{[n]}$ be r-cross 1-intersecting shifted co-complexes with $\bigcap \mathscr{F}_j = \varnothing$ if and only if $1 \leqslant j \leqslant i$.

We proceed exactly as in the proof of Theorem 7.5. However, in equality (7.5) we can't estimate directly the term $m(\varnothing)$, because the families $\mathscr{F}_1(1), ..., \mathscr{F}_r(1)$ need not be r-cross intersecting.

However, these families have the following property.

> Taking all families except $\mathscr{F}_j(1)$, $1 \leqslant j \leqslant i$, these families are $(r-1)$-cross intersecting. (7.9)

Indeed, otherwise $1 \in \bigcap \mathscr{F}_j$ would follow.

Now we prove that for $(i, r) \neq (2, 3)$ the property (7.9) implies the following upperbound.

$$m(\varnothing) = \prod_{1 \leqslant j \leqslant r} |\mathscr{F}_j(1)| \leqslant (i+1)^i \, 2^{r(n-i-1)}. \tag{7.10}$$

Using induction on n, Theorem 7.5 implies

$$m(A) \leqslant (i+1)^{i-|A|} \, 2^{r(n-i-1)}.$$

Now using this inequality and (7.10) from (7.5) it follows that $\prod_{1 \leqslant j \leqslant r} |\mathscr{F}_j| \leqslant 2^{r(n-i-1)} \sum_{0 \leqslant g \leqslant i} \binom{i}{g} (i+1)^{i-g} = 2^{r(n-i-1)} (i+2)^i$, as desired.

PROPOSITION 7.12. *Let $\mathscr{G}_1, ..., \mathscr{G}_r \subset 2^Y$ such that $\bigcap \mathscr{G}_j = \varnothing$ for $1 \leqslant j \leqslant i$ and taking all families except $\mathscr{G}_j$, $1 \leqslant j \leqslant i$, these families are $(r-1)$-cross intersecting. Assume also that Theorem 7.11 is proved for $n = |Y|$. Then $|\mathscr{G}_1| \cdots |\mathscr{G}_r| \leqslant (i+1)^i \, 2^{r(|Y|-i)}$ holds for $r \geqslant 3$, $i \geqslant 1$, and $(r, i) \neq (3, 2)$.*

Unfortunately, we did not find a unified proof for this and therefore we postpone the somewhat lengthy argument together with the proof of validity of (7.1) for $(i, r, t) = (2, 3, 1)$ until after the end of the proof of Theorem 7.11.

Thus we will show first that (7.2) holds for all (i, t) with $t \geqslant 2$.

(a) $i + t \leqslant 2^r - r - 1$. The RHS of (7.2) becomes $\sum_{0 \leqslant g \leqslant i} \binom{i}{g} (i+t)^{i-g} = (i+t+1)^i$, that is, (7.2) holds with equality unless $i + t = 2^r - r - 1$. In this case the inequality is strict because $(2^r - r)^i < (2^r - r)^r \, \alpha(r)^{r(r-i)}$ holds in view of (7.7).

(b) $i + t \geqslant 2^r - r$. Note that now $m(i-j, t-1+j) = m(i, t)(2\alpha(r))^{-r} \alpha(r)^{rj}$ holds for $0 \leqslant j \leqslant i$. Thus equality follows for $i = r$ and strict inequality for $0 \leqslant i < r$ because of $\sum_{0 \leqslant g \leqslant i} \binom{i}{g} \alpha(r)^{r(i-g)} = (1 + \alpha(r)^r)^i = (2\alpha(r))^i \leqslant (2\alpha(r))^r$. ∎

Proof of Proposition 7.12. Let us introduce the notation $x_j = |\mathscr{G}_j| / 2^{|Y|-i}$. Using the induction assumption and Proposition 7.10 we infer

$$\prod_{j \in A} x_j \leqslant (i+1)^{i-1} \quad \text{for all } A \subset [r] \text{ satisfying } |A \cap [i]| = i-1, \ |A| \geqslant 2, \text{ and } (|A|, i) \neq (2, 2). \tag{7.11}$$

In most cases one can deduce (7.10), that is,

$$x_1 \cdots x_r \leqslant (i+1)^i \quad \text{from (7.11)}. \tag{7.12}$$

However, we have to distinguish a few cases.

(i) $i=r$. Take the product of (7.11) over all $A \in \binom{[r]}{r-1}$.

(ii) $i=r-1 \geqslant 3$. Taking the product of (7.11) over all $A \in \binom{[r-1]}{r-2}$ leads to $x_1 \cdots x_{r-1} \leqslant r^{(r-1)}$. Similarly, using $A = A' \cup \{r\}$ over all $A' \in \binom{[r-1]}{r-2}$ leads to $(x_1 \cdots x_{r-1})^{r-2} x_r^{r-1} \leqslant r^{(r-1)(r-2)}$. The product of these inequalities gives $(x_1 \cdots x_r)^{r-1} \leqslant (r^{r-1})^{r-1}$, which is equivalent to (7.12).

(iii) $i=1$. Consider $A=[r-1]$ to obtain $x_1 \cdots x_{r-1} \leqslant 1$. Noting $x_r \leqslant 2^i = 2$, (7.12) follows.

(iv) $i \geqslant 3$, $r-i \geqslant 2$. Take the product of (7.11) over all $A \subset [r]$ $|A \cap [i]| = i-1$, first with $|A| = i$, then with $|A| = r-1$. We obtain after taking appropriate powers

$$(x_1 \cdots x_i)^{(i-1)(r-i)} (x_{i+1} \cdots x_r)^i \leqslant (i+1)^{(i-1)i(r-i)}$$

$$(x_i \cdots x_i)^{(i-1)((i-1)(r-i)-i)} (x_{i+1} \cdots x_r)^{i((i-1)(r-i)-i)}$$
$$\leqslant (i+1)^{(i-1)i((i-1)(r-i)-i)}.$$

The product of these inequalities is $(x_1 \cdots x_r)^{i(i-1)(r-i-1)} \leqslant (i+1)^{i^2(i-1)(r-i-1)}$, yielding (7.12).

(v) $i=2$, $r \geqslant 4$. Suppose by symmetry that $x_1 \leqslant x_2$. If $x_1 \leqslant 3$ then using this and (7.11) applied with $A=[2, r]$ gives $x_1 x_2 \cdots x_r \leqslant 3 \cdot 3 = 3^2$, as desired.

Suppose next $x_1 > 3$. Using (7.8) gives now $x_i + x_j \leqslant 4$ for $i=1, 2$ and $j=3, ..., r$. Consequently, $x_1 x_3 \leqslant 3$, $x_2 x_4 \leqslant 3$, and $x_j \leqslant 1$ for $5 \leqslant j \leqslant r$. Taking products (7.12) follows. ∎

Finally, we have to prove (7.1) for the case $r=3$, $i=2$.

PROPOSITION 7.13. *Suppose that Theorem* 7.11 *holds for* $n-1$. *Then*

$$b^{(2)}(n, 3, 1)^3 = 4^2 2^{3(n-3)}. \tag{7.13}$$

Proof. Let $\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3 \subset 2^{[n]}$ be 3-cross intersecting shifted co-complexes with $\bigcap \mathscr{F}_1 = \varnothing = \bigcap \mathscr{F}_2$ but $\bigcap \mathscr{F}_3 \neq \varnothing$.

Using the notation of the proof of Theorem 7.5,

$$m(A) \leqslant 3 \qquad \text{follows for } A = \{i\},\ i=1, 2, \text{and } m([2]) \leqslant 1 \text{ also.} \tag{7.14}$$

Thus $m(\varnothing) \leqslant 9$ would imply $|\mathscr{F}_1|\, |\mathscr{F}_2|\, |\mathscr{F}_3| \cdot 2^{-3(n-3)} \leqslant 16$, as desired. Consequently, we may assume that

$$|\mathscr{F}_1(1)|\, |\mathscr{F}_2(1)|\, |\mathscr{F}_3(1)| > 9 \cdot 2^{3(n-3)}.$$

We claim that this implies $|\mathscr{F}_3(1)| > 2^{n-3}$. Indeed, otherwise setting $x_i = |\mathscr{F}_i(1)|/2^{n-3}$ and using (7.8) gives

$$x_1 + x_3 \leqslant 4$$

$$x_2 + x_3 \leqslant 4 \qquad \text{and thus}$$

$$x_1 x_2 x_3 \leqslant x_3(4 - x_3)^2 \leqslant 9 \qquad \text{follows for} \quad x_3 \leqslant 1.$$

If $x_3 > 1$ then, $x_1 x_2 x_3 \leqslant x_3(4 - x_3)^2$ implies only

$$x_1 x_2 x_3 \leqslant \left(\frac{8}{3}\right)^2 \frac{4}{3}. \tag{7.15}$$

However, $|\mathscr{F}_3(1)| > 2^{n-3}$ implies $|\bigcap \mathscr{F}_3(1)| < 2$.

Consequently, $\mathscr{F}_1(\bar{1})$, $\mathscr{F}_2(\bar{1})$, $\mathscr{F}_3(1)$ are 3-cross 2-intersecting with $|\bigcap \mathscr{F}_3(1)| \leqslant 1$. In view of the induction assumption (7.4) applied with $r = 3$, $t = 2$, $i = 1$ we obtain the improved upper bound $m([2]) \leqslant 1/2$.

Combining this with (7.14) and (7.15) gives $|\mathscr{F}_1|\,|\mathscr{F}_2|\,|\mathscr{F}_3| \leqslant 2^{3(n-3)}(1/2 + 6 + 256/27) < 15.99 \cdot 2^{3(n-3)}$, completing the proof of (7.13). ∎

COROLLARY 7.14. *$b(n, r, t) = 2^{n-t}$ holds for $t \leqslant 2^r - r - 2$. Moreover, the only optimal families are $\mathscr{F}_1 = \cdots = \mathscr{F}_r = \{F \subset [n] : T \subset F\}$, where $T \in \binom{[n]}{t}$.*

Proof. The upper bound follows directly from Theorems 7.5 and 7.11. The uniqueness is a consequence of $b_0(n, r, t) < b_1(n, r, t)$ for the corresponding values of the parameters. ∎

8. THE CASES $r = 5$ AND 6 OF CONJECTURE 1.6

We use the notation of Section 4.

Suppose that $\mathscr{F} \subset 2^{[n]}$ is a dually r-wise intersecting complex, $\delta(\mathscr{F}) > 2^{n-r-1}$. We have to derive a contradiction.

LEMMA 8.1. *Suppose that there is a hole of size 3. Then $t(3) \leqslant 4$ for $r = 6$ and $t(3) \leqslant 2$ for $r = 5$.*

Proof. Let $[3]$ be the hole. Then $\mathscr{F}(1)$, $\mathscr{F}(2)$, and $\mathscr{F}(3)$ are 3-cross $t(3)$-intersecting on $[4, n]$. If $t(3) \geqslant 5$ then Corollary 7.14 implies

$$\min_{1 \leqslant i \leqslant 3} |\mathscr{F}_i(1)| \leqslant 2^{n-6}\alpha(3)^2 < 2^{n-7},$$

contradicting $\delta(\mathscr{F}) > 2^{n-7}$ for $r = 6$.

Similarly, for $t(3) \geqslant 3$ we have

$$\min_{1 \leqslant i \leqslant 3} |\mathscr{F}_i(1)| \leqslant 2^{n-6},$$

contradiction if $r = 5$. ∎

LEMMA 8.2.

$$4 \leqslant t(3) \leqslant 5 \quad \text{if} \quad r = 5$$

and

$$6 \leqslant t(3) \leqslant 7 \quad \text{if} \quad r = 6.$$

Proof. Since $\delta(\mathscr{F}) \leqslant |\mathscr{F}|/2$, $|\mathscr{F}| \geqslant 2^{n-r}$. This and Corollary 7.14 imply the upper bounds.

To prove the lower bounds, recall $t(3) \geqslant r-2$ for all dually r-intersecting families with $\delta(\mathscr{F}) > 0$. Since $t(3) = r-2$ implies the existence of a hole of size $r-2$, we get a contradiction for $r = 5$. For $r = 6$ we have a hole (say $[4]$) of size four if $t(3) = 4$. Then $\mathscr{F}(1)$, $\mathscr{F}(2)$, $\mathscr{F}(3)$ are 3-cross 3-intersecting on $[5, n]$. This would yield again $\min_{1 \leqslant i \leqslant 3} |\mathscr{F}(i)| \leqslant 2^{n-7}$.

The only remaining case is $r = 6$ and $t(3) = 5$. However, in view of Propositions 4.2 and 4.5 this would imply the existence of a hole of size 3, contradicting Lemma 8.1. ∎

Let us deal with the remaining four cases one by one.

(a) $r = 6$, $t(3) = 6$. Suppose that $F_1 \cup F_2 \cup F_3 = [7, n]$ and consider $\mathscr{G} = \mathscr{F}|_{[6]}$. Since no hole of size 3 exists $\mathscr{G}$ contains no sets of size 3 or more.

Define, as before, $\mathscr{F}(6, A) = \{F - A : F \in \mathscr{F}, F \cap [6] = A\} \subset 2^{[7, n]}$. If $|\mathscr{F}(6, \{i\})| < 2^{n-7}/6$ holds for some $1 \leqslant i \leqslant 6$ then we infer

$$|\mathscr{F}(i)| = |\mathscr{F}(6, \{i\})| + \sum_{i \neq j \in [6]} |\mathscr{F}(6, \{i, j\})| \leqslant 6\, |\mathscr{F}(6, \{i\}| < 2^{n-7}.$$

Otherwise, using Theorems 7.5 and 7.11 for the 3-cross 4-intersecting families $\mathscr{F}(6, \{i\})$, $\mathscr{F}(6, \{j\})$, and $\mathscr{F}(6, \{k, l\})$, where $\{i, j, k, l\} \in \binom{[6]}{4}$ gives

$$|\mathscr{F}(6, \{k, l\})| \leqslant \left(2^{n-10} \cdot 5 \cdot \left(\frac{\sqrt{5}-1}{2}\right)^3\right)^3 \Big/ (2^{n-7}/6)^2 < 2^{n-10}$$

for all 2-element sets $\{k, l\} \subset [6]$.

On the other hand the families $\mathscr{F}(6, \{i\})$, $1 \leqslant i \leqslant 3$, are 3-cross 3-intersecting. By Corollary 7.14 and by symmetry we may assume that $|\mathscr{F}(6, \{1\})| \leqslant 2^{n-9}$.

In view of the preceding inequality, $|\mathscr{F}(1)| \leqslant 2^{n-9}+5\cdot 2^{n-10}<2^{n-7}$, a contradiction follows.

(b) $r=6$, $t(3)=7$. Suppose that $F_1\cup F_2\cup F_3=[8,n]$ and define $\mathscr{G}=\mathscr{F}|_{[7]}$. The families $\mathscr{F}(i)|_{[8,n]}$, $i=1,2,3$, are dually 3-cross 3-intersecting. By Corollary 7.14 and symmetry we may assume that $|\mathscr{F}(1)|_{[8,n]}| \leqslant 2^{n-10}$.

Now $|\mathscr{F}(1)|>2^{n-7}$ implies that the degree $|\mathscr{G}(1)|$ of 1 in $\mathscr{G}$ is at least 9. Consequently there are some 3-element sets in $\mathscr{G}$. By symmetry suppose $[3]\in\mathscr{G}$. Now $\mathscr{H}=\mathscr{F}|_{[4,7]}$ is dually intersecting and has no hole of size 3. The only possibility is that $\mathscr{H}$ contains exactly three 2-element sets which form a triangle. Let i be the fourth element of $[4,7]$. Then $\mathscr{F}(i)$ is a dually 3-wise 4-intersecting family on $[3]\cup[8,n]$. Consequently, $|\mathscr{F}(i)| \leqslant 2^{n-7}\alpha(3)<2^{n-7}$ by Corollary 4.10.

In the remaining two cases $r=5$. Let us suppose that $F_1\cup F_2\cup F_3=[t(3)+1,n]$ and define $\mathscr{G}=\mathscr{F}|_{[t(3)]}$.

Set also $\mathscr{G}^{(2)}=\{G\in\mathscr{G}: |G|=2\}$.

(c) $r=5$, $t(3)=5$. If $\mathscr{G}$ contains some 3-element set then let i, j be the remaining two elements of $[5]$. Now $\mathscr{F}(i)$, $\mathscr{F}(i)$, $\mathscr{F}(j)$ are dually 3-cross 5-intersecting on $[n]-\{i,j\}$. By Corollary 4.10,

$$\min\{|\mathscr{F}(i)|, |\mathscr{F}(j)|\} \leqslant 2^{n-5}\alpha(3)^2<2^{n-6}$$

follows.

Thus $|G|\leqslant 2$ for all $G\in\mathscr{G}$.

For $B\subset[5]$ define $f(B)=|\{F\in\mathscr{F}: F\cap[5]=B\}|$, also $f(i)=f(\{i\})$.

By Corollary 7.10 we have the inequality

$$f(B_1)\,f(B_2)\,f(B_3)\leqslant (2^{n-8}\alpha(3)^{|B_1\cup B_2\cup B_3|-3})^3. \tag{8.1}$$

If for some $i\in[5]$ we had $f(i)\leqslant 2^{n-6}/5$, then $f(B)\leqslant f(i)$ for $i\in B$ would imply

$$|\mathscr{F}(i)|=f(i)+\sum_{j\in([5]-\{i\})} f(\{i,j\})\leqslant 5f(i)\leqslant 2^{n-6},$$

a contradiction.

Applying (8.1) with $|B_1|=|B_2|=1$ and $|B_3|=2$ and $|B_1\cup B_2\cup B_3|=4$ gives

$$f(B_3)\leqslant 25\cdot 2^{n-12}\alpha(3)^3<2^{n-9} \qquad \text{for all} \quad B_3\in\binom{[5]}{2}. \tag{8.2}$$

Using (8.1) with B_1, B_2, and B_3 distinct 1-element sets ensures the existence of $i\in[5]$ with $f(i)\leqslant 2^{n-8}$.

Combined with (8.2) this gives

$$|\mathscr{F}(i)| \leqslant 2^{n-8} + 4 \cdot 2^{n-9} < 2^{n-6}$$

the final contradiction.

(d) $r = 5$, $t(3) = 4$. Since $\mathscr{G} \subset 2^{[4]}$ is dually 2-wise 1-intersecting, and since there is no hole of size 3 (by Lemma 8.1), we may suppose that $\mathscr{G}^{(2)} = \binom{[3]}{2}$.

Using the same notation as in (c) we obtain

$$f(i)\, f(B)\, f(4) \leqslant (2^{n-7})^3\, \alpha(3)^{3(|B|-1)}$$

$$\text{for} \quad \varnothing \neq B \subset [3], \qquad 1 \leqslant i \leqslant 3, \qquad i \notin B. \qquad (8.3)$$

Recall $|\mathscr{F}(4)| = f(4) > 2^{n-6}$. Also, if $f(i) \leqslant 2^{n-6}/3$, then $|\mathscr{F}(i)| \leqslant 3f(i) \leqslant 2^{n-6}$ would follow. Thus (8.3) implies

$$f(B) \leqslant 3 \cdot 2^{n-9} \alpha(3)^3 < 3 \cdot 2^{n-10} \qquad \text{for all} \quad B \in \binom{[3]}{2}.$$

Now applying (8.3) with $|B| = 1$ shows the existence of $j \in [3]$ with $f(j) \leqslant 2^{n-7-1/2} < 2^{n-7}$. This in turn gives

$$|\mathscr{F}(j)| < 2^{n-7} + 6 \cdot 2^{n-10} < 2^{n-6},$$

the contradiction concluding the whole proof. ∎

9. Bounds on the Minimum Degree in Cross-Intersecting Families

Throughout this section $\mathscr{F}_1, \ldots, \mathscr{F}_r \subset 2^{[n]}$ are fixed, dually r-cross t-intersecting complexes. We shall denote by δ the minimum of their minimum degrees:

$$\delta = \min_{1 \leqslant j \leqslant r} \min_{i \in [n]} |\mathscr{F}_j(i)|.$$

Definition 9.1. For $2 \leqslant s \leqslant r$ let $t(s)$ be the maximum integer such that $\mathscr{F}_{i_1}, \ldots, \mathscr{F}_{i_s}$ are dually s-cross $t(s)$-intersecting for all choices of $1 \leqslant i_1 < \cdots < i_s \leqslant r$.

Recall the definition of a hole from Section 4. For $A \subset [n]$ define

$$H(A) = \{j \colon A \text{ is a hole in } \mathscr{F}_j\}, \qquad h(A) = |H(A)|.$$

In most cases we can assume without loss of generality that $H(A) =$

$[h(A)]$, and we shall often use this assumption without referring to it—for notational convenience. Also, we often suppose that $A = [|A|]$.

PROPOSITION 9.2. *Suppose that* $s \geqslant 2$ *and* $A \subset [n]$ *satisfy* $|A| \geqslant s$, $h(A) \geqslant s$. *Then*

$$\delta \leqslant b(n - |A|, s, t(s) - |A| + s). \tag{9.1}$$

Proof. Suppose that $[s] \subset A$ and $[s] \subset H(A)$. Consider $\mathscr{F}_i(i) \subset 2^{[n]-A}$, $1 \leqslant i \leqslant s$. Then these families are dually s-cross $(t(s) - |A| + s)$-intersecting on $[n] - A$. By definition,

$$\prod_{1 \leqslant i \leqslant s} |\mathscr{F}_i(i)| \leqslant b(n - |A|, s, t(s) - |A| + s)^s,$$

yielding (9.1). ∎

PROPOSITION 9.3. *If* $\delta > 0$ *then* $t(s) \geqslant t(s+1) + 1$ *holds for* $2 \leqslant s < r$. *Moreover, in case of equality there is a* $t(s)$*-element set* A, *satisfying* $h(A) \geqslant r - s$.

Proof. Suppose that $F_i \in \mathscr{F}_i$ and $|F_1 \cup \cdots \cup F_s| = n - t(s)$. Set $A = [n] - (F_1 \cup \cdots \cup F_s)$. Choose $F \in (\mathscr{F}_{s+1} \cup \cdots \cup \mathscr{F}_r)$ with $|F \cap A|$ as large as possible. Then $t(s+1) \leqslant t(s) - |F \cap A|$ holds. By $\delta \geqslant 1$ we have $F \cap A \neq \varnothing$. Moreover, $|F \cap A| = 1$ means that A is a hole in $\mathscr{F}_{s+1}, \ldots, \mathscr{F}_r$. ∎

The main result of this section establishes Conjecture 1.8 in a wide range of cases in the more general setting of dually r-cross t-intersecting families.

THEOREM 9.4. $\delta \leqslant 2^{n-r-t}$ *holds in each of the following cases*:

(i) $t \leqslant 2^{\lceil (r-p/2 \rceil} - r - \lceil (r-p)/2 \rceil - 1$ *where* p *is the largest integer satisfying* $2^p - 2 < r$

(ii) $t = 1$, $r \geqslant 7$.

Proof. Arguing indirectly we assume

$$\delta > 2^{n-r-t}. \tag{9.2}$$

CLAIM 9.5. *Let* $d \geqslant s \geqslant 2$ *be integers with* $2^s - s - 2 \geqslant r + t - d$ *then there is no* d*-set* D *with* $h(D) \geqslant s$.

Proof of the Claim. Suppose for contradiction that $H([d]) \supset [s]$. By Proposition 9.3 the families $\mathscr{F}_i(i)$, $1 \leqslant i \leqslant s$, are dually s-cross $(t + r - d)$-intersecting on $[d+1, n]$. From Corollary 7.14 we infer $\prod_{1 \leqslant i \leqslant s} |\mathscr{F}_i(i)| \leqslant 2^{s(n-r-t)}$, contradicting (9.2). ∎

CLAIM 9.6. *Suppose that* $b(n, s, q) \leqslant 2^{n-r-t+1}$ *then* $t(s) < q$.

Proof. Otherwise we may suppose that $|\mathscr{F}_1| \leqslant 2^{n-r-t+1}$ and since $\mathscr{F}_1$ is a complex, $\delta(\mathscr{F}_1) \leqslant (1/2)\,|\mathscr{F}_1| \leqslant 2^{n-r-t}$ follows. This contradicts (9.2). ∎

Clearly we may assume that $t(r) = t$ holds. Let P be the largest integer satisfying

$$t(r-p) = t + p. \tag{9.3}$$

This implies the existence of a $(t+p)$-element set D with $h(D) \geqslant p$. Thus Claim 9.5 implies that

$$2^p - 2 < r \qquad \text{holds.} \tag{9.4}$$

Using the definition of p and Proposition 9.3 we infer

$$t(r-p-b) \geqslant t + p + 2b \qquad \text{for} \quad 1 \leqslant b \leqslant r-p-2. \tag{9.5}$$

Consequently, for $b = \lceil (r-p-1)/2 \rceil$ we have $t(r-p-b) \geqslant t + r - 1$. If $2^{r-p-b} - (r-p-b) - 2 \geqslant t + r - 1$, then using Claim 9.6 and Corollary 7.14 we get a contradiction. Consequently,

$$2^{\lceil (r-p)/2 \rceil} - \lceil (r-p)/2 \rceil - 2 < t + r - 1 \qquad \text{holds.} \tag{9.6}$$

This is impossible in case (i), thus for that case the proof is finished.

Since the cases $t = 1$, $r \geqslant 10$ are covered by (i), we may suppose that $t = 1$, $7 \leqslant r \leqslant 9$ for the rest of the proof.

In view of (9.3)–(9.5) we have $t(4) \geqslant 4$ for $r = 7$, $t(r) \geqslant 6$ for $r = 8$, and $t(r) \geqslant 8$ for $r = 9$.

On the other hand applying Claim 9.6 for $s = 4$ gives $t(4) \leqslant r - 1$ in all three cases.

These considerations restrict the value of $(r, t(r))$ to the following 6 possibilities which we now examine. First we get rid of half of the cases.

Suppose that in (9.3) the value of p is 3, the maximum permitted by (9.4). That is, $t(r-3) = 4$.

Then, we may assume that $[3] \subset H([4])$ and thus $\mathscr{F}_1(1)$, $\mathscr{F}_2(2)$, $\mathscr{F}_3(3)$ are 3-cross $(t(3)-1)$-intersecting on $[5, n]$. Now $p \leqslant 3$ implies $t(3) \geqslant 6$ for $r = 7$, $t(3) \geqslant 8$ for $r = 8$, and $t(3) \geqslant 10$ for $r = 9$. These assumptions lead to the respective contradictions

$$\min_{1 \leqslant i \leqslant 3} |\mathscr{F}_i(i)| \leqslant 2^{n-7}\alpha(3)^2 < 2^{n-8} \qquad (r = 7)$$

$$\min_{1 \leqslant i \leqslant 3} |\mathscr{F}_i(i)| \leqslant 2^{n-7}\alpha(3)^4 < 2^{n-9} \qquad (r = 8)$$

$$\min_{1 \leqslant i \leqslant 3} |\mathscr{F}_i(i)| \leqslant 2^{n-7}\alpha(3)^6 < 2^{n-11} < 2^{n-10} \qquad (r = 9).$$

This proves $t(r-3)\geqslant 5$ for all cases, yielding $t(4)\geqslant 7$ for $r=8$ and $t(4)\geqslant 9$ for $r=9$. This last inequality contradicts $t(4)\leqslant 8$ for $r=9$, and concludes the proof for that case. For the case $r=8$ only one case remains ($t(4)=7$, $t(5)=5$). From now on $r=7$ or 8. Note that $t(3)\geqslant t(4)+2$ holds.

Recall the definition of $\mathscr{F}_i(b, B)$:

$$\mathscr{F}_i(b, B)=\{F-B\colon F\in\mathscr{F}_i,\ F\cap[b]=B\}\subset 2^{[b+1,n]}.$$

CLAIM 9.7. *Let $d\geqslant 1$ be an integer, $j\in[b]$. If*

$$|\mathscr{F}_i(b,\{j\})|\leqslant 2^{n-r-1}\Big/\sum_{l<d}\binom{b-1}{l}$$

then there is some D with $j\in D\in\binom{[b]}{d+1}$ satisfying $\mathscr{F}_i(b, D)\neq\varnothing$.

Proof. Suppose the contrary. Then we have

$$|\mathscr{F}_i(j)|=\sum_{j\in D\subset[b]}|\mathscr{F}_i(b, D)|\leqslant|\mathscr{F}_i(b,\{j\}|\sum_{l<d}\binom{b-1}{l}\leqslant 2^{n-r-1},$$

a contradiction. ∎

Let $3\leqslant s\leqslant r-3$ and suppose that $F_r\in\mathscr{F}_r, ..., \mathscr{F}_{s+1}\in\mathscr{F}_{s+1}$ satisfy

$$F_r\cup\cdots\cup F_{s+1}=[t(r-s)+1, n].$$

Then $\mathscr{F}_i(t(4), A_i)$, $1\leqslant i\leqslant s$, are dually 5-cross $(t(s)-t(r-s)+|A_1\cup\cdots\cup A_{r-4}|)$-intersecting. We shall refer to this as the *standard assumption* for s.

We distinguish three cases

(a) $r=8$, $t(4)=7$ or $r=7$, $t(r)=5$, $t(3)\geqslant 8$. In this case the standard assumption for $s=r-4$ and Claim 9.7 applied with $d=2$, $b=t(4)$ imply

$$|\mathscr{F}_i(t(4),\{j\})|>2^{n-r-1}/t(4)\qquad\text{for all}\quad 1\leqslant i\leqslant r-4,\qquad 1\leqslant j\leqslant t(4). \tag{9.7}$$

Now using the standard assumption for $r=8$ with a 2-element set and disjoint singletons gives

$$|\mathscr{F}_i(7, A)|<(2^{n-12})^4/(2^{n-9}/7)^3=7^3\cdot 2^{n-21}\qquad\text{for all } i \text{ and all}\quad A\in\binom{[7]}{2}. \tag{9.8}$$

By the standard assumption for $s=3$ and by symmetry we may assume that

$$|\mathscr{F}_1(7,\{1\})|\leqslant 2^{n-11}. \tag{9.9}$$

Combining (9.8) and (9.9) gives

$$\mathscr{F}_1(1) < 2^{n-11} + 6 \cdot 7^3 \cdot 2^{n-21} < 0.8 \cdot 2^{n-9},$$

a contradiction.

Similarly, in the case $r = 7$, $t(4) = 5$, $t(3) \geqslant 8$ we obtain

$$|\mathscr{F}_i(5, A)| < (2^{n-8}\alpha(3)^4)^3/(2^{n-8}/5)^2 < 2^{n-8}/12.$$

Combining with, say,

$$|\mathscr{F}_1(5, \{1\})| \leqslant 2^{n-8}\alpha(3)^3 < 2^{n-8}/4, \qquad |\mathscr{F}_1(1)| < 2^{n-8}$$

follows.

(b) $r = 7$, $t(3) \geqslant 7$, $t(4) = t(3) - 2$. Choose, by symmetry, $F_i \in \mathscr{F}_i$, $5 \leqslant i \leqslant 7$, such that

$$F_5 \cup F_6 \cup F_7 = [t(3) + 1, n].$$

Now $\mathscr{F}_i(t(3), \{i\})$, $i = 1, 2, 3$, are dually 3-cross 3-intersecting. By symmetry we may assume

$$|\mathscr{F}_1(t(3), \{1\})| \leqslant 2^{n-t(3)-3}. \tag{9.10}$$

Using $t(4) = t(3) - 2$, it follows that $|F \cap [t(3)]| \leqslant 2$ for all $F \in \mathscr{F}_1$. Consequently,

$$|\mathscr{F}_1(1)| \leqslant 2^{n-3} t(3)/2^{t(3)}.$$

The RHS is monotone decreasing in $t(3)$ and for $t(3) = 8$ its value is 2^{n-8}. Thus we may assume that $t(3) = 7$ and that

$$|\mathscr{F}_i(7, \{j\})| \geqslant 2^{n-8}/7 \qquad \text{for all} \quad 1 \leqslant i \leqslant 4, \qquad 1 \leqslant j \leqslant 7. \tag{9.11}$$

This assumption, as before, implies

$$|\mathscr{F}_i(7, A)| \leqslant (2^{n-10}\alpha(3))^3/(2^{n-8}/7)^2 < 2^{n-8}/5.$$

If we can find a 4-element set $\{j_1, j_2, j_3, j_4\} \subset [7]$ such that

$$|\mathscr{F}_i(7, \{j_i\})| \geqslant 2^{n-8}/5 \qquad \text{for} \quad 1 \leqslant i \leqslant 4$$

then for every $A \in \binom{[7]}{2}$ at least 2-elements out of $j_1, ..., j_4$ are outside A. Considering $\mathscr{F}_i(7, A)$ with the corresponding 2 families $\mathscr{F}_{i'}(7, \{j_{i'}\})$ gives

$$|\mathscr{F}_i(7, A)| < (2^{n-10}\alpha(3))^3/(2^{n-8}/5)^2 < 2^{n-8}/10.$$

In analogy with (9.10) we may suppose that

$$|\mathscr{F}_{i_1}(7, \{j_1\})| \leqslant 2^{n-10}.$$

Combining these inequalities gives

$$|\mathscr{F}_{i_1}(j_1)| < 2^{n-8}\left(\frac{7}{10}+\frac{1}{4}\right) < 2^{n-8}.$$

If we can find distinct i_1, i_2 and distinct j_1, j_2 with $|\mathscr{F}_{i_l}(7, \{j_l\})| \geqslant 2^{n-8}/3$ for $l = 1, 2$ then with $i_3 \in ([4] - \{i_1, i_2\})$ and $j_3 \in ([7] - \{j_1, j_2\})$ we obtain

$$|\mathscr{F}_{i_3}(7, \{j_3\})| \leqslant (2^{n-10})^3/(2^{n-8}/3)^2 < 2^{n-8}/7,$$

contradicting (9.11).

To conclude the proof in this case we will find such i_1, i_2, j_1, j_2. Assume the contrary.

Suppose without loss of generality that $\mathscr{F}_4(7, \{6, 7\}) \neq \emptyset$. If $|\mathscr{F}_i(7, \{5\})| < 2^{n-8}/3$, then it follows that for some $1 \leqslant j \leqslant 4$, $\mathscr{F}_i(7, \{j, 5\}) \neq \emptyset$ holds. By symmetry we may assume that $\mathscr{F}_3(7, \{4, 5\}) \neq \emptyset$.

Now $|F \cap [3]| \leqslant 1$ follows from the dually 7-cross 1-intersecting property for $i = 1, 2$.

We may assume that $|\mathscr{F}_2(7, \{1\})| < 2^{n-8}/3$. Thus, $\mathscr{F}_2(7, \{1, b\}) \neq \emptyset$ holds for at least 3 choices of $b \in [2, 7]$.

Now $\mathscr{F}_1(7, \{2, c\}) = \emptyset$ follows for all $c \neq 2$, $c \in [7]$, because otherwise $[7]$ is covered by the union of 4 sets, one each from $\mathscr{F}_1, ..., \mathscr{F}_4$.

This means $|\mathscr{F}_1(7, \{2\})| = |\mathscr{F}_1(7, 2)| > 2^{n-8}$, leading to

$$|\mathscr{F}_3(7, \{3\})|\ |\mathscr{F}_4(7, \{4\})| < (2^{n-10})^3/2^{n-8} = (2^{n-11})^2.$$

Consequently

$$|\mathscr{F}_j(7, \{j\})| < 2^{n-8}/8 \qquad \text{holds for} \quad j = 3 \text{ or } 4,$$

in contradiction with (9.11).

(c) $t(4) = 6$, $r = 7$. In view of the preceding case we may assume that $t(3) \geqslant 9$. However, this contradicts Claim 9.6. ∎

10. Families Containing All Small Sets

For a positive integer s, a family $\mathscr{F} \subset 2^{[n]}$ is called s-complete if $\binom{[n]}{s} \subset \mathscr{F}$.

With this terminology the Brace–Daykin Theorem (Theorem 1.2) determines the maximum size of dually r-wise intersecting 1-complete families. In this context it is natural to make the following.

DEFINITION 10.1. For $n \geqslant rs+t$ let $m(n, r, t, s)$ denote max $|\mathscr{F}|$ where the maximum is over all s-complete, dually r-wise t-intersecting $\mathscr{F} \subset 2^{[n]}$.

For $n < rs+t$ no such family exists and even for $n = rs+t$, trivially, $m(n, r, t, s) = \sum_{i \leqslant s} \binom{n}{i}$ holds, because no such $\mathscr{F}$ can contain sets of size exceeding s.

Let us give two examples.

EXAMPLE 10.2. $\mathscr{B}_l(n, r, t)$ for $l \geqslant s$.

EXAMPLE 10.3. Let $s \leqslant q \leqslant (n-s-t)/2$ and set

$$\mathscr{G}_q(n, t, s) = \{G \subset [n] : |G \cap [2q+s+t]| \leqslant s\} \cup \{G \subset [t+1, n] : |G \cap [t+1, 2q+s+t]| \leqslant q\}.$$

Note that $\mathscr{G}_q(n, t, s)$ is dually 3-wise t-intersecting and that for fixed t and s and $q = q(n) \to \infty$ one has

$$\lim_{n \to \infty} |\mathscr{G}_q(n, t, s)|\, 2^{-n} = 2^{-t-1} \qquad \text{holds.} \tag{10.1}$$

This shows that the function $p(r, t, s) = \lim_{n \to \infty} m(n, r, t, s)\, 2^{-n}$ has a positive lower bound for $r = 3$, t fixed, independent of s. For $r \geqslant 4$ the situation is different.

PROPOSITION 10.4. *For $r \geqslant 4$ and t fixed one has*

$$\lim_{s \to \infty} p(r, t, s) = 0.$$

Proof. Just observe the fact that every s-complete dually r-wise t-intersecting family (is dually $(r-1)$-wise $(s+t)$-intersecting. Consequently, $p(r, t, s) \leqslant \alpha(r-1)^{t+s}$. ∎

Note that s-completeness is invariant under shifting and therefore we may assume throughout the proofs in this section that $\mathscr{F}$ is a complex satisfying $S_{ji}(\mathscr{F}) = \mathscr{F}$ for all $1 \leqslant i < j \leqslant n$; i.e., $\mathscr{F}$ is shifted to the right.

PROPOSITION 10.5. *The following inequalities hold.*

$$m(n, r, t, s) \leqslant m(n, r-l, t+ls, s) \qquad \textit{for} \quad 1 \leqslant l \leqslant r-2. \tag{10.2}$$

$$m(n, r, t, s) \leqslant m(n-1, r-1, t+s-1, s) + m(n-1, r, t+r-1, s-1). \tag{10.3}$$

Note that $\mathscr{B}_s(n, r, t) = \mathscr{B}_s(n, r-l, t+ls)$ holds for $1 \leqslant l \leqslant r-2$. Using this fact, Theorem 6.1, and (10.2) one can prove that in many cases $m(n, r, t, s) = |\mathscr{B}_s(n, r, t)|$ holds. This motivates the following.

Conjecture 10.6. $m(n, r, t, s) = \max_{s \leqslant l \leqslant (n-t)/r} |\mathscr{B}_l(n, r, t)|$ holds for $r \geqslant 4$.

For $r=3$ we believe that either Example 10.2 or Example 10.3 is best possible.

Conjecture 10.7.

$$m(n, 3, t, s) = \max \left\{ \max_{s \leqslant q \leqslant (n-s-t)/2} |\mathscr{G}_q(n, t, s)|, \max_{s \leqslant l \leqslant (n-t)/3} |\mathscr{B}_l(n, t, 3)| \right\}.$$

Note that, for s fixed and $t > t_0(s)$, e.g., $t > s2^r$ both conjectures would follow from Conjecture 1.1.

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