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Simplices with Given 2-Face Areas

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Let V be a point set in a Euclidean space. We prove that if $|V| \geq 5$ and all triangles, each spanned by a triple of V, have the same area >0 then V forms the vertex set of a regular simplex. Further, if $|V|$ is large and the triangles have $r \geq 2$ different areas then the number of different distances between the pairs of V is at most $2r^3(2r+1)$.

1. INTRODUCTION

Every graph G can be represented in a Euclidean space as a unit distance graph; that is, there is an embedding $\{\bar{v}: v \in V(G)\} \subset R^n$ of $V(G)$ such that $uv \in E(G)$ iff $|\bar{u} - \bar{v}| = 1$, see, e.g., [1, 2].

As a similar kind of representation for a 3-uniform hypergraph H (shortly, a 3-graph), we may consider an embbeding $\{\bar{v}: v \in V(H)\} \subset R^n$ such that $uvw \in E(H)$ iff the area of the triangle $\bar{u}\bar{v}\bar{w}$ is equal to 1. We call such an embedding a *unit-area representation* of H. Then the first problem we meet would be whether every 3-graph admits a unit-area representation.

Let V be a point set in a Euclidean space. For each integer $k \geq 2$, the *color* of a k-subset of V means the $(k-1)$-dimensional volume of the convex hull of that k-subset. Then it will be proved that if $|V| \geq 5$ and $\binom{V}{3}$ is monochromatic with positive color then $\binom{V}{2}$ is also monochromatic, and hence V forms the vertex set of a regular simplex. (The symbol $\binom{V}{k}$ denotes the collection of all k-subsets of V.) From this it follows easily that the 3-graph obtained from the complete 3-graph on $n \geq 6$ vertices by removing one hyper-edge admits no unit-area representation. It will also be proved that among the 3-graphs on n vertices, the proportion of those 3-graphs that admit unit-area representation tends to 0 as $n \to \infty$. Further, we will prove that if $|V|$ is sufficiently large and $\binom{V}{3}$ has $r \geq 2$ colors, then $\binom{V}{2}$ has at most $2r^3(2r+1)$ colors.

2. THE INTERSECTION OF CYLINDERS

The following lemma is fundamental in this paper.

LEMMA. *Let* ABC *be a triangle in* 3*-space. Let* Q_A *be the cylinder with axis* BC *and passing through* A. *Define* Q_B *and* Q_C *similarly. Then* $Q_A \cap Q_B \cap Q_C$ *consists of either three points or five points, three of them are on the plane* ABC, *and the other two* (*if they exist*) *are symmetric with respect to the plane* ABC.

PROOF. Introduce a co-ordinate system in the following way. The origin is at the point A, the x-axis is parallel to the line BC, and the z-axis is perpendicular to the plane ABC. We may suppose that the line BC cuts the y-axis at $y = -1$, as in Figure 1. It is then easy to see that the three points X, Y and Z indicated in Figure 1 are common to Q_A, Q_B and Q_C. Now, the cylinder Q_A is described by the equation

$$(y+1)^2 + z^2 = 1. \tag{1}$$

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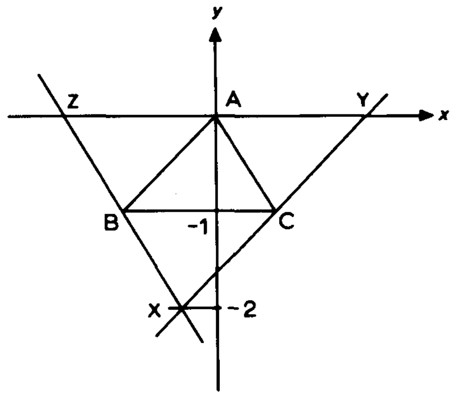

FIGURE 1.

Let d be the length of the edge BC. Then since (AC)sin $C = 1$, and since AC/sin B = BC/sin A, it follows that

$$d = (\sin A)/(\sin B \sin C).$$

Since the cylinder Q_C is obtained from the cylinder $y^2 + z^2 = d^2 \sin^2 B$ by a rotation around the z-axis by the angle $\angle B$, the equation of Q_C is

$$(-x \sin B + y \cos B)^2 + z^2 = d^2 \sin^2 B. \tag{2}$$

Similarly, the equation of Q_B is

$$(x \sin C + y \cos C)^2 + z^2 = d^2 \sin^2 C. \tag{3}$$

Now, eliminating x and z from (1), (2), (3), we obtain the equation

$$ay^4 + by^3 + cy^2 = 0, \tag{4}$$

where

$$a = (\sin^2 A - \sin^2 B - \sin^2 C)^2 - 4 \sin^2 B \sin^2 C,$$
$$b = 4(\sin^2 C - \sin^2 B)^2 - 4 \sin^2 A(\sin^2 B + \sin^2 C),$$
$$c = 4(\sin^2 C - \sin^2 B)^2 - 4 \sin^4 A.$$

(It is not difficult to see that $a \neq 0$.) Since the three points X, Y and Z are common to the three cylinders, the solutions of (4) are $y = 0$ (double root), $y = -2$ and $y = -c/(2a)$ $(=\alpha)$.

For $y = 0$ and -2, we have $z = 0$, and the corresponding points of $Q_A \cap Q_B \cap Q_C$ are the three points X, Y and Z indicated in Figure 1. It is clear that there are no other points of $Q_A \cap Q_B \cap Q_C$ on the plane ABC. Hence any possible other point (x, y, z) of $Q_A \cap Q_B \cap Q_C$ satisfies $z \neq 0$ and $y = \alpha$, and hence it must lie on a pair of generators of Q_A which are symmetric with respect to the x–y plane. That is, those points of $Q_A \cap Q_B \cap Q_C$ which are not on the plane ABC must lie on a symmetric pair of generators of Q_A with respect to the plane ABC. Hence those points must lie on a symmetric pair of generators of Q_B, and on that of Q_C as well. Therefore, $Q_A \cap Q_B \cap Q_C$ either has two more points that are symmetric with respect to the plane ABC, or has no point except X, Y, Z. □

Let V be a point set in a Euclidean space. For each integer $k > 1$, let $c:\binom{V}{k} \to R$ denote the coloring; that is, c is the map which assigns to each k-subset of V the $(k-1)$-dimensional volume of the convex hull spanned by the k-subset. Thus the color of $v_1v_2 \ldots v_k \in \binom{V}{k}$ is $c(v_1v_2 \ldots v_k)$. If V consists of $n+1 \geq 3$ points and $\binom{V}{n}$ is monochromatic, then we simply say that V is monochromatic. Thus, (the vertex set of) a triangle is monochromatic if it is an equilateral triangle, and (the vertex set of) a tetrahedron is monochromatic if its four faces have the same area.

COROLLARY. *Let* ABCD *and* A′B′C′D′ *be two monochromatic tetrahedra in* R^n. *If the triangles* ABC *and* A′B′C′ *are congruent, then the tetrahedra* ABCD *and* A′B′C′D′ *are congruent.*

PROOF. We may suppose that the two tetrahedra are contained in a same 3-space, i.e. we may assume $n = 3$. Let Q_A, Q_B and Q_C be the cylinders as in the above lemma. Then since ABC, BCD, CDA and DAB have the same area, the point D must be common to all Q_A, Q_B and Q_C. Since ABC is congruent to A′B′C′, we can put A′B′C′ on ABC. Then D′ is also common to all Q_A, Q_B and Q_C. Since D and D′ are not on the plane ABC, it follows either that D = D′ or that D and D′ are symmetric points with respect to the plane ABC, by the above lemma. Therefore, ABCD is congruent to A′B′C′D′ □

3. Point Sets With Monochromatic Triples

THEOREM 1. *Let V be a set in a Euclidean space. If $|V| \geq 5$ and $\binom{V}{3}$ is monochromatic with positive color, then $\binom{V}{2}$ is also monochromatic, i.e. V forms the vertex set of a regular simplex.*

PROOF. Since $\binom{V}{3}$ has positive color, V is not collinear. Furthermore, any four points span a tetrahedron. Indeed, if A, B, C and D lie on a plane then it is impossible to take another point P so that

$$c(\mathrm{PAB}) = c(\mathrm{PBC}) = c(\mathrm{PCD}) = c(\mathrm{PAD}) = c(\mathrm{PBD}) = c(\mathrm{PAC}),$$

which contradicts the assumption that V contains at least five points. Thus any four points of V span a monochromatic tetrahedron. Now, consider five points A, B, C, D and E of V. Then by the above corollary, ABCD is congruent to ABCE. Hence

$$c(\mathrm{AD}) = c(\mathrm{AE}), \qquad c(\mathrm{BD}) = c(\mathrm{BE}), \qquad c(\mathrm{CD}) = c(\mathrm{CE}).$$

Since the choice and order of A, B, C, D and E were arbitrary, we conclude that any two 'edges' from a fixed point have the same color, which implies that all edges have the same color, and hence $\binom{V}{2}$ is monochromatic. □

REMARK 1. In the above theorem, the assumption $n \geq 5$ is necessary, because the set of four vertices of a parallelogram is monochromatic.

REMARK 2. For every $n \geq 3$, there is a set V of $n+1$ points in R^n for which $\binom{V}{4}$ is monochromatic with positive color, but V does not span a regular simplex. Indeed, let V be the set of the following $n+1$ points in R^n:

$$\begin{aligned} P_0 &= (-1, 0, \ldots, 0), \\ P_1 &= (1, 0, \ldots, 0), \\ P_2 &= (0, 1, 0, \ldots, 0), \\ P_3 &= (0, 0, 1, 0, \ldots, 0), \\ &\vdots \\ P_n &= (0, \ldots, 0, 1). \end{aligned}$$

Then every four points of $V\setminus\{P_0\}$ span a regular simplex of side length $\sqrt{2}$, which has volume 1/3. Similarly, every four points of $V\setminus\{P_1\}$ span a regular simplex of side length $\sqrt{2}$. And, for any P_i, P_j $(j>i>1)$, P_0, P_1, P_i and P_j span a simplex congruent to $P_0P_1P_2P_3$, the volume of which is also 1/3.

As mentioned in Remark 1, there is an affinely dependent four-point set which is monochromatic with positive color. This is not the case for sets with an odd number of points.

THEOREM 2. *Let V be a point set in a Euclidean space and suppose that $n=|V|$ is odd ≥ 3. If V is monochromatic with positive color, then V is affinely independent (i.e. V spans $n-1$ dimensions).*

PROOF. Let c be the common color of all $(n-1)$-subsets of V. Suppose that V is contained in R^{n-2}, and let

$$v_i=(v_{i1},\ldots,v_{i(n-2)}), \qquad i=1,\ldots,n.$$

Let D_i be the determinant of the $(n-1)\times(n-1)$ matrix obtained from the $n\times(n-1)$ matrix

$$\begin{pmatrix} 1, v_{11},\ldots,v_{1(n-2)} \\ 1, v_{21},\ldots,v_{2(n-2)} \\ \vdots \\ 1, v_{n1},\ldots,v_{n(n-2)} \end{pmatrix}$$

by deleting the ith row. Then $c=|D_i/(n-2)!|$. Now, since

$$0=\det\begin{pmatrix} 1, 1, v_{11},\ldots,v_{1(n-2)} \\ \vdots \\ 1, 1, v_{n1},\ldots,v_{n(n-2)} \end{pmatrix}=\sum_{i=1}^{n}(-1)^{i+1}D_i,$$

we have

$$0=\varepsilon_1K+\cdots+\varepsilon_nK,$$

where $\varepsilon_i=\pm1$, $K=(n-2)!\,c>0$. However, since n is odd, this is clearly impossible. □

4. UNIT-AREA REPRESENTATIONS OF 3-GRAPHS

THEOREM 3. *There exists a 3-graph H which admits no unit-area representation.*

PROOF. Let $V=\{v_i: i=1,\ldots,6\}$, and let H be the complete 3-graph on V, i.e. $E(H)=\binom{V}{3}$. Let H' be the 3-graph obtained from H by removing a hyper-edge, say $v_1v_2v_3$. We show that H' admits no unit-area representation. Suppose $\bar{V}=\{\bar{v}_i: i=1,\ldots,6\}\subset R^n$ is a unit-area representation. Since $H'-v_1$ is a complete 3-graph on $\{v_i: i=2,\ldots,6\}$, $\bar{V}\setminus\{\bar{v}_1\}$ spans a regular 4-simplex, by Theorem 1. Similarly, since $H'-\{v_2\}$ and $H'-\{v_3\}$ are complete 3-graphs, it follows that $\bar{V}\setminus\{\bar{v}_2\}$ and $\bar{V}\setminus\{\bar{v}_3\}$ also span regular simplices. In this case, we must have $c(\bar{v}_1\bar{v}_2\bar{v}_3)=1$, which contradicts $v_1v_2v_3\notin E(H')$. □

Invoking Warren's Theorem [3] we can prove the following.

THEOREM 4. *Among all 3-graphs on n vertices, the proportion of those 3-graphs which admit unit-area representations tends to 0 as $n \to \infty$.*

First we recall Warren's theorem.

THEOREM (Warren [3]). *Let $p_1(x), \ldots, p_M(x)$ be real polynomials in N variables, each of degree at most $D \geq 1$. If $M \geq N$ then the number of sign sequences $(\operatorname{sgn} p_1(x), \ldots, \operatorname{sgn} p_M(x))$ that consist of terms $+1$, -1 does not exceed $(4eDM/N)^N$.*

PROOF OF THEOREM 4. Note that if a 3-graph on n vertices admits a unit-area representation, then it admits a unit-area representation in R^{n-1}. Let H^s, $s = 1, \ldots, m$ be those 3-graphs on $V = \{v_i: i = 1, \ldots, n\}$ that admit unit-area representations, and let $V^s = \{v_i^s: i = 1, \ldots, n\} \subset R^n$ be a unit-area representation of H^s, $s = 1, \ldots, m$.

Now, for each $1 \leq i < j < k \leq n$, define a polynomial $p_{ijk}(\mathbf{x})$ in $N = n^2$ variables $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) = (x_{11}, \ldots, x_{1n}, \ldots, x_{n1}, \ldots, x_{nn})$ by

$$p_{ijk}(\mathbf{x}) = (1/4)\det\begin{pmatrix} (\mathbf{x}_j - \mathbf{x}_i)\cdot(\mathbf{x}_j - \mathbf{x}_i) & (\mathbf{x}_j - \mathbf{x}_i)\cdot(\mathbf{x}_k - \mathbf{x}_i) \\ (\mathbf{x}_k - \mathbf{x}_i)\cdot(\mathbf{x}_j - \mathbf{x}_i) & (\mathbf{x}_k - \mathbf{x}_i)\cdot(\mathbf{x}_k - \mathbf{x}_i) \end{pmatrix},$$

where the dot denotes the inner product. Note that $p_{ijk}(\mathbf{x})$ is a polynomial of degree 4, and letting $\mathbf{v}^s = (v_1^s, \ldots, v_n^s) \in R^{n \times n}$, we have

$$c(v_i^s v_j^s v_k^s)^2 = p_{ijk}(\mathbf{v}^s).$$

Let ε be the minimum value of

$$[(p_{ijk}(\mathbf{v}^s)) - 1]^2$$

for all $p_{ijk}(\mathbf{v}^s) \neq 1$, $s = 1, \ldots, m$, $1 \leq i < j < k \leq n$. Then

$$[(p_{ijk}(\mathbf{v}^s)) - 1]^2 - \varepsilon/2 < 0 \qquad \text{if } v_i v_j v_j \in E(H^s),$$

and

$$[(p_{ijk}(\mathbf{v}^s)) - 1]^2 - \varepsilon/2 > 0 \qquad \text{if } v_i v_j v_j \notin E(H^s).$$

Let $q_{ijk}(\mathbf{x})$ be the polynomial

$$[(p_{ijk}(\mathbf{v}^s)) - 1]^2 - \varepsilon/2.$$

Then $q_{ijk}(\mathbf{x})$, $1 \leq i < j < k \leq n$, are $\binom{n}{3}$ polynomials of degree 8 in n^2 variables, and the sign sequences

$$(q_{ijk}(\mathbf{v}^s))_{ijk}, \qquad s = 1, \ldots, m$$

are all different. Hence by Warren's theorem, we have

$$m \leq (4eDM/N)^N < (16en/3)^{n^2}.$$

Since the number of distinct labeled 3-graphs on V is $2^{\binom{n}{3}}$, and by the above inequality

$$m/2^{\binom{n}{3}} \to 0 \qquad \text{as } n \to \infty,$$

the statement of the theorem follows. □

5. SETS WITH A BOUNDED NUMBER OF AREAS

THEOREM 5. *Let V be a point set in a Euclidean space. If $\binom{V}{3}$ has $r \geq 2$ colors and $|V|$ is sufficiently large, then*

$$\left|c\left(\binom{V}{2}\right)\right| \leq 2r^3(2r+1).$$

PROOF. First note that the assumption $r \geqslant 2$ implies that V is not collinear. Suppose that $|V|$ is sufficiently large, i.e. $|V|$ is greater than the Ramsey number $R_r^3(2r+2, \ldots, 2r+2)$. Then there exists a subset U of V such that $|U| \geqslant 2r+2$ and $\binom{U}{3}$ is monochromatic. In this case the color of $\binom{U}{3}$ is positive. Indeed, if U is on a line L then, taking a point v of V which is not on L, we can see that more than r colors appear in $\binom{U \cup \{v\}}{3}$. Thus by Theorem 1, $\binom{U}{2}$ is also monochromatic. Let b be the color of the pairs in $\binom{U}{2}$.

Here we note the following facts.

(1) There are at most two non-congruent triangles with given two sides and area.

(2) An isosceles triangle is determined by its base and area.

Now, for a $d > 0$, let

$$W(d) = \{w \in V : \text{for some } u_w \in U,\ c(u_w w) = d\}.$$

Then we claim the following.

(3) $|\{c(uw) : u \in U,\ w \in W(d)\}| \leqslant 2r+1$.

To see this consider a triangle $u_w u w$, $u \in U$. It has two sides of length b and d, and its area is one of the r values of $c(\binom{V}{3})$. Hence by (1), $c(uw)$ can take at most $2r$ different values. Therefore, taking the value d into account, we have (3).

Since $|U| > 2r+1$, for any v of V there exist two distinct points $u_1, u_2 \in U$ such that $c(vu_1) = c(vu_2)$. Then the triangle vu_1u_2 is isosceles with base b and area one of the r values of $c(\binom{V}{3})$. Hence by (2), the common value $c(vu_1) = c(vu_2)$ is one of certain r values, say $d_1, \ldots, d_r$. Thus, for any v of V there exists a point u_v of U such that

$$c(u_v v) \in \{d_1, \ldots, d_r\}.$$

Therefore, $V = W(d_1) \cup \cdots \cup W(d_r)$, and hence

$$|\{c(uv) : u \in U,\ v \in V\}| \leqslant r(2r+1).$$

Now we can evaluate the number of possible colors of vw, $v, w \in V$. Let u be a point of U such that $c(uv) \in \{d_1, \ldots, d_r\}$, and consider the triangle uvw. Then $c(uw)$ takes one of the $r(2r+1)$ values. Hence, by (1), $c(vw)$ can take at most

$$2r \times r \times r(2r+1) = 2r^3(2r+1)$$

different values. Therefore

$$\left|c\left(\binom{V}{2}\right)\right| \leqslant 2r^3(2r+1). \quad \square$$

REMARK 3. It would be worthwhile to determine the best possible bound in Theorem 5; that is, the maximum number, $m(r)$, of pairwise distances in a non-collinear point set with at most r areas. In view of the lemma and Theorem 1 one can prove that $m(1) = 6$.

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