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A LOWER BOUND ON THE SIZE OF A COMPLEX GENERATED BY AN ANTICHAIN

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A short proof of the following result of Kleitman is given: the total number of sets contained in some member of an antichain of size $\binom{n}{k}$ over the n-set is at least $\binom{n}{0}+\cdots+\binom{n}{k}$ for $0\leq k\leq \frac{1}{2}n$. An equally short proof of Harper's isoperimetric theorem is provided as well.

1. Introduction

Let $[n]=\{1,2,\ldots,n\}$ be an n-element set. A family $\mathscr{F}\subset 2^{[n]}$ is called an *antichain* if $F, F'\in\mathscr{F}$, $F\subset F'$ imply $F=F'$. A family $\mathscr{C}$ is called a complex if $\emptyset\in\mathscr{C}$ and $E\subset F\in\mathscr{C}$ implies $E\in\mathscr{C}$. There is a 1-1 correspondence between nonempty antichains and complexes. Namely if $\mathscr{C}$ is a complex then define the family of maximal sets in $\mathscr{C}$ by

$$\mathscr{A}(\mathscr{C})=\{A\in\mathscr{C}:\nexists B\in\mathscr{C},\ B\neq A,\ A\subset B\}.$$

Clearly, $\mathscr{A}(\mathscr{C})$ is an antichain and

$$\mathscr{C}=\{C\subset[n]:\exists A\in\mathscr{A}(\mathscr{C}),\ C\subset A\}.$$

We call $\mathscr{I}(\mathscr{C})=\mathscr{C}-\mathscr{A}(\mathscr{C})$ the *interior* of $\mathscr{C}$. Recall that $|\mathscr{F}|$ is called the *size* of $\mathscr{F}$. The main result of this note is the following.

Theorem 1.1. *Suppose that $\mathscr{C}\subset 2^{[n]}$ is a complex of size at least $\binom{n}{0}+\cdots+\binom{n}{k}+\binom{x}{k+1}$ for some $1\leq k+1\leq x\leq n$. Then*

$$|\mathscr{I}(\mathscr{C})|\geq\binom{n}{0}+\cdots+\binom{n}{k-1}+\binom{x}{k}\ \textit{holds.}\tag{1}$$

For a family $\mathscr{F}\subset 2^{[n]}$ define its *boundary* $\sigma(\mathscr{F})$ by

$$\sigma(\mathscr{F})=\{E\subset[n]:|E\Delta F|\leq 1 \text{ for some } F\in\mathscr{F}\},$$

where Δ denotes the symmetric difference.

The strongest version of the isoperimetric theorem of Harper can be stated as follows (cf. [6, 9] on [4]).

Harper's Theorem. *Suppose that* $\mathcal{F} \subset 2^{[n]}$,

$$|\mathcal{F}| = \binom{n}{n} + \cdots + \binom{n}{k+1} + \binom{a_k}{k} + \cdots + \binom{a_s}{s},$$

where $n \geq a_k > \cdots > a_s \geq s \geq 1$. *Then*

$$|\sigma(\mathcal{F}) \geq \binom{n}{n} + \cdots + \binom{n}{k} + \binom{a_k}{k-1} + \cdots + \binom{a_s}{s-1}.$$

In Section 3 we give a short proof of this result.

Let us state now the result mentioned in the abstract.

Kleitman's Theorem ([10]). *Suppose that* $\mathcal{C} \subset 2^{[n]}$ *is a complex with* $|\mathcal{A}(\mathcal{C})| \geq \binom{n}{k}$, $0 \leq k \leq \frac{1}{2}n$. *Then*

$$|\mathcal{C}| \geq \binom{n}{0} + \cdots + \binom{n}{k} \text{ holds.} \tag{2}$$

Before deriving this result from Theorem 1.1 let us mention that the original proof was incomplete. A full version, due to A.M. Odlyzko appears in [5]. The theorem was extended to multisets by Clements [1] who proves best possible inequalities even if $|\mathcal{A}(\mathcal{C})|$ is not of the form $\binom{n}{k}$.

Suppose for contradiction that

$$|\mathcal{C}| = \binom{n}{0} + \cdots + \binom{n}{i} + \binom{x}{i+1} \text{ holds with } i < k \text{ and } x < n.$$

Then (1) gives

$$|\mathcal{A}(\mathcal{C})| \leq \binom{n}{i} + \binom{x}{i+1} - \binom{x}{i} < \binom{n}{i+1} \leq \binom{n}{k},$$

a contradiction. (We used the inequality

$$\binom{n}{i+1} - \binom{n}{i} = \frac{n-2i-1}{i+1}\binom{n}{i} > \frac{x-2i-1}{i+1}\binom{x}{i} = \binom{x}{i+1} - \binom{x}{i}$$

which is true by $n > 2(i+1)$, $n > x$ and the monotonicity of $\binom{y}{i}$ for $y \geq i$.)

2. Proof of the Theorem 1.1

Let us introduce the notation

$$\partial\mathcal{F} = \{G : \exists F \in \mathcal{F}, G \subset F, |F - G| = 1\}.$$

Note that if $\mathcal{F}$ is a complex then $\mathcal{I}(\mathcal{F}) = \partial\mathcal{F}$ holds.

Let us recall the definition of the *shifting operator* S_{ij} for $i \leqslant i < j \leqslant n$, which goes back to Erdős-Ko-Rado [2].

$$S_{ij}(\mathscr{F}) = \{S_{ij}(F) : F \in \mathscr{F}\}$$

where

$$S_{ij}(F) = \begin{cases} F' = (F - \{j\}) \cup \{i\} & \text{if } i \notin F, j \in F, F' \notin \mathscr{F} \\ F & \text{otherwise.} \end{cases}$$

The following simple but important proposition goes back to Katona [7] (see also [3], where it was used to give a short proof of the Kruskal–Katona Theorem).

Proposition 2.1. *$\partial(S_{ij}(\mathscr{F})) \subset S_{ij}\partial\mathscr{F}$ holds for all $1 \leqslant i < j \leqslant n$.*

This proposition shows that in proving the theorem we may replace $\mathscr{C}$ repeatedly by $S_{ij}(\mathscr{C})$. Doing so repeatedly for $\{i, j\} = \{1, 2\}, \ldots, \{1, n\}$ will leave us with a family $\mathscr{F}$ satisfying $|\mathscr{F}| = |\mathscr{C}|$, $|\partial\mathscr{F}| \leqslant |\partial\mathscr{C}|$ and $S_{1j}(\mathscr{F}) = \mathscr{F}$ for $2 \leqslant j \leqslant n$.

Define $\mathscr{F}(1) = \{F - \{1\} : 1 \in F \in \mathscr{F}\}$ and $\mathscr{F}(\bar{1}) = \{F \in \mathscr{F} : 1 \notin \mathscr{F}\}$.

Claim 2.2.

(i) $|\partial\mathscr{F}| = |\mathscr{F}(1)| + |\partial\mathscr{F}(1)|$

(ii) $\partial\mathscr{F}(\bar{1}) \subset \mathscr{F}(1)$.

Proof of Claim. First we prove (ii). If $G \in \partial\mathscr{F}(\bar{1})$ then for some $1 < j \leqslant n$ and $j \notin G$ we have $G \cup \{j\} \in \mathscr{F}$. Since $1 \notin G$ and $S_{1j}(\mathscr{F}) = \mathscr{F}$, $(G \cup \{1\}) \in \mathscr{F}$, i.e. $G \in \mathscr{F}(1)$ follows.

Now (i) follows from $|\partial\mathscr{F}| = |\partial\mathscr{F}(1)| + |\mathscr{F}(1) \cup \partial\mathscr{F}(\bar{1})|$ which is valid for all families $\mathscr{F}$. □

Now we are ready to prove Theorem 1.1 by induction on n. We distinguish two cases

(a) $$|\mathscr{F}(1)| \geqslant \binom{n-1}{0} + \cdots + \binom{n-1}{k-1} + \binom{x-1}{k}.$$

By the induction hypothesis $|\partial\mathscr{F}(1)| \geqslant \binom{n-1}{0} + \cdots + \binom{n-1}{k-2} + \binom{x-1}{k-1}$. Thus the statement follows from Claim (i).

(b) $$|\mathscr{F}(1)| < \binom{n-1}{0} + \cdots + \binom{n-1}{k-1} + \binom{x-1}{k}.$$

Now

$$|\mathcal{F}(\bar{1})| > \binom{n-1}{0} + \cdots + \binom{n-1}{k} + \binom{x-1}{k+1}.$$

We want to apply the induction hypothesis to $\mathcal{F}(\bar{1}) \subset 2^{\{2,\ldots,n\}}$. There is a slight technical difficulty, namely $x-1 < k+1$ might happen. However, in that case we can replace x by $k+2$ and the following argument remains valid.

$$|\partial\mathcal{F}(\bar{1})| \geq \binom{n-1}{0} + \cdots + \binom{n-1}{k-1} + \binom{x-1}{k},$$

which contradicts Claim (ii). □

Just as in [3], the same proof would work to give the following best possible result. Suppose

$$|\mathcal{F}| = \binom{n}{0} + \cdots + \binom{n}{k} + \binom{a_{k+1}}{k+1} + \binom{a_k}{k} + \cdots + \binom{a_s}{s}$$

for some integers $1 \leq s \leq a_s < \cdots < a_{k+1} < n$. Then

$$|\partial\mathcal{F}| \geq \binom{n}{0} + \cdots + \binom{n}{k-1} + \binom{a_{k+1}}{k} + \cdots + \binom{a_s}{s-1}.$$

Note the relation with the Kruskal–Katona Theorem [8, 11].

The exact form permits to give an exact answer to the problem given in $1 \leq m \leq \binom{n}{\lfloor n/2 \rfloor}$, minimize $|\mathcal{C}|$, where $\mathcal{C}$ is a complex with $|\mathcal{A}(\mathcal{C})| = m$, i.e. $\mathcal{C}$ is generated by an antichain of size m. This problem was solved by Clements [1].

3. The size of the exterior of co-complexes and Harper's theorem

Recall that $\mathcal{F} \subset 2^{[n]}$ is called a co-complex if $\{[n] - F : F \in \mathcal{F}\}$ is a complex.

Theorem 3.1. *Let $\mathcal{F} \subset 2^{[n]}$ be a co-complex,*

$$|\mathcal{F}| \geq \binom{n}{n} + \cdots + \binom{n}{k+1} + \binom{x}{k}, \quad k \leq x \leq n,$$

x real. Then

$$|\partial\mathcal{F}| \geq \binom{n}{n-1} + \cdots + \binom{n}{k} + \binom{x}{k-1}. \tag{3.1}$$

Proof. The proof is very similar to that of Theorem 1.1, therefore we shall be somewhat sketchy.

In view of Proposition 2.1 we may assume that $\mathcal{F}$ is shifted. Apply induction on

n, the case $n=1$ being trivial. We distinguish two cases again.

(a) $$|\mathcal{F}(1)| \geq \binom{n-1}{n-1} + \cdots + \binom{n-1}{k} + \binom{x-1}{k-1}.$$

By the induction hypothesis we have

$$|\partial\mathcal{F}(1)| \geq \binom{n-1}{n-2} + \cdots + \binom{n-1}{k-1} + \binom{x-1}{k-2}$$

and (3.1) follows from Claim 2.2(i).

(b) $$|\mathcal{F}(1)| < \binom{n-1}{n-1} + \cdots + \binom{n-1}{k} + \binom{x-1}{k-1}.$$

Now

$$|\mathcal{F}(\bar{1})| > \binom{n-1}{n-1} + \cdots + \binom{n-1}{k+1} + \binom{x-1}{k}$$

and thus by the induction hypothesis

$$|\partial\mathcal{F}(\bar{1})| \geq \binom{n-1}{n-2} + \cdots + \binom{n-1}{k} + \binom{x-1}{k-1} \text{ follows.}$$

Since $[2,n] \in (\mathcal{F}(1) - \partial\mathcal{F}(\bar{1}))$, Claim 2.2(ii) gives the contradiction

$$|\mathcal{F}(1) \geq \binom{n-1}{n-1} + \binom{n-1}{n-2} + \cdots + \binom{n-1}{k} + \binom{x-1}{k-1}. \quad \square$$

The same proof gives the following, more exact version.

Theorem 3.2. *Let $\mathcal{F} \subset 2^{[n]}$ be a co-complex,*

$$|\mathcal{F}| = \binom{n}{n} + \cdots + \binom{n}{k+1} + \binom{a_k}{k} + \binom{a_k-1}{k-1} + \cdots + \binom{a_s}{s},$$

$n \geq a_k > a_{k-1} > \cdots > a_s \geq s \geq 1$. *Then*

$$|\partial\mathcal{F}| \geq \binom{n}{n-1} + \cdots + \binom{n}{k} + \binom{a_k}{k-1} + \cdots + \binom{a_s}{s-1}. \quad \square$$

Recall the definition of the pushing-up operation T_i, $1 \leq i \leq n$.

$T_i(\mathcal{F}) = \{T_i(F) : F \in \mathcal{F}\}$, where

$$T_i(F) = \begin{cases} F' = F \cup \{i\} & \text{if } i \notin F,\ F' \notin \mathcal{F} \\ F & \text{otherwise.} \end{cases}$$

The following lemma is easy to prove.

Proposition 3.3 [4]. *$\sigma T_i(\mathcal{F}) \subset T_i(\sigma\mathcal{F})$ for all $\mathcal{F} \subset 2^{[n]}$.*

Applying $T_1, \ldots, T_n$ consecutively to a family produces a co-complex of the same size whose boundary is not larger. Noting that $\sigma\mathcal{F} = \{[n]\} \cup \partial\mathcal{F}$ holds for a co-complex $\mathcal{F}$, Harper's theorem follows from Theorem 3.2.

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