Solution of the Littlewood-Offord problem in high dimensions

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Abstract

Consider the 2^n partial sums of arbitrary n vectors of length at least one in d-dimensional Euclidean space. It is shown that as n goes to infinity no closed ball of diameter Δ contains more than $(\lfloor\Delta\rfloor + 1 + o(1))\binom{n}{\lfloor n/2\rfloor}$ out of these sums and this is best possible. For $\Delta - \lfloor\Delta\rfloor$ small an exact formula is given.

1. Introduction

Investigating the number of zeros of random polynomials, Littlewood and Offord [14] were led to the following problem. Let $d \geq 1$ and $\mathbf{R}^d$ be d-dimensional Euclidean space. Further let $V = \{v_1, \dots, v_n\}$ be a set of n non-necessarily distinct vectors in $\mathbf{R}^d$; $|v_i|$, the length of v_i, is supposed to be at least one, $1 \leq i \leq n$. Consider ΣV, the collection of all 2^n partial sums

$$\sum_{i=1}^{n} \varepsilon_i v_i \text{ with } \varepsilon_i = 0 \text{ or } 1.$$

For a positive real Δ, let

$$m(V, \Delta) = \max\{|S \cap \Sigma V| : S \text{ is a closed ball of diameter } \Delta\}.$$

Now, the famous Littlewood-Offord problem is to determine or estimate

$$m(n, \Delta) = m_d(n, \Delta) = \max\{m(V, \Delta): V \subset \mathbf{R}^d \text{ is a set of } n \text{ vectors of length at least one}\}.$$

In 1945 Erdös [1] determined $m_d(n, \Delta)$ for $d = 1$ and arbitrary Δ. Set $s = \lfloor\Delta\rfloor + 1$.

THEOREM 1.1 (Erdös). *$m_1(n, \Delta)$ is the sum of the largest s binomial coefficients $\binom{n}{i}$ with $0 \leq i \leq n$.*

We will outline his proof in Section 4. To see the lower bound part, one can take $v_1 = v_2 = \cdots = v_n = 1$. Note that for fixed Δ and $n \to \infty$, $m_1(n, \Delta) = (\lfloor\Delta\rfloor + 1 + o(1))\binom{n}{\lfloor n/2\rfloor}$.

There has been a lot of research related to this problem for $d \geq 2$. In particular, Katona [7] and Kleitman [9] showed that $m_2(n, \Delta) = \binom{n}{\lfloor n/2 \rfloor}$ holds for $\Delta < 1$. This was extended by Kleitman [10] to arbitrary $d \geq 2$.

Their proofs led to the creation of a new area in extremal set theory, to the so-called M-part Sperner theorems; see e.g., Füredi [2], Griggs, Odlyzko and Shearer [5].

These results were used to give upper bounds on $m_d(n, \Delta)$. To mention a few, Kleitman [12] showed that $m_2(n, \Delta)$ is upper-bounded by the sum of the $2\lfloor \Delta/\sqrt{2} \rfloor$ largest binomial coefficients in n.

Griggs [3] proved

$$m_d(n, \Delta) \leq 2^{2^{d-1}-2} \lceil \Delta\sqrt{d} \rceil \binom{n}{\lfloor n/2 \rfloor}.$$

Sali [16], [17] improved this bound to

$$m_d(n, \Delta) \leq 2^d \lceil \Delta\sqrt{d} \rceil \binom{n}{\lfloor n/2 \rfloor}.$$

Let us mention also that Griggs et al. [4] proved that for $\Delta > n/\sqrt{d}$ and for $n > n_0(d)$ one has $m_d(n, \Delta) = 2^n$. This shows that for large d and Δ, $m_d(n, \Delta)/m_1(n, \Delta)$ can be arbitrarily large. Here we prove:

THEOREM 1.2. *For fixed d and Δ,*

$$m_d(n, \Delta) = (\lfloor \Delta \rfloor + 1 + o(1)) \binom{n}{\lfloor n/2 \rfloor} \tag{1.1}$$

whenever $n \to \infty$.

One might think that Theorem 1.1 holds for arbitrary d, Δ and $n > n_0(d, \Delta)$. However, this is not true for $d \geq 2$ and $(s-1)^2 + 1 < \Delta^2 < s^2$, $s \geq 2$, arbitrary.

Example 1.3 ([13]). Let $v_1 = v_2 = \cdots = v_{n-1}$ be unit vectors and v_n a unit vector orthogonal to v_1. Take the sphere S of diameter Δ centered at $(v_1 + \cdots + v_n)/2$. Suppose that $n + s$ is even. Then

$$|\Sigma V \cap S| = 2 \sum_{n-s/2 \leq i \leq n+s/2} \binom{n-1}{i} > m_1(n, \Delta).$$

Our second result says that if $\Delta - \lfloor \Delta \rfloor$ is very small then the bound of Theorem 1.1 is valid.

THEOREM 1.4. *Suppose that $s - 1 \leq \Delta \leq s - 1 + 1/10s^2$; then*

$$m_d(n, \Delta) = m_1(n, \Delta) \text{ holds for } n > n_0(d, \Delta).$$

We need some geometric preliminaries as well. By a *cone* C we mean always a circular closed double cone with vertex at the origin. Thus if the *axis* of the cone is a line L and the *angle* of the cone is α then C consists of the points of those lines through the origin which have angle at most $\alpha/2$ with L. A cone is the union of two *halfcones*.

Let S_0 denote the *unit sphere* centered at the origin. Then $S_0 \cap C$ is a *spherical (double) cap* of angle α. Let $\zeta(d, \alpha)$ denote the minimum number of double caps of angle α needed to cover S_0. Let us recall the following upper bound on $\zeta(d, \alpha)$ from [15]: If $\alpha < \pi/2$ then

$$\zeta(d, \alpha) < d^2\left(\sin\frac{\alpha}{2}\right)^{-d+1}.$$

For two disjoint cones C, D (that is, $C \cap D$ consists of the origin only), considering their intersection with the plane P determined by the two axes, we can define (see Figure 1, next page) the angles α, β as the angles of the two open cones whose union is $P - (C \cup D)$. Call $\min\{\alpha, \beta\}$ the angle between C and D. Note that if C has angle γ and D has angle δ, then $\alpha + \beta + \gamma + \delta = \pi$ holds.

2. The main lemmas

By vectors we shall always mean vectors of length at least one in $\mathbf{R}^d$. For a set V of vectors let ΣV denote the set of all $2^{|V|}$ sums $\sum_{v \in V} \varepsilon(v) v$ with $\varepsilon(v) = 0$ or 1. Recall that

$$m(V, \Delta) = \max_{\substack{S \text{ a ball of} \\ \text{diameter } \Delta}} |S \cap \Sigma V|.$$

Of course $m(V, \Delta) = m(V - \{u\} \cup \{-u\}, \Delta)$ for any $u \in V$; i.e., we can reverse a vector. Sometimes the Littlewood-Offord problem is reformulated in the following way:

$$m(V, \Delta) = \max\Big\{|S \cap \{\Sigma\varepsilon(v)v\colon \text{where } \varepsilon(v) = \pm 1,\ v \in V\}|\colon \\ S \subset \mathbf{R}^d \text{ a ball of radius } \Delta\Big\}.$$

Because of Kleitman's theorem we will suppose that $\Delta \geq 1$ (i.e., $s \geq 2$), $d \geq 2$.

Define also

$$p(V, \Delta) = m(V, \Delta)/2^{|V|}.$$

Our first proposition says that $p(V, \Delta)$ is monotone decreasing.

PROPOSITION 2.0. *Let $W \subset V$ be sets of vectors. Then*

$$p(V, \Delta) \leq p(W, \Delta) \tag{2.0}$$

holds for all $\Delta > 0$.

Proof. Let S be an arbitrary ball of diameter Δ. Then

$$|S \cap \Sigma V| \leq \sum_{u \in \Sigma(V-W)} |S \cap (u + \Sigma W)| \leq 2^{|V-W|} m(W, \Delta),$$

yielding

$$m(V, \Delta) \leq 2^{|V-W|} m(W, \Delta).$$

Dividing both sides by $2^{|V|}$, we see that (2.0) follows. □

LEMMA 2.1. *Let C, D be disjoint cones in $\mathbf{R}^d$ with respective angles γ, δ. Let α and β be the two angles between the cones (see Figure 1). Let h be a positive integer, $\Delta > 0$, real such that*

$$h \min\left\{\sin\frac{\alpha}{2}, \sin\frac{\beta}{2}\right\} > \Delta. \tag{2.1}$$

Suppose further that $|C \cap V| = c$, $|D \cap V| = d$. Then

$$p(V, \Delta) \leq h^2/\sqrt{cd} \tag{2.2}$$

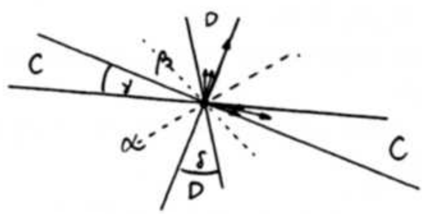

FIGURE 1

Proof. Let $v_1, \ldots, v_c$ and $w_1, \ldots, w_d$ be the vectors from V, contained in C and D, respectively. When we apply Proposition 2.0 with $W = \{v_1, \ldots, v_c, w_1, \ldots, w_d\} \subset V$, it follows that it is sufficient to prove (2.2) for W. Without loss of generality, we may assume that all vectors are in the same halfcone as shown in Figure 1. Let S be an arbitrary sphere of diameter Δ. We denote $\{1, 2, \ldots, i\}$ by $[i]$, and the set of all permutations of $[i]$ by $S_{[i]}$. Let us define the family $\mathcal{F}$ by:

$$\mathcal{F} = \left\{(A, B): A \subset [c], B \subset [d], \sum_{i \in A} v_i + \sum_{j \in B} w_j \in S\right\}.$$

Let (π, ζ) be a random element of $S_{[c]} + S_{[d]}$. Consider the rectangle R, defined by

$$R = \{(\pi([i]), \zeta([j])): 1 \leq i \leq c, 1 \leq j \leq d\}.$$

Claim 2.2. $|R \cap \mathcal{F}| \le h^2$.

Proof. Define $I = \{i: \exists j, (\pi([i]), \zeta([j])) \in R \cap \mathcal{F}\}$; that is, I is the "projection" on the side of the points in that rectangle. The set J is defined analogously, with the roles of i and j interchanged. If we prove $|I| \le h$, $|J| \le h$, then the claim follows. Suppose the contrary and let, e.g., $|I| \ge h + 1$. Then we can choose $i_1, i_2 \in I$ with $i_1 - i_2 \ge h$. Choose $j_1, j_2 \in J$ such that

$$(\pi([i_t]), \zeta([j_t])) \in R \cap \mathcal{F}, \; t = 1, 2.$$

Let u_1, u_2 be the corresponding sum of vectors. Suppose first that $j_1 \ge j_2$ and let L be a perpendicular line to the bisector of the angle β. Then both the vectors v_i and w_j have projection of length at least $\sin(\beta/2)$ on L.

Consequently,

$$u_1 - u_2 = \sum_{i_2 < i \le i_1} v_{\pi(i)} + \sum_{j_2 < j \le j_1} w_{\zeta(j)}$$

has projection of length at least

$$((i_1 - i_2) + (j_1 - j_2))\sin(\beta/2) \ge h \sin(\beta/2) > \Delta,$$

in contradiction with $u_1, u_2 \in S$.

If $j_2 > j_1$ then we argue in the same way except for the perpendicular to the bisector of the angle α. □

To conclude the proof of Lemma 2.1 we show that there is a choice of $\pi \in S_{[c]}$, $\zeta \in S_{[d]}$ with

$$|R \cap \mathcal{F}| > |\mathcal{F}|\sqrt{cd}\, 2^{-c-d}. \tag{2.3}$$

Let $(A, B) \in \mathcal{F}$ be arbitrary, $|A| = a$, $|B| = b$. Then the probability $p(A, B)$ of $(A, B) \in R$ satisfies

$$p(A, B) = 1 \Big/ \left(\binom{c}{a}\binom{d}{b}\right) \ge \left(\binom{c}{\lfloor \frac{c}{2} \rfloor}\right)^{-1} \left(\binom{d}{\lfloor \frac{d}{2} \rfloor}\right)^{-1} > \frac{\pi}{2}\sqrt{cd}\, 2^{-c-d} > \sqrt{cd}\, 2^{-c-d}.$$

Thus, the expected size $E(|R \cap \mathcal{F}|)$ of $R \cap \mathcal{F}$ satisfies

$$E(|R \cap \mathcal{F}|) = \sum_{(A, B) \in \mathcal{F}} p(A, B) > |\mathcal{F}|\sqrt{cd}\, 2^{-c-d},$$

proving (2.3). □

LEMMA 2.3. *Suppose that $W \subset C$ is a set of vectors, C is a cone with angle γ and Δ, Δ' are positive reals with $\Delta'\cos(\gamma/2) > \Delta$. Then*

$$m(W, \Delta) \le m_1(|W|, \Delta'). \tag{2.4}$$

Proof. Suppose without loss of generality that the axis of C is the real line. Set $|W| = r$ and let $x_1, \ldots, x_r$ be the projections of the vectors $w \in W$ on the axis. Set $y_i = x_i/\cos(\gamma/2)$. Then $|y_i| \ge 1$ for $i = 1, \ldots, r$. By definition

$$m_1(\{x_1, \ldots, x_r\}, \Delta) = m_1(\{y_1, \ldots, y_r\}, \Delta') \le m_1(r, \Delta')$$

holds. On the other hand,

$$m_d(W, \Delta) \le m_1(\{x_1, \ldots, x_r\}, \Delta)$$

is obvious, proving (2.4). □

For our final lemma we need to prove first a geometric proposition. For vectors $v_1, \ldots, v_r$ and w define

$$A(v_1, \ldots, v_r; w) = \{v_1 + \cdots + v_i + \varepsilon w : 0 \le i \le r,\ \varepsilon = 0, 1\}.$$

PROPOSITION 2.4. *Let β and α be positive reals, $\beta > \alpha$, $\alpha \le \pi/3$, and $s \ge 2$ a positive integer satisfying*

$$s - 1 \le \Delta < (s-1)\cos\frac{\alpha}{2} + \frac{\sin^2\frac{\beta - \alpha}{2}}{4(s-1)\cos\frac{\alpha}{2}}. \tag{2.5}$$

Let $v_1, v_2, \ldots, v_r$ be vectors of at least unit length in a halfcone C with angle α and let w, $|w| \ge 1$ be a vector having angle at least $\beta/2$ and at most $\pi - \beta/2$ with the axis. Then for every ball S of diameter Δ,

$$|S \cap A(v_1, \ldots, v_r; w)| \le 2s - 1.$$

Proof. Denote by $A(i)$ the sum $v_i + v_2 + \cdots + v_i$ $(A(0) = 0)$, and let $B(j) = A(j) + w$ for $0 \le i, j \le r$. We may suppose that $\beta \le \pi/2$. Let S be a ball with diameter Δ and suppose on the contrary that it contains at least $2s$ vectors from $A(v_1, \ldots, v_r; w)$. Let $I = \{i\colon A(i) \in S\}$ and $J = \{j\colon B(j) \in S\}$. Consider a line c through the center of S and parallel to the axis of C. Consider the projections $A'(i)$ and $B'(j)$ of the points $A(i)$ and $B(j)$ on the line c. Now

$$|A'(i)A'(i')| \ge |i - i'| \cdot \cos\frac{\alpha}{2}$$

holds. As the right-hand side of (2.5) is smaller than $s\cos(\alpha/2)$ we have that $|I|$ (and $|J|$) is at most s. So if S contains $2s$ vectors from $A(v_1, \ldots, v_r, w)$ then there exist k and l such that $A(i) \in S$, $B(j) \in S$ for $k \le i \le k + s - 1$, $l \le j \le l + s - 1$. Consider a plane P orthogonal to c which cuts a piece from S with width $\Delta - (s-1)\cos(\alpha/2)$. Denote this piece by H. Then $A(k), B(l) \in H$.

The diameter of H is

$$2\sqrt{\left((s-1)\cos\frac{\alpha}{2}\right)\left(\Delta-(s-1)\cos\frac{\alpha}{2}\right)} \le \sin\frac{\beta-\alpha}{2}. \tag{2.6}$$

So $|A(k)B(l)| < 1$, implying $l \neq k$. Suppose, say, $l < k$ and consider the $A(l)B(l)A(k)$ triangle. We have $|A(l)B(l)| \ge 1$, $|A(l)A(k)| \ge 1$, and the angle at $A(l)$ is at least $(\beta - \alpha)/2$. Hence the length of the side $A(k)B(l)$ is at least $2\sin((\beta-\alpha)/4)$, which contradicts (2.6). So S cannot contain $2s$ elements from $A(v_1, \ldots, v_r; w)$. □

LEMMA 2.5. *Let α, β, s and Δ be as in Proposition 2.4. Let W be a set of vectors contained in a cone C of angle α and let w be a vector having angle at least $\beta/2$ with the axis of the cone. Set $r = |W|$. Then*

$$m(W \cup \{w\}, \Delta) \le (2s-1)\binom{r}{\lfloor r/2 \rfloor}. \tag{2.7}$$

Proof. We can reverse the directions of the vectors; so we can suppose that W is contained in a halfcone of C and the angle of W, and the axis of C is at most $\pi/2$. Let S be a fixed sphere of diameter Δ. Let us consider a random ordering $v_1, v_2, \ldots, v_r$ of the elements of W. As in the proof of Lemma 2.1, there exists an ordering with

$$|S \cap A(v_1, \ldots, v_r; w)| \ge |S \cap \Sigma(W \cup \{w\})| \Big/ \binom{r}{\lfloor r/2 \rfloor}.$$

On the other hand, Proposition 2.4 implies

$$|S \cap A(v_1, \ldots, v_r; w)| \le 2s - 1, \text{ which proves (2.7)}$$ □

3. Proof of Theorems 1.2 and 1.4

Set $s = \lfloor \Delta \rfloor + 1$ and choose $0 < \alpha < \pi/2$ such that

$$s\cos\frac{\alpha}{2} > \Delta. \tag{3.1}$$

Recall the definition of $\zeta(d, \alpha)$ from the introduction and set $t = \zeta(d, \alpha/5)$. Let $C_1, \ldots, C_t$ be cones with angle $\alpha/5$ which cover $\mathbf{R}^d$. Suppose by symmetry that

$$|V \cap C_1| \ge |V|/t \text{ holds.} \tag{3.2}$$

Consider the cone C (of angle α) which has the same axis as C_1. Define $k = 2t^2(\lceil(\Delta+1)/\sin(\alpha/10)\rceil)^4/\Delta$.

If $|C \cap V| \geq n - k$, then Proposition 2.0 and Lemma 2.3 imply

$$m(V, \Delta) \leq 2^k s \binom{n-k}{\left\lfloor \frac{n-k}{2} \right\rfloor} = (1 + o(1)) s \binom{n}{\left\lfloor \frac{n}{2} \right\rfloor},$$

as desired.

Suppose next $|V - C| > k$. Note that if a vector $v \in V - C$ is contained in C_i, $2 \leq i \leq t$, then C_1 and C_i are disjoint and the angle between them is at least 0.3α. Suppose by symmetry, that

$$|(V - C) \cap C_2| \geq k/t. \tag{3.3}$$

Applying Lemma 2.1 to C_1 and C_2 with $h = \lceil (\Delta + 1)/\sin(\alpha/10) \rceil$ and using (3.2) and (3.3) we obtain

$$p(V, \Delta) < h^2 t/\sqrt{nk} < s/\sqrt{\pi n/2} \tag{3.4}$$

for our choice of h and k, which concludes the proof of Theorem 1.2.

In the case of Theorem 1.4 we first note that (3.4) implies for $n > n_0(d, \Delta)$ that $m(V, \Delta) < m_1(n, \Delta)$, as desired. Choose α positive but very small (e.g., $\sin(\alpha/2) = 1/2s^2$). Then we may assume that

$$|V - C| \leq k.$$

Let β be a small angle satisfying $\cos(\beta/2) = 1 - (1/2s)$. Then

$$s \cos \frac{\beta}{2} > \Delta, \tag{3.5}$$

Let D be the cone with angle β and the same center as C. If $V \subset D$, then Lemma 2.3 concludes the proof. Thus we may suppose that there is a vector $w \in (V - D)$.

Setting $W = V \cap C$, using $s - 1 \leq \Delta < s - 1 + 1/10s^2$, we see that Proposition 2.0 and Lemma 2.5 imply

$$p(V, \Delta) \leq p(W \cup \{w\}, \Delta) \leq \frac{2s - 1 + o(1)}{2} \binom{n}{\left\lfloor \frac{n}{2} \right\rfloor} \Big/ 2^n < m_1(n, \Delta),$$

which concludes the proof. □

4. The case when the diameter is an integer

We call a family of vectors *optimal* if $m_d(n, \Delta) = m(V, \Delta)$. In the case of $s - 1 < \Delta < s - 1 + (1/10s^2)$ we obviously have infinitely many optimal families, because we can perturb slightly the set of vectors $V = \{n$ copies of the same vector of length $\Delta/(s - 1)\}$.

Theorem 4.1. *Suppose Δ is an integer, $n > n_0(d, \Delta)$. Then the only optimal family V consists of n copies of a unit vector.*

For the proof of 4.1 we need the following theorem of Erdös. He noticed the connection of the Littlewood-Offord problem to extremal set theory.

Definitions. 2^X denotes the power set of X; $\mathcal{F}(\subset 2^X)$ denotes a family of sets and is called a *k-Sperner* family if it does not contain $k+1$ members $F_1, \ldots, F_{k+1} \in \mathcal{F}$ such that $F_1 \subsetneqq F_2 \subsetneqq \cdots \subsetneqq F_{k+1}$.

Theorem 4.2 (Erdös [1] and Sperner [18] for $k = 1$). *Let $\mathcal{F}$ be a k-Sperner family over an n element set X. Then*

$$|\mathcal{F}| \leq \text{sum of the largest } k \text{ binomial coefficients } \binom{n}{i}.$$

Here equality holds if and only if $\mathcal{F}$ consists of all the subsets of X of sizes $\lfloor (n-k+1)/2 \rfloor, \ldots, \lfloor (n-k+1)/2 \rfloor + (k-1)$ or $\lceil (n-k+1)/2 \rceil, \ldots, \lceil (n-k+1)/2 \rceil + (k-1)$ (i.e., for $n-k$ odd there exists only one optimal family; in case $n-k$ is even there are two optimal families).

With a set of vectors V and a ball S we associate a family $\mathcal{F} = \mathcal{F}(V, S) = \{I \subset \{1, 2, \ldots, n\}: \sum_{i \in I} v_i \in S\}$. A consequence of 4.2 and the proof of 1.4 is the following.

Lemma 4.3. *Suppose that $n > n_0(d, \Delta)$, V is an optimal family of vectors, Δ is an integer, S is a ball of diameter Δ with $|S \cap \Sigma V| = m_1(d, \Delta)$. Then there are a direction w and a small $\beta > 0$ (e.g., $\cos^2(\beta/2) = 1 - (1/2s)$) such that every $v \in V$ is contained in a cone of angle α and axis w. If all $v \in V$ are contained in a halfcone of that cone then for every sequence of vectors $\{v_1, \ldots, v_n\} = V$,*

$$v_1 + \cdots + v_j \in S \text{ for } n_1 \leq j \leq n_1 + \Delta$$

where $n_1 = n_1(S) = \lfloor (n - \Delta)/2 \rfloor$ or $\lceil (n-\Delta)/2 \rceil$.

We need one more proposition.

Proposition 4.4. *Let $w, u_1, \ldots, u_n \in \mathbf{R}^d$ be vectors $0.4n < n_1 \leq n/2$, and suppose that $|\sum_{i \in I} u_i - n_1 w| \leq r$ for every $I \subset \{1, \ldots, n\}$ with $|I| = n_1$. Then*

$$\sum |u_i - w|^2 \leq 5r^2.$$

Proof. Define $w_i = u_i - w$. We have $|\sum_{i \in I} w_i| \leq r$ for every $I \subset [n]$, $|I| = n_1$, and we have to prove that $\sum w_i^2 \leq 5r^2$. The standard calculation is the following:

$$\binom{n}{n_1} r^2 \geq \sum_I \Big(\sum_{i \in I} w_i \Big)^2 = \binom{n-2}{n_1-2} \Big(\sum w_i \Big)^2 + \binom{n-2}{n_1-1} \Big(\sum w_i^2 \Big)$$

$$\geq \binom{n-2}{n_1-1} \Big(\sum w_i^2 \Big) \qquad \square$$

Proof of 4.1. Suppose that $n > 20\Delta^3$. Lemma 4.3 implies that $|\sum_{i \in I} v_i| \leq \Delta$ holds for every $I \subset \{1, \ldots, n\}$, $|I| = \Delta$. Suppose that $I \subset \{1, \ldots, n\}$, $|I| = \Delta$ such that for $u = \sum\{ v_i \colon i \in I\}$, $|u| = \Delta - x$ is maximal. Then all the sums of n_1 vectors from $\{ v_i \colon i \notin I\}$ are in $S \cap (S - u)$ which is contained in a sphere of radius $\sqrt{\frac{1}{2}x\Delta - \frac{1}{4}x^2}$. Let 0_1 be the center of $S \cap (S - u)$, and $n_1 w_1 = \overrightarrow{00_1}$. Then 4.4 gives

$$\sum_{i \notin I} |v_i - w_1|^2 \leq \tfrac{5}{2} x\Delta.$$

Then one can choose $J \subset \{1, \ldots, n\} - I$, $|J| = \Delta$ in such a way that

$$\sum_{j \in J} |v_j - w_1|^2 < \tfrac{5}{2} x\Delta(\Delta/n - \Delta) < (x/4\Delta).$$

Then all the v_j $(j \in J)$ have components to direction w_1 with length at least $1 - x/4\Delta$. Hence $|\sum v_j| \geq \Delta - x/2$, a contradiction if $x \neq 0$. If $x = 0$, then it easily follows that all the vectors are the same unit vector. $\square$

5. Concluding remarks

Let us mention that the proof of Theorem 1.2 actually gives $m_d(n, \Delta) \leq m_1(n, \Delta)(1 + (c(d, \Delta)/n))$ where $c(d, \Delta)$ is a constant depending only on d and Δ.

Next we describe a construction showing that for $\lceil \Delta \rceil - \Delta$ small and d large there exists a positive constant $c'(d, \Delta)$ such that $m_d(n, \Delta) \geq m_1(n, \Delta)(1 + (c'(d, \Delta)/n))$ holds.

Moreover, $c'(d, \Delta) \to \infty$ if $d \to \infty$, $\Delta \to \infty$ and $\lceil \Delta \rceil - \Delta \to 0$.

Example 5.1. Let n, k, s be positive integers and suppose for convenience that $n + s - k$ is even. Let $v_1 = v_2 = \cdots = v_{n-k}$, $w_1, \ldots, w_k$ be unit vectors where $v_1, w_1, \ldots, w_k$ are pairwise orthogonal. Consider the sphere, S of diameter $(k + s^2)^{1/2}$ centered around $((n - k)v_1 + w_1 \cdots + w_k)/2$. Then S contains all partial sums from $\Sigma(\{ v_1, \ldots, v_{n-k}, w_1, \ldots, w_k\})$ involving at least

$(n-k-s)/2$ and at most $(n-k+s)/2$ out of $v_1, \ldots, v_{n-k}$. That is,

$$m_{k+1}\left(n,(k+s^2)^{1/2}\right) \geq 2^k \sum_{(n-k-s/2) \leq i \leq (n-k+s/2)} \binom{n-k}{i}$$
$$= \left(1 + \frac{k+o(1)}{2n}\right) m_1\left(n,(k+s^2)^{1/2}\right)$$

holds for $k + s^2 < (s+1)^2$, i.e., $k \leq 2s$.

A sharpened version of Proposition 2.4 (we did not use that the points $A(l)$, $A(k)$, $A(k+s-1)$ lie almost on a line) gives that Theorem 1.3 holds for a slightly larger interval, especially for $s = 2$ if $1 \leq \Delta < \sqrt{2}$, and for $s = 3$ if $2 \leq \Delta < \sqrt{5}$. So we can construct a new proof for some theorems of Katona [8] and Kleitman [11], [13]. But the length of our interval is only $O(1/s^2)$. Now we have the following:

Conjecture 5.2. For $n > n_0(d, \Delta)$, if $s - 1 \leq \Delta < \sqrt{(s-1)^2 + 1}$, then $m_d(n, \Delta) = m_1(n, \Delta)$.

Let us consider now *open* spheres. Let $f_d(n, \Delta) = \max\{|S \cap \Sigma V| : S \subset \mathbf{R}^d$ is an open sphere of diameter Δ and V is a set of n vectors of length at least one$\}$.

COROLLARY 5.3. *For fixed d and Δ and $n \to \infty$, if Δ is not an integer then*

$$f_d(n, \Delta) = (\lfloor \Delta \rfloor + 1 + o(1)) \binom{n}{\lfloor n/2 \rfloor}.$$

Similarly, Theorem 1.3 *gives the value of $f_d(n, \Delta)$ for $n > n_0(d, \Delta)$, $s - 1 < \Delta < s - 1 + 1/10s^2$.*

Problem 5.4. Determine (if it exists) $\lim_{n \to \infty} f_d(n, \Delta) \binom{n}{\lfloor n/2 \rfloor}^{-1}$ for d, Δ fixed, Δ an integer. □

Finally we would like to mention that Katona formulated an interesting generalization of the Littlewood-Offord problem. L. Jones [6] answered some of his questions.

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