The Shifting Technique in Extremal Set Theory

Peter Frankl
CNRS
Paris, France

1. Introduction and the Erdős-Ko-Rado Theorem.

Let X be a finite set. If not said otherwise we assume $X = \{1, 2, \ldots, n\}$, n a positive integer. For $0 \leqslant k \leqslant n$ we set $2^X = \{F : F \subset X\}$, $\binom{X}{k} = \{F \in 2^X : |F| = k\}$. A *family* $\mathscr{F}$ is just a collection of subsets of X, i.e., $\mathscr{F} \subset 2^X$. If $\mathscr{F} \subset \binom{X}{k}$ then it is called *k-uniform*, or a *k-graph*. A family $\mathscr{F}$ is called *intersecting* if $F \cap F' \neq \varnothing$ holds for all $F, F' \in \mathscr{F}$. The simplest intersection theorem is the following:

Theorem 1.0. If $\mathscr{F}$ is intersecting then $|\mathscr{F}| \leqslant 2^{n-1}$ holds.

Proof. Partition the 2^n subsets of X into 2^{n-1} pairs where each subset F is paired with its *complement* $X - F$. Since $F \cap (X - F) = \varnothing$, at most one set out of each pair is in $\mathscr{F}$. Thus $|\mathscr{F}| \leqslant 2^{n-1}$. ■

Once an inequality is proved, one is interested in which families attain equality — such families are called *optimal*.

In case of Theorem 1.0 there are many optimal families, in fact, it is not hard to show the following:

Proposition 1.1. Given an intersecting family $\mathscr{F} \subset 2^X$, there exists another intersecting family $\mathscr{G} \subset 2^X$, satisfying $\mathscr{F} \subset \mathscr{G}$ and $|\mathscr{G}| = 2^{n-1}$. ■

The first intersection theorem was proved by Erdős, Ko and Rado in the late 1930s, however it was published only in 1961. Before giving its statement one more definition: A family $\mathscr{F}$ is called *t*-intersecting ($t \geqslant 1$, integer) if $|F \cap F'| \geqslant t$ holds for all $F, F' \in \mathscr{F}$.

Theorem 1.1. (The Erdős-Ko-Rado theorem, [EKR]). Given $n \geqslant k \geqslant t > 0$ and a t-intersecting family $\mathscr{F} \subset \binom{X}{k}$ then for $n \geqslant n_0(k, t)$,

$$|\mathscr{F}| \leqslant \binom{n-t}{k-t} \text{ holds.} \tag{1}$$

To see that the inequality (1) is best possible, i.e., $\max |\mathscr{F}| \geqslant \binom{n-t}{k-t}$, consider the family consisting all k-subsets of X which contain t fixed elements. For $n < (k-t+1)(t+1)$ a larger t-intersecting family was constructed in [F1] and [EKR]. (See next page). Denote by $n_0(k, t)$ the *least* integer such that (1) holds.

For $n < 2k$ any two k-subsets have nonempty intersection, that is $\binom{X}{k}$ is intersecting; and it was shown in [EKR] that $n_0(k, 1) = 2k$. Hilton and Milner [HM] proved that for $t = 1$ and $n > 2k$ the optimal family is unique.

In fact we have $n_0(k, t) = (k-t+1)(t+1)$ for all t and k as it was proved for $t \geqslant 15$ in [F1], and for all t by Wilson [W]. Moreover for $n > n_0(k, t)$ there is only one optimal family.

However, for $t \geqslant 2$ one may ask, what is the maximum size of a t-intersecting family $\mathcal{F}$, $\mathcal{F} \subset \binom{X}{k}$ for $2k - t < n < n_0(k, t)$. Denote this maximum by $m(n, k, t)$.

For $0 \leqslant i \leqslant (n-t)/2$ define the family

$$\mathcal{A}_i = \left\{A \in \binom{X}{k} : |A \cap \{1, 2, \ldots, t+2i\}| \geqslant t + i\right\}.$$

Clearly, $\mathcal{A}_i$ is t-intersecting. One can also check that $|\mathcal{A}_1| \overset{>}{\underset{<}{=}} |A_0| = \binom{n-t}{k-t}$ according as $n \overset{<}{\underset{>}{=}} (k-t+1)(t+1)$.

Conjecture 1.2. [F1]

$$m(n, k, t) = \max_i |\mathcal{A}_i|.$$

In [F1] it is shown that for $t \geqslant 15$ and $0.8(k-t+1)(t+1) < n < (k-t+1)(t+1)$ the conjecture is true and $\mathcal{A}_1$ is the only optimal family (up to permutation of the elements). In the case $n = 4n_0$, $k = 2n_0$, $t = 2$ the above conjecture reduces to

$$m(4n_0, 2n_0, 2) \leqslant |\mathcal{A}_{n_0-1}| = \left|\left\{F \in \binom{X}{2n_0} : |F \cap \{1, 2, \ldots, 2n_0\}| \geqslant n_0 + 1\right\}\right|.$$

This was already conjectured in [EKR]; however, it appears to be very difficult.

2. SHIFTING

Sets have little structure, and this often makes it hard to deal with them. For certain kind of extremal problems shifting permits one to overcome this difficulty.

For integers $1 \leqslant i < j \leqslant n$ and a family $\mathcal{F}$ define the (i,j)-shift S_{ij} as follows

$$S_{ij}(F) = \begin{cases} (F - \{j\}) \cup \{i\} \text{ if } i \notin F,\ j \in F,\ ((F - \{j\}) \cup \{i\}) \notin \mathcal{F} \\ F \text{ otherwise} \end{cases};$$

$$S_{ij}(\mathcal{F}) = \{S_{ij}(F) : F \in \mathcal{F}\}.$$

The next proposition collects some easy but important properties of shifting.

Proposition 2.1.

(i) $|S_{ij}(\mathcal{F})| = |\mathcal{F}|$

(ii) If $\mathcal{F}$ is k-uniform then so is $S_{ij}(\mathcal{F})$

(iii) If $\mathcal{F}$ is t-intersecting then so is $S_{ij}(\mathcal{F})$.

Proof. (i) and (ii) are evident. To prove (iii) choose $A_1, A_2 \in S_{ij}(\mathcal{F})$. Let B_1, B_2 be the corresponding sets in $\mathcal{F}$, i.e., $S_{ij}(B_\nu) = A_\nu$ for $\nu = 1, 2$. Since $|A_1 \cap A_2| \geqslant |B_1 \cap B_2|$ would imply $|A_1 \cap A_2| \geqslant t$, we may assume $|A_1 \cap A_2| < |B_1 \cap B_2|$. This implies $j \in B_1 \cap B_2$ and $\{i, j\} \cap A_1 \cap A_2 = \emptyset$. Say $j \notin A_1$. Then $A_1 = S_{ij}(B_1) = (B_1 - \{j\}) \cup \{i\}$. On the other hand $A_2 = B_2$. Why did we not shift B_2 when $i \notin B_2$ and $j \in B_2$? The only possible reason is $B_3 = (B_2 - \{j\}) \cup \{i\} \in \mathcal{F}$. Consequently, $|A_1 \cap A_2| = |B_1 \cap B_3| \geqslant t$ ■

It is not hard to see that if we keep on shifting then finally we end up with a *stable* or *shifted* family $\mathcal{G}$, i.e. $S_{ij}(\mathcal{G}) = \mathcal{G}$ for all $1 \leq i < j \leq n$. Let us show that $\binom{n}{2}$ shiftings are sufficient if we do them in the right order.

To do this let us first take a different look at shifting.

For $1 \leq i \leq n$ and a family $\mathcal{F} \subset 2^X$ define $\mathcal{F}(i) = \{F - \{i\}: i \in F \in \mathcal{F}\}$. Then $S_{ij}(\mathcal{F})$ is the unique family $\mathcal{G}$ satisfying $\mathcal{G}(i) = \mathcal{F}(i) \cup \mathcal{F}(j)$, $\mathcal{G}(j) = \mathcal{F}(i) \cap \mathcal{F}(j)$ and $H \in \mathcal{F}$ if and only if $H \in \mathcal{G}$ whenever $|H \cap \{i,j\}| \neq 1$.

Now, it is easy to see that $\mathcal{F}$ is shifted if and only if $\mathcal{F}(j) \subset \mathcal{F}(i)$ holds for all $1 \leq i < j \leq n$. Since we shall never use Proposition 2.2, its proof will be somewhat sketchy.

Proposition 2.2. Let $\mathcal{F} \subset 2^X$ be a family and suppose that we perform in succession all $\binom{n}{2}$ shifts S_{ij}, $1 \leq i < j \leq n$ exactly once, in an order where S_{ij} precedes $S_{i'j'}$ whenever $j' < j$. Then the resulting family is shifted.

Proof. Apply induction on n. The statement is trivial for $n \leq 2$. By the assumptions the $n-1$ shifts S_{in}, $1 \leq i < n$ are performed first. Let $\mathcal{G}$ be the family after these shifts. Then $\mathcal{G}(n) \subset \mathcal{G}(i)$ can be checked easily. Moreover, this property is maintained during later shifts.

Set $\mathcal{G}(\bar{n}) = \{G \in \mathcal{G}: n \notin \mathcal{G}\}$.

The remaining $\binom{n-1}{2}$ shifts transform $\mathcal{G}(n)$ and $\mathcal{G}(\bar{n})$ independently and the statement follows by induction. ∎

Proposition 2.3. Suppose $\mathcal{F}$ is k-uniform, t-intersecting and shifted. Then for all $F_1, F_2 \in \mathcal{F}$

$$|F_1 \cap F_2 \cap [1, 2k-t]| \geq t \quad \text{holds.}$$

Proof. Take a counter-example maximizing $|F_1 \cap [1, 2k-t]|$. Since $\mathcal{F}$ is t-intersecting there is some $j \in (F_1 \cap F_2)$, $j > 2k-t$. Thus $F_1 \cup F_2 \not\subset [1, 2k-t]$, hence we may choose $i \notin F_1 \cup F_2$, $i \leq 2k-t$ and replace F_1 by $(F_1 - \{j\}) \cup \{i\}$ (recall that $\mathcal{F}$ is shifted), to obtain a contradiction with the maximal choice of $|F_1 \cap [1, 2k-t]|$. ∎

Let us now prove the Erdös-Ko-Rado theorem for $t = 1$, $n \geq 2k$. Apply induction on k — the statement is trivial for $k = 1$.

a) $n = 2k$. If $F \in \mathcal{F}$ then $|X - F| = n - k = k$, and $(X - F) \notin \mathcal{F}$. Thus $|\mathcal{F}| \leq \frac{1}{2}\binom{2k}{k} = \binom{2k-1}{k-1}$, as desired.

b) $n \geq 2k$. In view of Proposition 2.1, we may assume that $\mathcal{F}$ is shifted. Define $\mathcal{F}_i = \{F \cap [1, 2k]: F \in \mathcal{F}, |F \cap [1, 2k]| = i\}$.

In view of Proposition 2.3 $\mathcal{F}_i$ is intersecting. By induction $|\mathcal{F}_i| \leq \binom{2k-1}{i-1}$ for $i = 0, \ldots, k-1$, the same holds for $i = k$ by a). Given $G \in \mathcal{F}_i$ there are at most $\binom{n-2k}{k-i}$ sets $F \in \mathcal{F}$ with $F \cap [1, 2k] = G$. We infer

$$|\mathcal{F}| \leqslant \sum_{1 \leqslant i \leqslant k} |\mathcal{F}_i| \binom{n-2k}{k-i} \leqslant \sum_{1 \leqslant i \leqslant k} \binom{2k-1}{i-1} \binom{n-2k}{k-i} = \binom{n-1}{k-1} \quad ■$$

Let us say that the Erdös-Ko-Rado theorem is true for (n, k, t) if $\binom{n-t}{k-t}$ is the maximum size of all t-intersecting families $\mathcal{F} \subset \binom{X}{k}$.

Proposition 2.4. Suppose the Erdös-Ko-Rado theorem is true for (n_0, j, t), n_0, t fixed and all j, $t \leqslant j \leqslant k$. Then it holds for (n, k, t) for all $n > n_0$.

Proof. Let us suppose $\mathcal{F}$ is a t-intersecting family of maximum size, $\mathcal{F} \subset \binom{X}{k}$, $\mathcal{F}$ is *stable*. For $i \leqslant k$ define $\mathcal{F}_j = \{F \cap [1, n_0]: F \in \mathcal{F}, |F \cap [1, n_0]| = j\}$. In view of Proposition 2.3 $\mathcal{F}_j$ is t-intersecting and thus $|\mathcal{F}_j| \leqslant \binom{n_0 - t}{j - t}$ holds. This implies

$$|\mathcal{F}| \leqslant \sum_{t \leqslant j \leqslant k} |\mathcal{F}_j| \binom{n-n_0}{k-j} \leqslant \sum_{0 \leqslant i \leqslant k-t} \binom{n_0 - t}{i} \binom{n-n_0}{k-t-i} = \binom{n-t}{k-t} \quad ■$$

Proposition 2.5. Let $\mathcal{G}$ be a shifted t-intersecting family. Then for each $G \in \mathcal{G}$ there exists $i = i(G)$ so that $|G \cap \{1, 2, \ldots, t + 2i\}| \geqslant t + i$.

Proof. Let $G = \{x_1, x_2, \ldots, x_{r+t}\}$ with $x_1 < x_2 < \cdots < x_{r+t}$. Suppose that the proposition is not true for $\mathcal{G}$. Note that $r \geqslant 0$ since $\mathcal{G}$ is t-intersecting and $x_t \geqslant t + 1$, $x_{t+1} \geqslant t + 3, \ldots, x_{t+r} \geqslant t = 2r + 1$. Adding the trivial $x_1 \geqslant 1, \ldots, x_{t-1} \geqslant t - 1$ and using that $\mathcal{G}$ is shifted, we infer

$$G_1 = \{1, 2, \ldots, t-1, t+1, t+3, \ldots, t+2r+1\} \in \mathcal{G}\,.$$

Using shiftedness again, it follows that

$$G_2 = \{1, 2, \ldots, t-1, t, t+2, \ldots, t+2r\} \in \mathcal{G}\,.$$

However, $|\mathcal{G}_1 \cap \mathcal{G}_2| = t - 1$, a contradiction. ■

Let us give now a geometric interpretation of both shifting and this proposition. For this we associate a walk in the plane with each subset F of $\{1, 2, \ldots, n\}$.

We start from the origin and at the i-th step we move one unit up if $i \in F$ and one step to the right if $i \notin F$, $1 \leqslant i \leqslant n$. Let us note that this defines a 1−1 correspondence between 2^X and all walks of length n. For a set F (a walk w) let $F(w)$ $(w(F))$ be the corresponding walk (set), respectively. It is clear that if $i \notin F$, $(i + 1) \in F$, then replacing $i + 1$ by i will change the corresponding part of the walk from ⌋ to ⌈ . Therefore if G can be obtained from F by such shifts then $w(G)$ is lying above $w(F)$. It is easy to see that for stable families $F \in \mathcal{F}$, $w(G)$ and $w(F)$ end in the same point and $w(G)$ lies above $w(F)$ imply $G \in \mathcal{F}$.

Let us now give a geometric proof of Proposition 2.5. Draw the line $y = x + t$. The integer points of this line have the form $(i, i + t)$. If a walk has some point above this line then it ought to have a point on the line too. But how do we get to the point $(i, i + t)$? Only if our set contains $i + t$ out of the first $2i + t$ elements. Thus to prove Proposition 2.5 we have to show that for a stable, t-intersecting family $\mathcal{F}$ none of the walks $w(F)$, $F \in \mathcal{F}$ lies entirely under the line. However, there is a unique highest walk under $y = x + t$; this corresponds to the set $H_1 = \{1, 2, \ldots, t-1, t+1, t+3, \ldots, t+2s+1, \ldots\}$ Therefore if $w(F)$ is under $y = x + t$ for some $F \in \mathcal{F}$ then some subset F_1 of H_1 is in $\mathcal{F}$, too. Using stability we infer that $F_2 \in \mathcal{F}$ for some $F_2 \subset H_2 = \{1, 2, ..., t, t+2, ..., t+2s, ...\}$. Since $H_1 \cap H_2 = \{1, 2, ..., t-1\}$, $|F_1 \cap F_2| \leqslant t - 1$, a contradiction.

3. SHADOWS: THE KRUSKAL-KATONA THEOREM

Suppose $\mathcal{F}$ is a family of k-element sets $|\mathcal{F}| = m$. Note that we do not require $\mathcal{F} \subset \binom{X}{k}$. What is the minimum number of $(k-\ell)$-element sets contained in some member of $\mathcal{F}$, as a function of k and m? This problem was solved independently by Kruskal [Kr] and Katona [Ka1] more than 20 years ago. To state this result we need some definitions.

For a family $\mathcal{F}$ define its *ℓ-th shadow*, $\partial_\ell(\mathcal{F})$ by $\partial_\ell(\mathcal{F}) = \{G : \exists F \in \mathcal{F}, G \subset F, |F - G| = \ell\}$. For $\ell = 1$ we write simply $\partial(\mathcal{F})$.

Let us define a linear order on the k-subsets of $\{1, 2, ..., n, ...\}$. We say $F < G$ if $F \neq G$ and $\max\{i : i \in F - G\} < \max\{j : j \in G - F\}$ holds. This ordering is called reverse-lexicographic. Suppose m is a given positive integer. Let us take the family of the first m k-element sets in this ordering. Denote it by $\mathcal{R}(k, m)$. From the definition it is clear that for some integer $a_k \geqslant k$, $\mathcal{R}(k, m) \subseteq \binom{\{1, ..., a_k\}}{k}$ holds, and moreover, all the remaining sets in $\mathcal{R}(k, m)$ (if there are any) contain $a_k + 1$. Now there is some $a_{k-1} \geqslant k - 1$ so that for all $G \in \binom{\{1, ..., a_{k-1}\}}{k-1}$ one has $(G \cup \{a_k + 1\}) \in \mathcal{R}(k, m)$. Clearly $a_k > a_{k-1}$, because $\binom{\{1, a_k + 1\}}{k} \not\subset \mathcal{R}(k, m)$. Now the remaining sets in $\mathcal{R}(k, m)$ contain both $a_k + 1$ and $a_{k-1} + 1$. Continuing in this way we find elements $a_k > a_{k-1} > \cdots > a_t \geqslant t \geqslant 1$ so that a set F is in $\mathcal{R}(k, m)$ if and only if $F < \{a_t + 1, ..., a_k + 1\}$ holds. This implies $m = \binom{a_k}{k} + \cdots + \binom{a_t}{t}$. This is called the *$k$-cascade representation of m*.

Proposition 3.1. (i) Every positive integer m has a unique k-cascade representation $m = \binom{a_k}{k} + \cdots + \binom{a_t}{t}$ with $a_k > a_{k-1} > \cdots > a_t \geqslant t \geqslant 1$.

(ii) $\partial_\ell(\mathcal{R}(k, m)) = \mathcal{R}\left(k - \ell, \binom{a_k}{k-\ell} + \cdots + \binom{a_t}{t-\ell}\right)$ (where $\binom{a}{b}$ is understood to be zero for $b < 0$).

We leave the easy proof to the reader.

Theorem 3.2. (Kruskal-Katona theorem) Suppose $\mathcal{F}$ is a family of k-sets, $|\mathcal{F}| = m$ and $m = \binom{a_k}{k} + \cdots + \binom{a_t}{t}$ is the k-cascade representation of m. Then for all ℓ, $1 \leqslant \ell \leqslant k$

(3.1) $$|\partial_\ell(\mathcal{F})| \geqslant \binom{a_k}{k-\ell} + \cdots + \binom{a_t}{t-\ell}$$ holds, or equivalently

(3.2) $$|\partial_\ell(\mathcal{F})| \geqslant |\partial_\ell(\mathcal{R}(k, m))| .$$

Because of the k-cascade representation, Theorem 3.2 is often clumsy for applications. Lovász proposed the following weaker, but handier version. Recall that $\binom{x}{a} = \frac{x(x-1)\cdot\ldots\cdot(x-a+1)}{a!}$ can be defined for all real values of x.

Theorem 3.3. ([L]). Suppose $\mathcal{F}$ is a family of k-sets, $|\mathcal{F}| = m$ and $x \geqslant k$ is defined by $m = \binom{x}{k}$. Then for all $1 \leqslant \ell \leqslant k$ one has

$$(3.3) \qquad |\partial_\ell(\mathcal{F})| \geqslant \binom{x}{k-\ell}.$$

Following [F2] we give a unified argument yielding both results.

Proof of Theorems 3.2 and 3.3. First we note that it is sufficient to settle the case $\ell = 1$ (and then iterate the result ℓ-times) — this is, in fact, trivial from Proposition 3.1 (ii) and the monotonicity of $\binom{x}{a}$ for $x \geqslant a$, respectively.

Our next observation is that for all $1 \leqslant i < j$ one has

$$\partial(S_{ij}(\mathcal{F})) \subset S_{ij}(\partial(\mathcal{F}))$$

— a fact which can be proved by a simple but somewhat tedious case by case analysis. Therefore, in proving (3.1) and (3.3) we may assume that $\mathcal{F}$ is stable (i.e., $S_{ij}(\mathcal{F}) = \mathcal{F}$ for all $1 \leqslant i < j$). We apply induction on m and for given m on k. Note that both statements are trivial for $k = 1$, m arbitrary.

Let us define two new families

$$\mathcal{F}_0 = \{F \in \mathcal{F} : 1 \notin F\}$$

$$\mathcal{F}_1 = \{F - \{1\} : 1 \in F \in \mathcal{F}\}.$$

Clearly,

$$(3.4) \qquad |\mathcal{F}_0| + |\mathcal{F}_1| = |\mathcal{F}|.$$

Since $\mathcal{F}$ is stable, we have

$$(3.5) \qquad \partial\mathcal{F}_0 \subset \mathcal{F}_1.$$

We claim

$$(3.6) \qquad |\mathcal{F}_1| \geqslant \binom{x-1}{k-1}.$$

In fact, $|\mathcal{F}| = \binom{x}{k} = \binom{x-1}{k} + \binom{x-1}{k-1}$ and (3.4) would imply that if (3.6) were not true then $|\mathcal{F}_0| > \binom{x-1}{k}$, so that by induction $|\partial\mathcal{F}_0| > \binom{x-1}{k-1}$, contradicting (3.5). Therefore (3.6) is true. Now note that $\partial\mathcal{F} \subset \mathcal{F}_1 \cup \{\{1\} \cup G : G \in \partial\mathcal{F}_1\}$. By induction $|\partial\mathcal{F}_1| \geqslant \binom{x-1}{k-2}$, and thus

$$|\partial\mathcal{F}| \geqslant \binom{x-1}{k-1} + \binom{x-1}{k-2} = \binom{x}{k-1}, \text{ proving (3.3) .}$$

To prove (3.1) we first show that one can assume

$$(3.7) \qquad |\mathcal{F}_1| \geqslant \binom{a_k - 1}{k-1} + \cdots + \binom{a_t - 1}{t-1}.$$

If this were not the case then (3.4) would imply

$$(3.8) \qquad |\mathcal{F}_0| \geqslant \binom{a_k - 1}{k} + \cdots + \binom{a_t - 1}{t} + 1 .$$

If $a_t - 1 \geqslant t$, then we can forget about the +1 in (3.8) and deduce from the induction hypothesis that

$$|\partial\mathcal{F}_0| \geqslant \binom{a_k - 1}{k-1} + \cdots + \binom{a_t - 1}{t-1}, \text{ contradicting (3.5) .}$$

If $a_t = t$ then let s be the largest integer so that $a_s = s$ holds, $k \geqslant s \geqslant t$. Then (3.8) can be rewritten as

$$|\mathcal{F}_0| \geqslant \binom{a_k - 1}{k} + \cdots + \binom{a_{s+1} - 1}{s+1} + \binom{s}{s}.$$

From the induction hypothesis we infer

$$|\partial\mathcal{F}_0| \geqslant \binom{a_k - 1}{k-1} + \cdots + \binom{a_{s+1} - 1}{s} + \binom{s}{s-1}$$

$$\geqslant \binom{a_k - 1}{k-1} + \cdots + \binom{a_{s+1} - 1}{s} + (s - t + 1)$$

$$= \binom{a_k - 1}{k-1} + \cdots + \binom{a_t - 1}{t-1},$$

again in contradiction to (3.5). Therefore we can assume that (3.7) is true. We conclude the proof of (3.1) as that of (3.3), i.e., using $|\partial\mathcal{F}| \geqslant |\partial\mathcal{F}_1| + |\mathcal{F}_1|$. By the induction hypothesis and (3.7) $|\partial\mathcal{F}_1| \geqslant \binom{a_k - 1}{k-2} + \cdots + \binom{a_t - 1}{t-2}$. Adding this inequality to (3.7), (3.1) follows. ■

Proposition 3.4. Suppose that $\mathcal{F}$ is a family of k sets, $|\mathcal{F}| = m \geqslant 1$ and $x \geqslant k$ is defined by $m = \binom{x}{k}$. Suppose further that $|\partial_\ell(\mathcal{F})| = \binom{x}{k-\ell}$ holds for some $1 \leqslant \ell < k$. Then x is an integer and $\mathcal{F} = \binom{X_0}{k}$ holds for some x-element set X_0.

Proof. Suppose first that $\ell = 1$ and $\mathcal{F}$ is stable. Recall the proof of Theorem 3.3.

We conclude that equality must hold in (3.6). That is, (3.9) $$|\mathscr{F}_1| = \binom{x-1}{k-1}.$$

Since $\binom{x}{k} = \frac{x}{k}\binom{x-1}{k-1}$ and both $\binom{x}{k}$ and $\binom{x-1}{k-1}$ are integers, x must be rational. Now the fact, that $\binom{x}{k}$ is an integer, implies that x is an integer too.

Apply induction on x. For $x = k$ the statement is trivially true.

From (3.9) we infer

(3.10) $$|\mathscr{F}_0| = \binom{x-1}{k}.$$

By (3.5) and Theorem 3.3 we infer from (3.9) and (3.10) that

(3.11) $$\partial\mathscr{F}_0 = \mathscr{F}_1 \text{ and } |\partial\mathscr{F}_0| = \binom{x-1}{k-1}.$$

By the induction hypothesis $\mathscr{F}_0 = \binom{Y}{k}$ holds for some set Y with $|Y| = x - 1$. Now $\mathscr{F} = \binom{Y \cup \{1\}}{k}$ follows from (3.11).

To settle the case $\ell = 1$ have to deal with non-shifted families, as well. By the above argument we may assume that x is an integer. We have to show that $|\cup\mathscr{F}| = x$.

We know, that this must hold for the shifted family. Therefore we may indirectly assume that there exist $1 \leqslant i < j \leqslant n$ such that $|\cup\mathscr{F}| = x + 1$ but $|\cup S_{ij}(\mathscr{F})| = x$.

That is, the (i, j)-shift removes all sets containing j from $\mathscr{F}$. Also, $|\mathscr{F}| = \binom{x}{k}$ implies $S_{ij}(\mathscr{F}) = \binom{\cup S_{ij}(\mathscr{F})}{k}$. Set $Y = \cup S_{ij}(\mathscr{F})$ and note $j \notin Y$. Consider the following two families:

$$\mathscr{G} = \{F - \{i\}: i \in F \in \mathscr{F}\},\ \mathscr{H} = \{F - \{j\}: j \in F \in \mathscr{F}\}.$$

Then $G \cap \mathscr{H} = \varnothing$ and $\mathscr{G} \cup \mathscr{H} = \binom{Y - \{i\}}{k-1}$ hold. Since $|\cup\mathscr{F}| = x + 1$, $\mathscr{G} \neq \varnothing \neq \mathscr{H}$ follows. Consequently, there exists some $B \in \binom{Y - \{i\}}{k-2}$ with $B \in (\partial\mathscr{G} \cap \partial\mathscr{H})$.

Therefore $B \cup \{j\}$ does not change when one applies the (i, j)-shift to $\partial\mathscr{F}$. Thus

$$\partial(S_{ij}(\mathscr{F})) \subsetneq S_{ij}(\partial(\mathscr{F})) \text{ holds, i.e., } |\partial(\mathscr{F})| > \binom{x}{k-1}.$$

Let now $\ell \geqslant 2$. If $|\partial_1(\mathscr{F})| = \binom{x}{k-1}$, then we are done by the preceding case. If $|\partial_1(\mathcal{F})| = \binom{y}{k-1}$ where $y > x$, then Theorem 3.3 implies $|\partial_\ell(\mathcal{F}) = \partial_{\ell-1}(\partial(\mathcal{F}))| \geqslant \binom{y}{k-\ell}$, a contradiction. ∎ Let us mention that recently Füredi-Griggs [FG] and Mörs [M] characterized those triples (m, k, ℓ) for which $\mathcal{R}(k, m)$ is the unique optimal family in the Kruskal-Katona Theorem.

Corollary 3.5. (Sperner [S]) Suppose that $\emptyset \neq \mathcal{F} \subset \binom{X}{k}$. Then $\partial_\ell(\mathcal{F})/|\mathcal{F}| \geqslant \binom{n}{k-\ell}/\binom{n}{k}$ holds with equality if and only if $\mathcal{F} = \binom{X}{k}$.

Proof. Note that $\binom{x}{k-\ell}/\binom{x}{k} = k!/(k-\ell)!(x-k+\ell)\cdot\ldots\cdot(x-k+1)$ is monotone decreasing and apply Theorem 3.3 together with Proposition 3.4. ∎

Note that this corollary can be easily proved by a direct double-counting argument, too.

4. SHADOWS OF t-INTERSECTING FAMILIES

If one assumes that $\mathcal{F} \subset \binom{X}{k}$ is t-intersecting then the bound of the Kruskal-Katona theorem for $|\partial_\ell \mathcal{F}|$ can be improved. In particular, we shall show $|\partial_\ell \mathcal{F}| \geqslant |\mathcal{F}|$ for $\ell \leqslant t$.

Let us first consider $\mathcal{A} = \binom{[1, 2k-t]}{k}$. Clearly $\mathcal{A}$ is t-intersecting and $|\partial_\ell \mathcal{A}| = \binom{2k-t}{k-\ell}$. The next theorem shows that $\mathcal{A}$ is the "worst example".

Theorem 4.1 (Katona [Ka2]) Suppose that $\mathcal{F}$ is a k-uniform, t-intersecting family. Then for $1 \leqslant \ell \leqslant t$

(4.1)
$$|\partial_\ell \mathcal{F}| \geqslant |\mathcal{F}| \frac{\binom{2k-t}{k-\ell}}{\binom{2k-t}{k}}$$

holds.

Proof of Theorem 4.1. (cf. [F3]). In view of Propositions 2.1 and 2.4 we may assume that $\mathcal{F}$ is shifted. Then in view of Proposition 2.5 for each $F \in \mathcal{F}$ there exists i so that $|[1, t+2i] \cap F| \geqslant t+i$ holds. Let $i(F)$ denote the maximum value of i for which this holds, i.e., for all $j > i(F)$ one has $|F \cap [1, t+2j]| < t+j$ and, consequently, $|F \cap [1, t+2i(F)]| = t+i(F)$. This makes it possible to partition $\mathcal{F}$ according to $i(F)$ and $F \cap [t+2i(F)+1, n]$.

First define for $A \in \binom{[2t+i+1, n]}{k-t-i}$ $\mathcal{F}_A = \{F \in \mathcal{F} : i(F) = i,$ $F \cap [2t+i+1, n] = A\}$. Then we have the partition

$$\mathcal{F} = \bigcup_{0 \leqslant i \leqslant k-t} \; \bigcup_{A \in \binom{[2i+t+1,n]}{k-t-i}} \mathcal{F}_A .$$

Define $\overline{\mathcal{F}}_A = \{F - A : F \in \mathcal{F}_A\}$ and note that $\overline{\mathcal{F}}_A \subset \binom{[1, t+2i]}{t+i}$. Thus by Corollary 3.5 and the fact that $\binom{t+2i}{t+i-\ell} / \binom{t+2i}{t+i}$ is monotone increasing as a function of i for fixed $0 \leqslant \ell \leqslant t$ we have

(4.2) $$|\partial_\ell \overline{\mathcal{F}}_A| \geqslant |\overline{\mathcal{F}}_A| \binom{t+2i}{t+i-\ell} / \binom{t+2i}{t+i} \geqslant |\overline{\mathcal{F}}_A| \binom{2k-t}{k-\ell} / \binom{2k-t}{k} .$$

Define $\tilde{\partial}_\ell \mathcal{F}_A = \{G \cup A : G \in \partial_\ell \overline{\mathcal{F}}_A\}$. It is immediate that $\tilde{\partial}_\ell \mathcal{F}_A \subset \partial_\ell \mathcal{F}_A$ holds. We claim that for A, A' distinct $\tilde{\partial}_\ell \mathcal{F}_A \cap \tilde{\partial}_\ell \mathcal{F}_{A'} = \emptyset$.

Suppose $|A| = k - t - i$, $|A'| = k - t - i'$, $i \leqslant i'$. Let us first consider the case $i = i'$. Since for $H \in \tilde{\partial}_\ell \mathcal{F}_A (\tilde{\partial}_\ell \mathcal{F}_{A'})$ one has $H \cap [t + 2i + 1, n] = A \, (A')$, respectively, we see that the same H cannot be in both families.

Suppose next $i < i'$, $H \in \tilde{\partial}_\ell \mathcal{F}_A$, $H' \in \tilde{\partial}_\ell \mathcal{F}_{A'}$, let F, F' be respective members of $\mathcal{F}_A$, $\mathcal{F}_{A'}$ satisfying $H \subset F$, $H' \subset F'$. Note that $i(F) = i$, $i(F') = i'$. Thus the definition of $i(F)$ implies $|F \cap [1, 2i + t]| < i' + t$, $|F' \cap [1, 2i' + t]| = i' + t$. Consequently $|H \cap [1, 2i' + t]| < i' + t - \ell = |H' \cap [1, 2i' + t]|$, showing $H \neq H'$. Therefore summing (4.2) over all $0 \leqslant i \leqslant k - t$ and all $A \in \binom{[2i+t+1, n]}{k-t-i}$ the inequality (4.1) follows. ∎

Remark 4.3. One can show, using the above approach, that in (4.1) equality holds only for $\mathcal{F} = \binom{[1, 2k-t]}{k}$. Moreover, the following result of Füredi and the author can be deduced.

Theorem 4.4. Suppose $\mathcal{F}$ is k-uniform, t-intersecting and $|\mathcal{F}| > m_0(k, t)$. Then for $1 \leqslant \ell < t$

(4.3) $$|\partial_\ell \mathcal{F}| / |\mathcal{F}| > \binom{2k-2-t}{k-1-\ell} / \binom{2k-2-t}{k-1} .$$

The inequality (4.3) is asymptotically best possible as is seen by considering $\mathcal{A}_{k-t-1} = \left\{F \in \binom{[1,n]}{k} : |F \cap [1, 2k-2-t]| \geqslant k-1\right\}$, n tending to infinity.

5. THE MAXIMUM SIZE OF t-INTERSECTING FAMILIES: THE KATONA-THEOREM

In Theorem 1.0 we showed that 2^{n-1} is the maximum size of an intersecting family $\mathcal{F} \subset 2^{[1,n]}$. What if $\mathcal{F}$ is t-intersecting? Let us first give examples of large t-intersecting families:

$$\mathcal{B}(n, t) = \left\{B \in X : |B| \geqslant \frac{n+t}{2}\right\} .$$

It is clear that $\mathcal{B}(n, t)$ is t-intersecting. However, for $n + t$ odd one can add $\frac{n+t-1}{2}$-element sets to $\mathcal{B}(n, t)$ and still have a t-intersecting family.

$$\mathcal{B}^*(n, t) = \mathcal{B}(n, t) \cup \left\{ B \in \binom{X}{\frac{n+t-1}{2}} : n \notin B \right\}.$$

It is easy to check that $\mathcal{B}^*(n, t) = \{B \subset X : B \cap [1, n-1] \in \mathcal{B}(n-1, t)\}$. Thus $|\mathcal{B}^*(n, t)| = 2|\mathcal{B}(n-1, t)|$ holds.

Theorem 5.1. (Katona ([Ka2])) Suppose $\mathcal{F} \subset 2^X$ is t-intersecting. Then one of the following two cases occurs.

a) $\quad n + t = 2s$ and $|\mathcal{F}| \leqslant |\mathcal{B}(n, t)| = \sum_{i=s}^{n} \binom{n}{i}$

b) $\quad n + t = 2s + 1$ and $|\mathcal{F}| \leqslant |\mathcal{B}^*(n, t)| = \binom{n-1}{s} + \sum_{i=s+1}^{n} \binom{n}{i}$.

Moreover, for $t \geqslant 2$ the only optimal families are $\mathcal{B}(n, t)$ and $\mathcal{B}^*(n, t)$, respectively.

Proof. First note that $|F \cap F'| \geqslant t$ implies that for $G \subset F$, $|F - G| = t - 1$ one still has $G \cap F' \neq \emptyset$. In particular, no member of $\partial_{t-1}\mathcal{F}$ can be the complement of a member of $\mathcal{F}$. To apply this observation define $\mathcal{F}^{(i)} = \{F \in \mathcal{F} : |F| = i\}$, $f_i = |\mathcal{F}^{(i)}|$. Then we have

$$|\partial_{t-1}\mathcal{F}^{(i)}| + |\mathcal{F}^{(n+t-1-i)}| \leqslant \binom{n}{n+t-1-i}, \quad t \leqslant i \leqslant \frac{n+t-1}{2}$$

Using (4.1) we infer

$$\text{(5.1)} \qquad \frac{i}{i-t+1} f_i + f_{n+t-1-i} \leqslant \binom{n}{n+t-1-i}, \quad t \leqslant i \leqslant \frac{n+t-1}{2}.$$

Summing (5.1) for $t \leqslant i < \frac{n+t-1}{2}$ and noting $f_n \leqslant 1$, $f_i = 0$ for $i < t$ we obtain in the case $n + t = 2s$

$$|\mathcal{F}| = \sum_{i=0}^{n} f_i \leqslant \left[\sum_{i=t}^{s-1} \frac{i}{i-t+1} f_i + f_{n+t-1-i} \right] + f_n \leqslant \sum_{j=s}^{n} \binom{n}{j}, \text{ proving}$$

the theorem for this case. If $t \geqslant 2$, then $i > i - t + 1$. Thus to have equality, one must have $f_i = 0$ for $i \leqslant s - 1 = \frac{n+t-2}{2}$, i.e., $\mathcal{F} \subset \mathcal{B}(n, t)$.

In the case $n + 1 = 2s + 1$, we obtain in the same way

$$|\mathcal{F}| \leqslant |\mathcal{B}(n, t)| + |\mathcal{F}^{(s)}|.$$

Applying (5.1) for $i = s = \frac{n+t-1}{2}$ gives $|\mathcal{F}^{(s)}| = f_s \leqslant \frac{n-s}{n}\binom{n}{s} = \binom{n-1}{s}$, yielding $|\mathcal{F}| \leqslant |\mathcal{B}^*(n, t)|$, as desired. The uniqueness of the optimal family can be argued similarly to case a). ■

Let us note that for $F, F' \subset X$ $|F \cap F'| \geqslant t$ is equivalent to $|F \cup F'| \leqslant n - t$. Thus the Katona Theorem can be restated as follows.

Theorem 5.2. Suppose that $\mathcal{F} \subset 2^X$ satisfies for all $F, F' \in \mathcal{F}$, (5.1) $$|F \cup F'| \leqslant b < n .$$

Then one of the following holds

(i) $$b = 2k \text{ and } |\mathscr{F}| \leqslant \sum_{i=0}^{k} \binom{n}{i}.$$

(ii) $$b = 2k-1 \text{ and } |\mathscr{F}| \leqslant \sum_{0 \leqslant i < k} \binom{n}{i} + \binom{n-1}{k-1}.$$

Moreover, for $b \leqslant n-2$, the optimal families are unique.

Note that in case (ii) for every intersecting family $\mathscr{G} \subset \binom{X}{k}$ the family $\mathscr{G} \cup \{F \subset X : |F| < k\}$ satisfies (5.1). Thus Theorem 5.2(ii) implies that $|\mathscr{G}| \leqslant \binom{n-1}{k-1}$, which is the Erdös-Ko-Rado Theorem. Also, the uniqueness of the optimal families implies that for $n > 2k$ there is a unique optimal family in the Erdös-Ko-Rado Theorem, as well.

6. THE HILTON-MILNER THEOREM.

As we saw in the preceding section, the Katona Theorem implies the Erdös-Ko-Rado Theorem together with the uniqueness of the optimal families for $n > 2k$. Hilton and Milner described the next to optimal families.

Define $$\mathscr{H} = \left\{H \in \binom{[1,n]}{k} : 1 \in H,\ [2,k+1] \cap H \neq \emptyset\right\} \cup \{[2,k+1]\}.$$

Clearly, $\mathscr{H}$ is intersecting and $\cap \mathscr{H} = \emptyset$. Define also $\mathscr{G} = \left\{G \in \binom{[1,n]}{k} : |G \cap [1,3]| \geqslant 2\right\}$. Note that $\mathscr{G}$ is intersecting, $\cap \mathscr{G} = \emptyset$ and for $k = 2$, $\mathscr{G} = \mathscr{H}$ holds.

Theorem 6.1. (Hilton-Milner Theorem ([HM])) Suppose that $\mathscr{F} \subset \binom{X}{k}$ is intersecting, $n > 2k$ and $\cap \mathscr{F} = \emptyset$. Then

(6.1) $$|\mathscr{F}| \leqslant |\mathscr{H}| = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1 .$$

Moreover, equality holds in (6.1) if and only if $\mathscr{F}$ is isomorphic to $\mathscr{H}$, or $k = 3$ and $\mathscr{F}$ is isomorphic to $\mathscr{G}$.

The original proof of this theorem is rather involved. For other proofs cf. Mörs [M] and Alon [A]. The present proof is due to [FF].

Proof. We start by applying the (i,j)-shift to $\mathscr{F}$. Then either $S_{ij}(\mathscr{F})$ satisfies the assumptions of the theorem or i is contained in every member of $S_{ij}(\mathscr{F})$. In the first case we keep on shifting until, eventually, we obtain a shifted family satisfying the assumptions.

Suppose now that at some point the second possibility occurs. Without loss of generality suppose $i = 1$, $j = 2$. Since $1 \in F$ for all $F \in S_{ij}(\mathscr{F})$, $\{1,2\}$ intersects all members of $\mathscr{F}$. Taking $\mathscr{F}$ of maximal size we may assume that

$$(6.2)\qquad \left\{G : [1,2] \subset G \in \binom{X}{k}\right\} \subset \mathscr{F}.$$

Since $\cap \mathscr{F} = \varnothing$, we may assume that $\{1, 3, 4, \ldots, k+1\} \in \mathscr{F}$. Now, instead of S_{12} we keep applying the (i, j)-shift for $3 \leqslant i < j \leqslant n$. Then (6.2) implies that $\cap S_{ij}(\mathscr{F}) = \varnothing$. Eventually we obtain a family, which we denote by abuse of notation by $\mathscr{F}$, satisfying $S_{ij}(\mathscr{F}) = \mathscr{F}$ for all $3 \leqslant i < j \leqslant n$. Note that $G_\ell = \{\ell\} \cup [3, k+1]$ is the unique smallest set $G \in \binom{X}{k}$ satisfying $G \cap [1,2] = \{\ell\}$, $\ell = 1, 2$. Thus $\cap \mathscr{F} = \varnothing$ and the shiftedness of $\mathscr{F}$ implies G_1, $G_2 \in \mathscr{F}$. Together with (6.2) this yields

$$(6.3)\qquad \binom{[1, k+1]}{k} \subset \mathscr{F}.$$

Now we can apply an arbitrary (i, j)-shift, even with $i = 1, 2$, and $\binom{[1, k+1]}{k}$ will not change and therefore $\cap \mathscr{F} = \varnothing$ will be maintained.

Consequently, in proving (6.1) we may assume that $\mathscr{F}$ is a stable family. Now stability implies $[2, k+1] \in \mathscr{F}$ and thus (6.3) holds by stability.

We apply induction on n. Define $\mathscr{F}_i = \{F \cap [1, 2k] : F \in \mathscr{F}, |F \cap [1, 2k]| = i\}$, $0 \leqslant i \leqslant k$. In view of Proposition 2.3 the family $\bigcup_i \mathscr{F}_i$ is intersecting. Consequently, $\mathscr{F}_0 = \varnothing$. Also, (6.3) implies $\mathscr{F}_1 = \varnothing$.

Claim 6.2

$$(6.4)\qquad |\mathscr{F}_i| \leqslant \binom{2k-1}{i-1} - \binom{k-1}{i-1}, \quad 2 \leqslant i < k \ \text{ and}$$

$$(6.5)\qquad |\mathscr{F}_k| \leqslant \binom{2k-1}{k-1} - \binom{k-1}{k-1} + 1 \ \text{ hold}.$$

Proof. If $\cap \mathscr{F}_i \neq \varnothing$, then (6.3) implies (6.4). If $\cap \mathscr{F}_i = \varnothing$ then by the induction assumption

$$|\mathscr{F}_i| \leqslant \binom{2k-1}{i-1} - \binom{2k-i-1}{i-1} + 1 \leqslant \binom{2k-1}{i-1} - \binom{k-1}{i-1}, \text{ proving (6.4)}.$$

For $i = k$, $|\mathscr{F}_k| \leqslant \frac{1}{2}\binom{2k}{k} = \binom{2k-1}{k-1} - \binom{k-1}{k-1} + 1$ is trivial because $\mathscr{F}_k$ is intersecting. ∎

Given $A \subset [1, 2k]$, there are at most $\binom{n-2k}{k-|A|}$ sets $F \in \mathscr{F}$ with $F \cap [1, 2k] = A$. Thus Claim 6.2 implies:

$$|\mathscr{F}| \leq \sum_{i=1}^{k} \binom{n-2k}{k-i} |\mathscr{F}_i| \leq 1 + \sum_{i=2}^{k} \binom{n-2k}{k-i} \left(\binom{2k-1}{i-1} - \binom{k-1}{i-1} \right)$$

$$= 1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1},$$

proving (6.1).

If we have equality, then $|\mathscr{F}_2| = \max\{3, k\}$ follows. Since $\mathscr{F}_2$ is intersecting, either $\mathscr{F}_2 = \{\{1, j\} : 2 \leq j \leq k+1\}$ and consequently, $\mathscr{F} \subset \left\{ H \in \binom{X}{k} : H \cap A \neq \emptyset \text{ for all } A \in \mathscr{F}_2 \right\} = \mathscr{H}$. Or $k = 3$, $\mathscr{F}_2 = \binom{[1,3]}{2}$, and $\mathscr{F} \subset \{G \in \binom{X}{k} : |G \cap [1,3]| \geq 2\} = \mathscr{G}$ This proves the uniqueness of the optimal-families among stable families.

The general case follows from the fact — whose proof is easy and omitted — that if $\mathscr{F}$ is intersecting and $S_{ij}(\mathscr{F}) = \mathscr{G}$ or $\mathscr{H}$ then $\mathscr{F} = \mathscr{G}$ or $\mathscr{H}$. ■

Let us mention the following sharpening of the Hilton-Milner Theorem — the proof of which uses shifting as well.

For $3 \leq i \leq k+1$ define

$$\mathscr{H}_i = \left\{ H \in \binom{X}{k} : 1 \in H, [2,i] \cap H \neq \emptyset \right\} \cup \left\{ \binom{X}{k} : 1 \notin H, [2,i] \subset H \right\}.$$

Note that $\mathscr{H} = \mathscr{H}_{k+1}$ and $\mathscr{G} = \mathscr{H}_3$. Also for $n > 2k$ $|\mathscr{H}_3| = |\mathscr{H}_4| < |\mathscr{H}_5| < \cdots < |\mathscr{H}_{k+1}|$.

For $\mathscr{F} \subset \binom{X}{k}$ let $d(\mathscr{F})$ be the maximum degree of $\mathscr{F}$, i.e., $d(\mathscr{F}) = \max_{1 \leq i \leq n} |\{F \in \mathscr{F} : i \in \mathscr{F}\}|$.

Theorem 6.3 ([F4]) Suppose that $\mathscr{F} \subset \binom{X}{k}$ is intersecting and $d(\mathscr{F}) \leq d(\mathscr{H}_i)$ holds for some $3 \leq i \leq k+1$. Then $|\mathscr{F}| \leq |\mathscr{H}_i|$, moreover, equality holds if and only if either $\mathscr{F}$ is isomorphic to $\mathscr{H}_i$ or $i = 4$ and $\mathscr{F}$ is isomorphic to $\mathscr{H}_3$.

To obtain the Hilton-Milner Theorem just observe that $d(\mathscr{F}) > d(\mathscr{H}_{k+1})$ immediately implies $\cap \mathscr{F} \neq \emptyset$.

7. ON r-WISE t-INTERSECTING FAMILIES

A family $\mathscr{F} \subset 2^X$ is called r-wise t-intersecting if any r members of it intersect in at least t elements. Denote by $f(n, r, t)$ the maximum size of all r-wise t-intersecting families in 2^X.

Proposition 7.1.

(i) $\quad f(n+1, r, t) \geq 2f(n, r, t)$ for $n \geq t$,

(ii) $\quad p(r, t) = \lim_{n \to \infty} f(n, r, t)/2^n$ exists for all r, t.

Proof.

(i) If $\mathscr{F} \subset 2^X$ is r-wise t-intersecting then so is

$$\mathcal{F}^* = \{F \subseteq (X \cup \{n+1\}) : F \cap [1,n] \in \mathcal{F}\}.$$

(ii) In view of (i) the function $f(n, r, t)/2^n$ is monotone non-decreasing in n and it is clearly bounded above by 1. ■

Let us note that $p(r, t) \leqslant 1/2$ follows from Theorem 1.0. Consdering $\mathcal{B}(n, t)$ from Theorem 5.1 one sees that in fact

$$p(2, t) = \frac{1}{2} \text{ for all } t \geqslant 1.$$

Soon we will see that the situation is radically different for $r > 2$ and in fact $p(r, t)/p(r-1, t)$ tends to zero exponentially fast for r fixed and $t \to \infty$.

It is easy to see that Proposition 2.1.(iii) holds for r-wise t-intersecting families, therefore we will assume that $\mathcal{F}$ is a stable, r-wise t-intersecting family.

Proposition 7.2. For each $F \in \mathcal{F}$ there exists some $i \geqslant 0$ so that

(7.1) $$|F \cap [1, t + ri]| \geqslant t + (r-1)i \text{ holds}.$$

Proof. To prove (7.1) we apply the geometric approach of the proof of Proposition 2.5.

Then the statement is equivalent to saying that for $F \in \mathcal{F}$ $w(F)$ meets the line $y = t + (r-1)x$. Again, there is a unique maximal walk not meeting this line, corresponding to the (infinite) set

$$A_0 = \{1, 2, \ldots, t-1, t+1, \ldots, t+r-1, t+r+1, \ldots, t+2r-1, t+2r+1, \ldots\},$$

i.e., A_0 misses t, $t+r$, $t+2r$, ...

If Proposition 7.2. was not true then for some $F \in \mathcal{F}$ $w(F)$ would lie under $w(A_0)$. Using the stability of $\mathcal{F}$ one finds $F_0 \in \mathcal{F}$ satisfying $F_0 \subset A_0$.

Let us define $A_i = \{1, 2, \ldots, n, \ldots\} - \{t+i, t+i+r, t+i+2r, \ldots\}$, $1 \leqslant i < r$. Since $F_0 \subset A_0$ and A_i can be obtained from A_0 by shifting, there exist $F_i \subset A_i$, $F_i \in \mathcal{F}$ for all $0 \leqslant i < r$. However, $|F_0 \cap \cdots \cap F_{r-1}| \leqslant |A_0 \cap \cdots \cap A_{r-1}| = t - 1$, a contradiction. ■

Since a set uniquely corresponds to a (0,1)-vector, its characteristic vector, one can give a probabilistic interpretation for the ratio of sets satisfying (7.1): consider the infinite random walk in which at each step we move one unit with probability $\frac{1}{2}$ up or right. Then the above ratio is the probability $q(r, t)$ that we ever hit the line $y = t + (r-1)x$.

It is easy to see that $q(2, t) = 1$ for all t while $q(r, t) < 1$ for all $r \geqslant 3$.

Proposition 7.3. Let α_r denote the unique root in the interval $\left(\frac{1}{2}, 1\right)$ of the polynomial $z^r - 2z + 1$. Then

(i) $$q(r, t) = \alpha_r^t.$$

(ii) $$\frac{1}{2} < \alpha_r < \frac{1}{2} + \frac{1}{2^r}$$

Proof. First note that $q(r, t)$ satisfies the linear recursion $q(r, 0) = 1$, $q(r, t+1) = \frac{1}{2}q(r, t) + \frac{1}{2}q(r, t+r)$. Next we prove that $q(r, t)$ is multiplicative, i.e.,

(7.2) $$q(r, t+s) = q(r, t)q(r, s).$$

Let q_i be the probability that a walk hits the line $y = t + s + (r-1)x$ and the first place it hits $y = t + (r-1)x$ is at $x = i$. Obviously

(7.3) $$q(r, t+s) = \sum_{i \geq 0} q_i .$$

Let p_i be the probability that a walk hits the line $y = t + (r-1)x$ first in the place $x = i$. Clearly

$$q(r, t) = \sum_{i \geq 0} p_i \text{ and } q_i = p_i q(r, s).$$

Thus (7.3) implies

$$q(r, t+s) = \sum_{i \geq 0} p_i q(r, s) = q(r, t)q(r, s), \text{ proving (7.2)}.$$

In view of (7.2) $q(r, t) = q(r, 1)^t$ holds. Let us set $\beta_r = q(r, 1)$. Substituting this into the recurrence relation $2\beta_r = 1 + \beta_r^r$ follows. Thus β_r is a root of $(z^r - 2z + 1) = (z-1)(z^{r-1} + \cdots + z - 1)$. Since β_r is a positive real and $\beta_r < 1$ it is a positive root of $z^{r-1} + \cdots + z - 1$. This polynomial is monotone increasing for $z > 0$, thus it has only one positive root which is easily seen to be between $1/2$ and $1/2 + 1/2^r$. ∎

Note that Proposition 7.3(i) holds for $r = 2$ as well, $\alpha_2 = 1$.

Theorem 7.4 ([F5]) There exists an absolute constant c so that for all $t \geq 1$ and $r \geq 3$ one has

(6.4) $$\alpha_r^t \geq p(r, t) > c\alpha_r^t / t .$$

Proof. The first part follows directly from Propositions 7.2 and 7.3. To prove the second consider the family $W(n, i)$ consisting of those subsets F of X for which the corresponding walk $w(F)$ is going above the line $y = t + (r-1)x$ at $x = i$. That is $W(n, i) = \{F \subseteq X : |F \cap [1, t + ri]| \geq t + (r-1)i\}$. This family is clearly r-wise t-intersecting and with the previous notation $|W(n, i)| 2^{-n} \geq p_i$ holds for $n \geq t + ri$. Direct calculation shows that for $i > c_r t$ the ratio $|W(n, i)| 2^{-n}$ decreases exponentially fast as a function of i, where $c_r \to 0$ as $r \to \infty$. This shows that the maximal value of $|W(n, i)| 2^{-n}$ is greater than $\frac{c}{t} \Sigma p_i = c\alpha_r^t / t$ for some absolute constant c. ∎

Conjecture 7.5. ([F6])

(6.5) $$f(n, r, t) = \max_i |W(n, i)| .$$

Remark 7.6. In [F6] the above methods were used to prove this conjecture for $t \leq r2^r/150$. In the case $r = 2$ the conjecture is implied by the Katona Theorem (Theorem 5.1). Simple computation shows that on the RHS of (7.5) for $t \leq 2^r - r - 1$ is the family $W(n, 0) = \{F \subseteq X : [1, t] \subseteq F\}$ is maximal and in fact it is the only maximal one for $t < 2^r - r - 1$. It is tied with $W(n, 1)$ if $t = 2^r - r - 1$, while for $t > 2^r - r - 1$ one has $|W(n, 1)| > |W(n, 0)|$.

Let us give the proof of the following, relatively simple case.

Proposition 7.7. Suppose that $t < (\ln 2)2^{r-1} - 1$ then $f(n, r, t) = 2^{n-t}$ and $W(n, 0)$ is the only optimal family.

Proof. Suppose $\mathcal{F} \subset 2^X$ is r-wise t-intersecting. If for some $F_1, \ldots, F_{r-1} \in \mathcal{F}$ one has $|F_1 \cap \cdots \cap F_{r-1}| = t$, then necessarily this t-subset of X is contained in all members of $\mathcal{F}$ proving the statement. Thus we may assume that $\mathcal{F}$ is $(r-1)$-wise $(t+1)$-intersecting. Since $p(r, t)2^n \geqslant f(n, r, t)$, Theorem 7.4 implies

$$f(n, r-1, t+1) \leqslant 2^n(\alpha_{r-1})^{t+1} < 2^n \left(\frac{1}{2}\left(1 + \frac{1}{2^{r-1}}\right)\right)^{t+1} < 2^{n-t-1}e^{(t+1)/2^{r-1}}$$

which is smaller than 2^{n-t} for $t + 1 \leqslant 2^{r-1} \ln 2$. This yields $|\mathcal{F}| < 2^{n-1}$. ∎

8. CROSS-INTERSECTING FAMILIES

Let $\mathcal{F}_1, \ldots, \mathcal{F}_r$ be families of subsets of X. We say that they are *cross-wise t-intersecting* if for all choices of $F_j \in \mathcal{F}_j$, $1 \leqslant j \leqslant r$ $|F_1 \cap \cdots \cap F_r| \geqslant t$ holds.

It is easy to check that if we apply the (i, j)-shift simultaneously to $\mathcal{F}_1, \ldots, \mathcal{F}_r$ then the resulting families $S_{ij}(\mathcal{F}_1), \ldots, S_{ij}(\mathcal{F}_r)$ will be cross-wise t-intersecting.

Thus when we are interesting in bounds on $f(|\mathcal{F}_1|, \ldots, |\mathcal{F}_r|)$, we may suppose that $\mathcal{F}_1, \ldots, \mathcal{F}_r$ are stable. The next proposition is a sharpening of Proposition 7.2.

Proposition 8.1. Suppose that $\mathcal{F}_1, \ldots, \mathcal{F}_r$ are stable, cross-wise t-intersecting families, $F_j \in \mathcal{F}_j$ is arbitrary but fixed, $1 \leqslant j \leqslant r$. Then there exists $\ell \geqslant t$ such that

$$\sum_{1 \leqslant j \leqslant r} |F_j \cap [1, \ell]| \geqslant (r-1)\ell + t \text{ holds.} \tag{8.1}$$

Note that (8.1) is equivalent to $\sum |[1, \ell] - F_j| \leqslant \ell - t$ and therefore implies $|[1, \ell] \cap F_1 \cap \cdots \cap F_r| \geqslant t$.

Proof of (8.1). Suppose that (8.1) does not hold for some $F_1 \in \mathcal{F}_1, \ldots, F_r \in \mathcal{F}_r$ and among such F_j suppose that $F_1, \ldots, F_r$ is chosen so that $|F_1 \cap \cdots \cap F_r|$ is minimal.

Choose ℓ as the minimal integer satisfying

$$|F_1 \cap \cdots \cap F_r \cap [1, \ell]| = t \tag{8.2}$$

If there exists $1 \leqslant i < \ell$ such that i is not contained in at least two out of $F_1, \ldots, F_r$ then choose $1 \leqslant p < q \leqslant r$ with $i \notin F_p$, $i \notin F_q$ and define $\tilde{F}_p = (F_p - \{\ell\}) \cup \{i\}$, $\tilde{F}_s = F_s$ for $s \neq p$. Since $\mathcal{F}_p$ is stable, $\tilde{F}_p \in \mathcal{F}_p$ and

$$\tilde{F}_1 \cap \cdots \cap \tilde{F}_r = F_1 \cap \cdots \cap F_r - \{\ell\} \text{ holds.} \tag{8.3}$$

By minimality, there exists $\tilde{\ell} \geqslant 0$, such that

$$\sum_{1 \leqslant j \leqslant r} |\tilde{F}_j \cap [1, \tilde{\ell}]| \geqslant (r-1)\tilde{\ell} + t > \sum_{1 \leqslant j \leqslant r} |F_j \cap [1, \tilde{\ell}]| . \tag{8.4}$$

Comparing the extreme sides of (8.4) yields $\tilde{\ell} < \ell$. Consequently,

$$|\tilde{F}_1 \cap \cdots \cap \tilde{F}_r \cap [1, \ell]| \geqslant |\tilde{F}_1 \cap \cdots \cap \tilde{F}_r \cap [1, \tilde{\ell}]| \geqslant t .$$

This, however contradicts (8.2) and (8.3). ∎

Hujter observed that Proposition 8.1 has the following surprising corollary.

Proposition 8.2. (Hujter [Hu]) Suppose that $\mathcal{F}_1, \ldots, \mathcal{F}_r \subset 2^X$ are stable, cross-wise t-intersecting. Let $1 \leqslant j \leqslant r$ and let F_j, G_j be arbitrary sets satisfying $F_j - [1,t] = G_j - [1,t]$, $|F_j| = |G_j|$ and $F_j \in \mathcal{F}_j$. Then adding G_j to $\mathcal{F}_j$ will not destroy the cross-wise t-intersecting property.

Proof. Let $F_s \in \mathcal{F}_s$ be arbitrary, $1 \leqslant s \neq j \leqslant r$. Since $|F_j \cap [1,\ell]| = |G_j \cap [1,\ell]|$ for all $\ell \geqslant t$, Proposition 8.1 implies the existence of $\ell \geqslant t$ with

$$|G_j \cap [1,\ell]| + \sum_{s \neq j} |F_s \cap [1,\ell]| \geqslant (r-1)\ell + t \quad \text{and thus}$$

$$|G_j \cap (\bigcap_{s \neq j} F_s) \cap [1,\ell]| \geqslant t \quad \blacksquare$$

For a family $\mathcal{F} \subset 2^X$ define $\lambda(\mathcal{F}) = \max\{\ell : \forall F \in \mathcal{F}, \exists i \geqslant 0$ with $|F \cap [1, ri+\ell]| \geqslant (r-1)i + \ell\}$. Clearly, $\lambda(\mathcal{F}) \geqslant 0$. In the geometric language, ℓ is the largest integer such that no walk $w(F)$, $F \in \mathcal{F}$, lies entirely under the line $y = (r-1)x + \ell$.

The next proposition extends Proposition 7.2 in another way.

Proposition 8.3. Suppose that $\mathcal{F}_1, \ldots, \mathcal{F}_r \subset 2^X$ are stable and cross-wise t-intersecting. Then

(8.5) $$\lambda(\mathcal{F}_1) + \cdots + \lambda(\mathcal{F}_r) \geqslant rt \quad \text{holds}.$$

Proof. Set $\lambda_j = \lambda(\mathcal{F}_j)$. By stability for $1 \leqslant j \leqslant r$ we can choose $F_j \in \mathcal{F}_j$ satisfying

$$F_j \subset \{1, 2, \ldots, \lambda_j, \lambda_j + 2, \lambda_j + 3, \ldots, \lambda_j + r, \lambda_j + r + 2, \lambda_j + r + 3, \ldots\} \text{ i.e.,}$$

$$F_j \subset [1,n] - \{\lambda_j + 1 + ir : i \geqslant 0\}.$$

Note that for every $\ell \geqslant 0$

(8.6) $$|[1,\ell] - F_j| \geqslant \lceil (\ell - \lambda_j)/r \rceil \quad \text{holds}.$$

In view of Proposition 8.1 we can choose $\ell \geqslant 0$ such that

(8.7) $$\sum_j |[1,\ell] - F_j| \leqslant \ell - t$$

Summing (8.6) for $1 \leqslant j \leqslant r$ and using (8.7) gives

$$\ell - t \geqslant \sum_j |[1,\ell] - F_j| \geqslant \sum_{1 \leqslant j \leqslant r} \lceil (\ell - \lambda_j)/r \rceil \geqslant \ell - \sum_j \lambda_j / r .$$

Comparing the extreme sides (8.5) follows. $\blacksquare$

Recall the definition of α_r from Proposition 7.3.

Corollary 8.4 Suppose that $\mathcal{F}_1, \ldots, \mathcal{F}_r \subset 2^X$ are cross-wise t-intersecting. Then

(8.8) $$|\mathcal{F}_1| \cdot \ldots \cdot |\mathcal{F}_r| < 2^{nr} \alpha_r^{tr} .$$

Proof. In proving (8.8) we may assume that $\mathcal{F}_j$ is stable. Then Theorem 7.4 implies $|\mathcal{F}_j| < 2^n \alpha_r^{\lambda(\mathcal{F}_j)}$. Taking products over $1 \leqslant j \leqslant r$ and using (8.5) the inequality (8.8) follows.

9. SOME NUMERICAL EXAMPLES

In this section first we will bound the maximum size $f(n,3,t)$ of 3-wise t-intersecting families for $t \leqslant 6$. In particular, we will show that

$$f(n,3,t) = 2^{n-t} \text{ for } t \leqslant 3 .$$

Throughout this section $\mathscr{F}$ is a 3-wise t-intersecting stable family on $[1,n] = \{1,2,\dots,n\}$.

For sets $A \subset B$ let us set $\mathscr{F}(A,B) = \{F - B : F \cap B = A, F \in \mathscr{F}\}$.

Claim 9.1. Either $\cap \mathscr{F} \supset [1,t]$ and thus $|\mathscr{F}| \leqslant 2^{n-t}$ or

(9.1) $$|\mathscr{F}([1,t],[1,t])| \leqslant 2^{n-t-1}$$

Proof. If (9.1) does not hold, then by Theorem 1.0 we can find two sets $F, F' \in \mathscr{F}$ with $F \cap F' = [1,t]$ implying $[1,t] \subset F''$ for all $F'' \in \mathscr{F}$. ■

Recall the definition of α_3 from Proposition 7.3 and note that $\alpha_3 = (\sqrt{5}-1)/2$. Propositions 7.2 and 7.3 imply $f(n,3,t) < 2^n\alpha_3^t$. The next proposition gives a slight improvement.

Claim 9.2. For $t \geqslant 2$

(9.2) $$f(n,3,t) \leqslant 2^n(\alpha_3^t - 2^{-t-1}) \text{ holds} .$$

Proof. Since $\alpha_3^t - 2^{-t-1} > 2^{-t}$ for $t \geqslant 2$, we may suppose that $|\cap \mathscr{F}| \leqslant t-1$. By Proposition 7.2 every walk $w(F)$ with $F \in \mathscr{F}$ hits the line $y = 2x + t$. By Proposition 7.3 there are less than $\alpha_3^t 2^n$ subsets $F \in 2^{[1,n]}$ such that $w(F)$ hits this line. Among these sets 2^{n-t} contain $[1,t]$. But Claim 9.1 implies that at least 2^{n-t-1} out of these sets are not in $\mathscr{F}$. ■

Proposition 9.3. For $s \leqslant t$ and $A \subset [1,s]$ the family $\mathscr{F}(A,[1,s])$ is 3-wise $(t+2s-3|A|)$-intersecting.

Proof. Let F_1, F_2, F_3 be arbitrary members of $\mathscr{F}$ with $F_i \cap [1,s] = A$, $i = 1,2,3$. Choose ℓ from Proposition 8.1 and set $G_i = F_i \cap [s+1,\ell]$.

Then we infer

$$|G_1|+|G_2|+|G_3| = |F_1 \cap [1,\ell]| + |F_2 \cap [1,\ell]| + |F_3 \cap [1,\ell]| - 3|A|$$

$$\geqslant 2(\ell - s) + t + 2s - 3|A| ,$$

yielding $|G_1 \cap G_2 \cap G_3| \geqslant t + 2s - 3|A|$. ■

Theorem 9.4. The following equality and inequality hold.

(9.3) $$f(n,3,t) = 2^{n-t} \text{ for } i \leqslant t \leqslant 3,\ n \geqslant t .$$

(9.4) $$f(n,3,6) < 0.03149 \cdot 2^n .$$

Moreover, if $\mathscr{F}$ is 3-wise t-intersecting with $|\cap \mathscr{F}| < t$ then $|\mathscr{F}| < 2^{n-t}$ holds for $1 \leqslant t \leqslant 3$.

Proof. We prove all the upper bounds together, using induction on n. For $n \leqslant t$ all bounds are trivially true. Without loss of generality we may assume that $|\cap \mathscr{F}| < t$ and therefore we can apply (9.1).

Consider first the case $t = 3$. For $A \subset [1,3]$ set $f(A) = |\mathscr{F}(A,[1,3])|$. Clearly, we have

$$|\mathscr{F}| = \sum_{A \subset [1,3]} f(A) \tag{9.5}$$

In view of Claim 9.1 we have

$$f([1,3]) \leqslant 2^{n-4} . \tag{9.6}$$

For $A \subset [1,3]$, $\mathscr{F}(A, [1,3])$ is 3-wise $(9-3|A|)$-intersecting by Proposition 9.3. The induction hypothesis yields

$$f(A) \leqslant 2^{n-6} \text{ for all } A \subset [1,3],\ |A| = 2 \text{ and} \tag{9.7}$$

$$f(A) \leqslant 0.03149 \cdot 2^{n-3} \text{ for all } A \subset [1,3],\ |A| = 1 . \tag{9.8}$$

For $A = \varnothing$, (9.2) implies

$$f(\varnothing) \leqslant 0.01218 \cdot 2^{n-3} \tag{9.9}$$

Summing (9.6), (9.7), (9.8) and (9.9) it follows from (9.5) that $|\mathscr{F}| < 0.98165 \cdot 2^{n-3}$, as desired.
Now, $f(n,3,t) \geqslant f(n-1,3,t-1)$ implies (9.3) for $t = 1, 2$ as well.
To prove (9.4) we set $f(A) = |\mathscr{F}(A, [1,6])|$ for $A \subset [1,6]$. Then

$$|\mathscr{F}| = \sum_{A \subset [1,6]} f(A) \text{ holds} . \tag{9.10}$$

The next six inequalities follow from the induction hypothesis or from (9.2), using Proposition 9.3.

$$f([1,6]) \leqslant 2^{n-7}$$

$$f(A) \leqslant 2^{n-3} \qquad \text{for } A \subset [1,6],\ |A| = 5$$

$$f(A) \leqslant 0.03149 \cdot 2^{n-6} \qquad \text{for } A \in \binom{[1,6]}{4}$$

$$f(A) < 0.01218 \cdot 2^{n-6} \qquad \text{for } A \in \binom{[1,6]}{3}$$

$$f(A) < 0.00299 \cdot 2^{n-6} \qquad \text{for } A \in \binom{[1,6]}{2}$$

$$f(A) < 0.00072 \cdot 2^{n-6} \qquad \text{for } A \in \binom{[1,6]}{1} \text{ and}$$

$$f(\varnothing) < 0.00018 \cdot 2^{n-6}$$

Summing these inequalities yields in view of (9.10)

$$|\mathscr{F}| < 0.03149 \cdot 2^{n}, \text{ as desired} . \ \blacksquare$$

Remark 9.5. From the proof it is clear that if $\mathscr{F}$ is 3-wise 3-intersecting with $|\cap \mathscr{F}| < 3$ then actually, $|\mathscr{F}| < 0.98165 \cdot 2^{n-3}$ holds. With similar considerations one can show that if $\mathscr{F}$ is 3-wise 2-intersecting with $|\cap \mathscr{F}| < 2$ then $|\mathscr{F}| < 0.81 \cdot 2^{n-2}$. The same approach yields:

(9.11) $f(n, 5) \leqslant 0.79 \cdot 2^{n-4}$, which we will use in Section 12 .

10. PAIRWISE DISJOINT SETS

We start this section by a theorem of Erdös and Gallai on 2-uniform hypergraphs, that is ordinary graphs.

Theorem 10.1 ([EG]) Let $\mathcal{G} \subset \binom{X}{2}$ be a graph on n vertices, $s \geqslant 2$, $n \geqslant 2s$ and suppose that $\mathcal{G}$ does not contain s pairwise disjoint edges. Then

$$|\mathcal{G}| \leqslant \max\left\{\binom{2s-1}{2}, \binom{s-1}{2} + (s-1)(n-s)\right\} \tag{10.1}$$

Moreover, equality holds in (10.1) if and only if either $\mathcal{G} = \binom{Y}{2}$ for some $(2s-1)$-element set Y or $\mathcal{G} = \left\{G \in \binom{X}{2} : G \cap Z \neq \varnothing\right\}$ for some $Z \in \binom{X}{s-1}$.

Proof (Akiyama-Frankl [AF]). In proving (10.1) we may suppose again that $\mathcal{G}$ is stable. Since $\mathcal{G}$ contains no s pairwise disjoint edges, one of the following s subsets is not in $\mathcal{G}$.

$$G_i = \{i, 2s + 1 - i\}, \quad i = 1, \dots, s .$$

However, if $G_i \notin \mathcal{G}$, then the stability of $\mathcal{G}$ implies

$$\mathcal{G} \subset \mathcal{G}_i = \{G \in \binom{X}{2} : G \cap [1, i-1] \neq \varnothing \text{ or } G \subset [1, 2s-i]\}$$

Note that actually G_i contains no s pairwise disjoint edges. Now (10.1) follows from

$$\max_{1 \leqslant i \leqslant s} |\mathcal{G}_i| = |\mathcal{G}_1| \text{ or } |\mathcal{G}_s|$$

and equality holds only if

$$\mathcal{G} = \mathcal{G}_1 \text{ or } \mathcal{G} = \mathcal{G}_s .$$

Finally, note that if $\mathcal{G}$ contains no s pairwise disjoint edges and $S_{ij}(\mathcal{G})$ is isomorphic to $\mathcal{G}_\ell$ for some $1 \leqslant \ell \leqslant s$, then $\mathcal{G}$ is isomorphic to $\mathcal{G}_\ell$ as well. ■

The families corresponding to $\mathcal{G}_1$ and $\mathcal{G}_s$ for k-graphs with $k \geqslant 3$ are:

$$\mathcal{F}_1 = \left\{F \in \binom{X}{k} : F \cap [1, s-1] \neq \varnothing\right\} \text{ and } \mathcal{F}_k = \binom{[1, ks-1]}{k} .$$

Conjecture 10.2 (Erdös [E]) Suppose that $\mathcal{F} \subset \binom{X}{k}$, $n \geqslant ks$ and $\mathcal{F}$ contains no s pairwise disjoint sets. Then

$$|\mathcal{F}| \leqslant \max\left\{\binom{n}{k} - \binom{n-s+1}{k}, \binom{ks-1}{k}\right\} \text{ holds .} \tag{10.2}$$

Erdös [E] proved this conjecture for $n > n_0(k, s)$. The bounds on $n_0(k, s)$ were improved by Bollobás, Daykin and Erdös [BDS] who showed that (10.2) holds for $n > 2k^3s$. Füredi and the author (unpublished) proved (10.2) for $n > 100ks^2$, but to prove (10.2) in full generality appears to be a very difficult problem.

Let us prove an upper bound, which is not too far from (10.2) and holds for all $n \geqslant ks$.

Theorem 10.3. Suppose that $\mathcal{F} \subset \binom{X}{k}$, $n \geqslant ks$ and $\mathcal{F}$ contains no s pairwise disjoint edges. Then

$$|\mathcal{F}| \leqslant (s-1)\binom{n-1}{k-1} \text{ holds}. \tag{10.3}$$

Proof. Note that for $s = 2$, (10.3) reduces to the Erdös-Ko-Rado Theorem. In fact, our proof will be similar to that. First we prove (10.3) for $n = ks$. Let $X = G_1 \cup G_2 \cup \cdots \cup G_s$ be an arbitrary partition with $|G_1| = \cdots = |G_s| = k$. Out of these s sets at most $s-1$ can be in $\mathcal{F}$. Averaging over all partitions gives

$$|\mathcal{F}| \leqslant \frac{s-1}{s}\binom{ks}{k} = \binom{ks-1}{k}.$$

Now we apply induction on n and prove the statement simultaneously for all k with $ks \leqslant n$.

Again, we may assume that $\mathcal{F}$ is stable. Consider $\mathcal{F}(\bar{n}) = \{F \in \mathcal{F}: n \notin F\}$ and $\mathcal{F}(n) = \{F - \{n\}: n \in F \in \mathcal{F}\}$. We claim that neither of them contains s pairwise disjoint sets. Indeed, this is trivial for $\mathcal{F}(\bar{n}) \subset \mathcal{F}$. As to $\mathcal{F}(n)$, note that if $H_1, \dots, H_s \in \mathcal{F}(n)$ are pairwise disjoint then choosing s distinct elements $y_1, \dots, y_s$ from $[1,n] - (H_1 \cup \cdots \cup H_s)$, which has size $n - s(k-1) \geqslant s$, the stability of $\mathcal{F}$ implies $F_i = (H_i \cup \{y_i\}) \in \mathcal{F}$. However, $F_1, \dots, F_s$ are pairwise disjoint, a contradiction.

Now using the induction hypothesis we infer

$$|\mathcal{F}| = |\mathcal{F}(\bar{n})| + |\mathcal{F}(n)| \leqslant (s-1)\binom{n-2}{k-1} + (s-1)\binom{n-2}{k-2} = (s-1)\binom{n-1}{k-1}. \quad \blacksquare$$

11. ON r-WISE INTERSECTING FAMILIES

Recall that $\mathcal{F} \subset 2^X$ is called r-wise intersecting if $F_1 \cap \cdots \cap F_r \neq \emptyset$ holds for all $F_1, \dots, F_r \in \mathcal{F}$. If $|F_1| + \cdots + |F_r| > (r-1)n$, then necessarily $F_1 \cap \cdots \cap F_r \neq \emptyset$ holds. This shows that the assumptions of the next result are necessary.

Theorem 11.1 ([F8]) Suppose that $\mathcal{F} \subset \binom{X}{k}$ is r-wise intersecting, $rk \leqslant (r-1)n$. Then

$$|\mathcal{F}| \leqslant \binom{n-1}{k-1}. \tag{11.1}$$

Moreover, excepting the case $r = 2$, $n = 2k$ equality holds if and only if $\mathcal{F} = \{F \in \binom{X}{k} : x \in X\}$ holds for some $x \in X$.

Neither the original nor the present proof uses shifting. However, the present proof uses the Kruskal-Katona Theorem, which we proved by shifting.

First we prove a proposition which is due to Kleitman.

Proposition 11.2 ([Kl]) Suppose that $\mathcal{F}_i \subset \binom{X}{k_i}$, $i = 1, \ldots, r$, $k_1 + \cdots + k_r = n$. If there are no $F_i \in \mathcal{F}_i$, $1 \leqslant i \leqslant r$ with $F_1 \cup \cdots \cup F_r = X$, then

(11.2) $$\sum_{1 \leqslant i \leqslant r} |\mathcal{F}_i| / \binom{n}{k_i} \leqslant r - 1 \text{ holds}.$$

Moreover, equality holds if and only if for every ordered partition $X = G_1 \cup \cdots \cup G_r$ satisfying $|G_i| = k_i$ there is exactly one i, $1 \leqslant i \leqslant r$ with $G_i \notin \mathcal{F}_i$.

Proof. Consider all ordered partitions $X = G_1 \cup \cdots \cup G_r$ with $|G_i| = k_i$, $1 \leqslant i \leqslant r$. For a fixed $F \in \binom{X}{k_i}$ one has $F = G_i$ for a fraction $1/\binom{n}{k_i}$ of all these partitions. Thus $G_i \in \mathcal{F}_i$ holds for a fraction $|\mathcal{F}_i|/\binom{n}{k_i}$ of them. Since $G_i \notin \mathcal{F}_i$ must hold for at least one i, the statement follows. ∎

Proof of Theorem 11.1. Set $\mathcal{F}^c = \{X - F : F \in \mathcal{F}\}$. Choose numbers $k_1, \ldots, k_r$ satisfying $0 \leqslant k_i \leqslant n - k$, $k_1 + \cdots + k_r = n$.

Set $\mathcal{F}_i = \partial_{k-k_i}(\mathcal{F}) = \left\{G \in \binom{X}{k_i} : \exists F \in \mathcal{F}, G \subset F\right\}$.

Note that the fact that $\mathcal{F}$ is r-wise intersecting is equivalent to $\mathcal{F}^c$ not containing r sets whose union is X. Thus $\mathcal{F}_1, \ldots, \mathcal{F}_r$ satisfy the assumptions of Proposition 11.2.

Suppose $|\mathcal{F}| \geqslant \binom{n-1}{k-1} = \binom{n-1}{n-k}$. Then, by a consequence of the Kruskal-Katona Theorem (Corollary 3.4), one has $|\mathcal{F}_i| \geqslant \binom{n-1}{k_i}$, that is $|\mathcal{F}_i|/\binom{n}{k_i} \geqslant 1 - \frac{k_i}{n}$ for $1 \leqslant i \leqslant r$. Comparing with (11.2) gives that equality must hold for all i. Again by Corollary 3.4 if $k_i < n - k$ for some i, we infer $\mathcal{F}^c = \binom{X - \{x\}}{n-k}$ for some $x \in X$.

If $k_1 = \cdots = k_r = n - k$, and thus $(r-1)n = rk$, i.e., if $\mathcal{F}_1 = \cdots = \mathcal{F}_r = \mathcal{F}^c$, then by Proposition 11.2 $|\mathcal{F}^c| \leqslant \frac{r-1}{r}\binom{r(n-k)}{n-k} = \binom{n-1}{k-1}$ with equality only if there is one missing set from every partition. This implies that $\binom{X}{n-k} - \mathcal{F}^c$ is an intersecting family of size $\binom{n-1}{n-k-1}$.

Thus, for $r \geqslant 3$, that is, for $(n-k) \leqslant n/3$, the uniqueness of the optimal families in the Erdős-Ko-Rado Theorem implies $\binom{X}{n-k} - \mathcal{F}^c = \left\{G \in \binom{X}{n-k} : x \in G\right\}$ for some $x \in X$, and the uniqueness part of Theorem 11.1 follows. ■

12. A Helly-type theorem

From various previous theorems we know that if $\mathcal{F} \subset 2^X$ is r-wise intersecting with $|\mathcal{F}| \geq 2^{n-1}$, then for $r \geq 3$ $\mathcal{F}$ consists of all subsets through some fixed element. What if we bar this family? Define

$$\mathcal{H} = \mathcal{H}(n,r) = \{H \subset X: |H \cap [1,r+1]| \geq r\} .$$

Clearly, $\mathcal{H}$ is r-wise intersecting, $\cap \mathcal{H} = \emptyset$ and $|\mathcal{H}| = (r+2)2^{n-r-1}$ hold.

Theorem 12.1 (Brace and Daykin [BD]). Suppose that $\mathcal{F} \subset 2^X$ is r-wise intersecting, $\cap \mathcal{F} = \emptyset$ then $|\mathcal{F}| \leq |\mathcal{H}(n,r)|$; moreover, for $r \geq 3$ equality holds if and only if $\mathcal{F}$ is isomorphic to $\mathcal{H}(n,r)$.

The original proof of this powerful result did not use shifting. Kleitman (cf. [P]) gave a proof using shifting. Here we present an alternate proof which is based on the following.

Proposition 12.2. Suppose that $\mathcal{G} \subset 2^X$ is r-wise r-intersecting, $r \geq 3$. Then $|\mathcal{G}| \leq 2^{n-r}$ with equality holding if and only if $\mathcal{G} = \{G \subset X: T \subset G\}$ for some $T \in \binom{X}{r}$.

Proof. For $r = 3$ this result is contained in Theorem 9.4. For $r \geq 6$, the statement follows from Proposition 7.7. Suppose now that $r = 4$ or 5. If $\mathcal{G}$ is not $(r-1)$-wise $(r+1)$-intersecting, then we can find an r-element set T and $G_1, \ldots, G_{r-1} \in \mathcal{G}$ with $G_1 \cap \cdots \cap G_{r-1} = T$. Consequently, $T \subset G$ for every $G \in \mathcal{G}$.

Suppose next that $\mathcal{G}$ is $(r-1)$-wise $(r+1)$-intersecting. If $r = 4$, then (9.11) implies $|\mathcal{G}| < 0.79 \cdot 2^{n-4}$, and we are done. If $r = 5$, then Propositions 7.2 and 7.3 imply $|\mathcal{G}| < 2^n \alpha_4^6$. One can check that $\alpha_4 < 0.544$ and thus $|\mathcal{G}| < 2^{n-5}$ concluding the proof. ■

Proof of Theorem 12.1. We may assume that $\mathcal{G}$ is a filter, i.e., $G \subset H \subset X$ and $G \in \mathcal{G}$ imply $H \in \mathcal{G}$. Since $\cap \mathcal{G} = \emptyset$, $X - \{i\}$ is in $\mathcal{G}$ for all $1 \leq i \leq n$. As this is maintained by shifting, we may assume that $\mathcal{G}$ is stable. For $r = 2$ one has $(r+2)2^{n-r-1} = 2^{n-1}$. Thus the statement is trivially true. Apply induction and suppose that for $r-1$ the theorem is proved. Consider the families $\mathcal{G}(1)$ and $\mathcal{G}(\bar{1})$. Since $\mathcal{G}$ is r-wise intersecting, $\mathcal{G}(1)$ is $(r-1)$-wise intersecting on $X - \{1\}$ and $(X - \{i\}) \in \mathcal{G}$ for $2 \leq i \leq n$ implies $\cap \mathcal{G}(1) = \emptyset$. By the induction hypothesis we infer:

$$|\mathcal{G}(1)| \leq (r+1)2^{n-r-1} . \tag{12.1}$$

From Proposition 8.1 it follows, in the same way as Proposition 9.3, that $\mathcal{G}(\bar{1})$ is r-wise r-intersecting on $X - \{1\}$. Thus Proposition 12.2 implies:

$$|\mathcal{G}(\bar{1})| \leq 2^{n-r-1} . \tag{12.2}$$

Adding (12.1) and (12.2) we obtain $|\mathcal{G}| \leq (r+2)2^{n-r-1}$, as desired.

In case of equality, equality must hold in (12.2). Consequently, $[2,r+1] \in \mathcal{G}$, which implies $|H \cap [2,r+1]| \geq r-1$ for $H \in \mathcal{G}(1)$.

We infer $|G \cap [1,r+1]| \geq r$ for all $G \in \mathcal{G}$, i.e., $\mathcal{G} \subset \mathcal{H}(n,r)$, as desired. ■

Remark 12.3. Let us note that the present proof shows (via Remark 9.5) that if $\mathcal{G} \neq \mathcal{H}(n,r)$ then $|\mathcal{G}| < (r+1.982)2^{n-r-1}$.

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