

NOTE

COPS AND ROBBERS IN GRAPHS WITH LARGE GIRTH AND CAYLEY GRAPHS

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It is shown that if a graph has girth at least $8t-3$ and minimum degree greater than d , then more than d^t cops are needed to catch a robber. Some upper bounds, in particular for Cayley graphs of groups, are also obtained.

1. Introduction

In [1] Aigner and Fromme and in [10] Quilliot studied the following game, called cops and robbers. There is a finite, connected, undirected graph $G=(V, E)$, m cops and one robber. First the cops choose one vertex each as initial position. Next the robber makes his choice. Afterwards they move alternately (first the cops, then the robber) along the edges of the graph or stay. Denote by $c(G)$ the minimum value of m for which m cops have a winning strategy, i.e., they have an algorithm to catch the robber (get on the same vertex as he) no matter how he plays.

In [1] it is shown that $c(G)$ is at least the minimum degree in graphs with girth 5 or more.

Andreae [2] showed for every $d \geq 3$ the existence of regular graphs G of degree d and $c(G)$ arbitrarily large—solving a problem of [1].

The main result of this paper extends the Theorem of Andreae.

Theorem 1.1. *Suppose the minimum degree of G is greater than d and its girth is at least $8t-3$. Then $c(G) > d^t$.*

Note that for $t=1$ one obtains the bound of Aigner and Fromme. As to upper bounds, let us mention that Meyniel [8] conjectures $c(G) = O(\sqrt{|V|})$, which would be best possible.

We could only prove:

Proposition 1.2. $c(G) = o(|V|)$.

For graphs with large girth we have another upper bound. Let us set $n = |V|$ and let $N_h(x)$ denote the set of vertices at distance h from x ($x \in V$). Define further

$$n_h = n_h(G) = \min_{x \in V} |N_h(x)|, \quad m_h = \max_{x \in V} |N_h(x)|.$$

Theorem 1.3. *Suppose that the girth of G is at least $4h - 1$. Then $c(G) \leq (n/n_h)(1 + \log m_h)$ holds.*

Corollary 1.4. *Suppose G is $(d + 1)$ -regular with girth at least $4h - 1$. Then*

$$c(G) \leq \frac{n}{(d + 1)d^{h-1}} (1 + \log((d + 1)d^{h-1})).$$

Let us now suppose that G is a (connected) Cayley graph, i.e., there is a group structure on V , a generator set Γ so that $1 \notin \Gamma$ and $E = \{(v, v\gamma) : v \in V, \gamma \in (\Gamma \cup \Gamma^{-1})\}$. Clearly, G is $|\Gamma \cup \Gamma^{-1}|$ -regular. For $\gamma \in \Gamma$, $\langle \gamma \rangle$ denotes the cyclic group generated by γ . A Cayley graph is called full if Γ consists of full conjugacy classes, i.e., $\gamma \in \Gamma$, $g \in G$ imply $g^{-1}\gamma g \in \Gamma$.

Theorem 1.5. *Suppose G is a k -regular, full Cayley graph. Then $c(G) \leq k$.*

2. The lower bound

Let g be the girth of G and set $r = 2t - 2$. We may suppose without loss of generality that G is connected. Now, if the cops have a winning strategy from some initial position, then they can win from every initial position. Thus we may suppose that all cops are in a fixed vertex u and the robber is in a vertex v , adjacent to u ; moreover, it is the robber's turn to move.

The strategy of the robber is the following. He wants to be in a vertex v such that there is a neighbor u of v so that after the cops' move all geodesics connecting v to cops at distance not exceeding r pass through u . Note, that this initial position satisfies trivially these conditions.

Thus to conclude the proof it is sufficient to show that from such a vertex he can move in t steps to an other vertex satisfying the same conditions (note that this ensures that he is not caught on his way).

Suppose now that u, v are as above and consider the vertices at distance t from v . There are at least d^t vertices whose geodesic does not go through u . Draw a geodesic from each cop at distance at most $(g - 1)/2$ to v . These paths have one point each at distance t from v or they go through u . Thus we can find a vertex x at distance t , which does not lie on any of these geodesics.

Now the robber's strategy is to go straight (in t steps) from v to x . Let y be the vertex preceding x . We claim that x, y satisfy the desired conditions.

We have to verify that all geodesics from cops at distance at most r to the robber (in x) go through y . We distinguish 3 types of cops. Those whose distance to v was at most r , greater than r but less than $g/2$, at least $g/2$. These latter pose no threat, their distance to x will be at least $g/2 - 2t > r$.

As to those who were closest, they can be linked to x via a trail going through u , v and y and of total length at most $r + 2t < g/2$. Their geodesic path to x must go through y , otherwise there is a cycle of length at most $2(r + 2t) < g$, a contradiction.

Consider finally a cop whose distance to v was between $r + 1$ and $(g - 1)/2$. If he got within distance r , then his distance must have been at most $r + 2t$. Therefore, we have a trail from its present position to x , going through y and of total length at most $r + 4t$. If the path of length at most r from it to x misses y , then the graph contains a cycle of length at most $2r + 4t < g$, a contradiction. \square

3. Upper bounds

Proof of Proposition 1.2. It is sufficient to show (1) $c(G) < |V|/b + b^b$ for every positive integer b .

Use induction on $n = |V|$. The statement is trivial for $n < b^b$. Suppose G has a vertex, v of degree at least $b - 1$. Place a cop there to control its neighborhood. Let G_0 be the graph obtained by deleting the neighborhood of v together with v , and let G_1, \dots, G_r be the components of G_0 . Clearly, $c(G) \leq 1 + \max_{1 \leq i \leq r} c(G_i)$ holds, proving (1) via the induction assumption.

Next we may assume that G has maximum degree at most $b - 2$. As

$$n \geq b^b > 1 + (b - 2) + \dots + (b - 2)(b - 3)^{b-3},$$

G has diameter at least $b - 1$. Thus we may find two vertices x_1, x_b so that their distance is $b - 1$. Let x_1, x_2, \dots, x_b form a geodesic path between them. Let G_0 be the graph obtained by deleting $\{x_1, \dots, x_b\}$ from G and suppose G has components G_1, \dots, G_r . Then by Lemma 4 in [1], we have $c(G) \leq 1 + \max_{1 \leq i \leq r} c(G_i)$ again. Now (1) follows by induction. \square

Note that Proposition 1.2 gives

$$c(G) \leq (1 + o(1)) \frac{n \log \log n}{\log n}.$$

Proof of Theorem 1.3. Let us consider the hypergraph $H = \{N_h(v) : v \in V\}$. It has rank n_h and maximum degree m_h . Hence $\tau^*(H) \leq n/n_k$ (give weight $1/n_k$ to each vertex). By a theorem of Lovász and Stein (cf. [6]) one has $\tau(H) \leq (n/n_k)(1 + \log m_k)$. Moreover, one can obtain a covering of this size by the greedy algorithm.

Place the cops in the corresponding vertices. Suppose that the robber is in vertex x . For each $y \in N_{h-1}(x)$ let $c(y)$ be a cop in $N_h(y)$. Note that for $y \neq y'$ it might happen that $c(y) = c(y')$. Now the strategy of the cops is very simple, they just walk straight towards vertex x , stopping as soon as the cop is on the geodesic between the robber and x . We claim that the robber is caught after at most $2h - 1$ moves by the cops. Indeed, where could he be? Suppose that he is in z and let y be such that either y is on the geodesic between x and z or z is on the geodesic between x and y . Let $c(y)$ be in vertex u . If $c(y)$ did not stop then in $2h - 1$ steps he reached x . Thus z cannot be on the geodesic between u and x . By our stopping rule and because the girth is at least $4h - 1$, the robber must have gone through u before $c(y)$ got there. Let v be the first vertex in which the geodesics to x from y and the initial position of $c(y)$ meet. Its distance to $c(y)$ was only one longer than to x . Therefore u must be on the geodesic from v to x . Consequently, when $c(y)$ arrives to u , the robber cannot be on the geodesic linking x to y or z , and therefore it cannot get back there without being caught by $c(y)$. \square

4. Cayley graphs: Proof of Theorem 1.5

For $\gamma \in (\Gamma \cup \Gamma^{-1})$ let $c(\gamma)$ be the cop labelled with γ . At the beginning we put all cops at 1. Suppose the robber starts at x . Let us write x as a product of generators $x = g_1 g_2 \cdots g_m$ (obviously, we may suppose that $m \leq n/2$).

Let us make precise the strategy of the cop $c(h)$. Suppose that he is in vertex z and the robber is in vertex y , moreover $z^{-1}y = g_1 g_2 \cdots g_k h^i$ where all g are from $\Gamma \cup \Gamma^{-1}$ and k is as small as possible.

If the robber moves to the vertex yf where $f \notin \langle h \rangle$, then $c(h)$ moves to zf . Note that

$$(zf)^{-1}yf = (f^{-1}g_1f) \cdots (f^{-1}g_kf)(f^{-1}hf)^i.$$

If $f \in \langle h \rangle$ and $k = 0$, then $c(h)$ moves to zh . Finally if $f \in \langle h \rangle$ and $k \geq 1$, then $c(h)$ moves to zg_1 . Note that $(zg_1)^{-1}yf = g_2 \cdots g_k h^i$. This is called approaching move.

After the cops' move we change their labels: the one which had label h will get label $f^{-1}hf$ (note that this is a 1 to 1 map).

We claim that the robber is caught after at most $(m + n)|\Gamma|$ steps.

In fact, at each step of the robber the cops corresponding to that generator approach. If $c(\gamma)$ did m approaching moves, then its 'distance' to the robber will remain forever a power of his label γ . From this time on he and $c(\gamma^{-1})$ pursue the robber around the cyclic group $\langle \gamma \rangle$, where at non-approaching steps the whole cyclic group 'travels' without the relative position of the robber, $c(\gamma)$, $c(\gamma^{-1})$ being altered. As on the cyclic group the robber will be caught after less than $|\langle \gamma \rangle|$ steps, the proof is complete. \square

Remark. Since every Cayley graph of an Abelian group is full, we obtain if G is a

k -regular Cayley graph of an Abelian group, then $c(G) \leq k$ holds. However, for this case Hamidoune [4] obtained the upper bound $c(G) \leq \lceil 3k/4 \rceil$.

On the other hand the theorem does not hold for all Cayley graphs. In fact, Margulis [7] constructed for every $k \geq 2$, $2k$ -regular Cayley graphs whose girth is at least $c_k \log |V|$, where c_k is a positive constant. By Theorem 1.1 these graphs need at least $|V|^{\log(2k-1)c_k/8}$ cops. Imrich [5] improved Margulis' bounds. In particular, he constructed 3-regular Cayley graphs with girth at least $0.56 \log_2 |V| - 5$ and consequently, needing at least $|V|^{0.12}/2$ cops.

To conclude this paper, let us mention that most recently Andreae [3] obtained very nice upper bounds for $c(G)$ supposing that G is not contractible to a fixed graph H .

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Note added in proof

In [11] the often best possible bound $c(G) \leq \lceil (k+1)/2 \rceil$ is obtained for Abelian Cayley graphs of degree k .

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