JOURNAL OF NUMBER THEORY 23, 210–218 (1986)

# Union-Free Families of Sets and Equations over Fields

P. FRANKL

*C.N.R.S., 54 Bd. Raspail, Paris 75006, France*

AND

Z. FÜREDI

*Mathematics Institute, Hungarian Academy of Science, 1364 Budapest, Pf. 127, Hungary*

*Communicated by H. Zassenhaus*

Received May 23, 1984

Let $X$ be an $n$-element set and $\mathscr{F} \subset \binom{X}{k}$ such that all the $\binom{|\mathscr{F}|}{2}$ sets $F_1 \cup F_2$, $F_1, F_2 \in \mathscr{F}$ are distinct. Solving a problem of P. Erdős ("Proceedings, 8th Southeastern Conf. on Combinatorics, Graph Theory, and Computing, Baton Rouge, 1977," pp. 3–12) we show that there exist positive constants $c_k$, $c'_k$ such that $c_k n^{\lceil 4k/3 \rceil/2} \leqslant |\mathscr{F}| \leqslant c'_k n^{\lceil 4k/3 \rceil/2}$ holds. For the proof of the lower bound we need a theorem of independent interest which is of algebraic number-theoretic character (Theorem 1.4.). 

## 1. Introduction and Statement of the Results

Let $n, k$ be integers, $n > k > 0$. Suppose $X$ is an $n$-element set and $\mathscr{F} \subset \binom{X}{k}$, i.e., $\mathscr{F}$ is a $k$-uniform hypergraph on $X$. We say that $\mathscr{F}$ is *union-free* if all the $\binom{|\mathscr{F}|}{2}$ unions $F_1 \cup F_2$, $F_1, F_2 \in \mathscr{F}$, are distinct; $\mathscr{F}$ is called *weakly union-free* if for any four distinct sets $A, B, A', B' \in \mathscr{F}$ $A \cup B = A' \cup B'$ implies $\{A, B\} = \{A', B'\}$. That is, in a weakly union-free family $A \cup B = A \cup C$ is not excluded. Let us denote by $f_k(n)$ $(F_k(n))$ the maximum cardinality of a union-free (weakly union-free) family, respectively. The problem of determining $f_2(n)$ and $F_2(n)$ goes back to Erdős [1], 1938. For known bounds on these functions and related problems see [4]. Here we only mention that the exact value of $f_2(n)$ is unknown for $n > 10$ while the only infinite set of values for which $F_2(n)$ is known is $n = 4^s + 2^s + 1$, $s \geqslant 1$: $F_2(4^s + 2^s + 1) = 2^{s-1}(2^s + 1)^2$, which was proved by Füredi [5]. In [4] the present authors have shown that $f_3(n) = \lfloor n(n-1)/6 \rfloor$ for all $n \geqslant 1$ and $F_3(n) = n(n-1)/3$ for all $n \equiv 1 \pmod 6$, $n > n_0$.



0022-314X/86 $3.00

Our main result is

THEOREM 1.1. *There exist positive constants $c_k$, $c'_k$, such that*

$$c_k n^{\lceil 4k/3 \rceil/2} \leqslant f_k(n) \leqslant F_k(n) \leqslant c'_k n^{\lceil 4k/3 \rceil/2} \tag{1}$$

*holds.*

Let us denote by $H_k(n)$ the maximum possible size of $\mathscr{F}$ if it contains no four distinct sets $A, B, A', B'$ satisfying $A \cup B = A' \cup B'$ and $A' \cap B' = A \cap B$.

In view of the next proposition it is sufficient to deal with $H_k(n)$.

PROPOSITION 1.2. *For all $n \geqslant k \geqslant 1$, we have*

$$H_k(n) \geqslant F_k(n) \geqslant f_k(n) \geqslant \frac{k!}{k^k} H_k(n). \tag{2}$$

Thus Theorem 1.1 follows from the next theorem.

THEOREM 1.3.

$$\left(\frac{1}{k!} - o(1)\right) n^{2t+1} \leqslant H_{3t+1}(n) \leqslant \left(\frac{t!}{k!} + o(1)\right) n^{2t+1} \qquad \textit{if} \quad k = 3t+1, \tag{3}$$

$$\left(\frac{1}{k!} - o(1)\right) n^{2t} \leqslant H_{3t}(n) \leqslant \frac{\sqrt{\binom{2t}{t}} + 1 + o(1)}{\binom{3t}{t}(2t)!} n^{2t} \qquad \textit{if} \quad k = 3t, \tag{4}$$

$$\left(\frac{1}{(2k)!} - o(1)\right) n^{(4t+3)/2} 2^{(4t+3)/2} \leqslant H_{3t+2}(n) \leqslant \left(\frac{\sqrt{\binom{2t+1}{t+1}/(t+1)} + o(1)}{\binom{3t+2}{2t+1}(2t+1)!}\right) n^{(4t+3)/2} \qquad \textit{if} \quad k = 3t+2. \tag{5}$$

The lower bounds in (4) and (5) are consequences of the lower bounds in (3), as we shall show in Section 4.

To derive the lower bound in (3) we use the next theorem. To state it, we need some definitions.

Let $K$ be any field (not necessarily finite). For any subset $Y = \{y_1, \ldots, y_f\}$ of $K$ and any integer $i$, $i \geqslant 0$, we denote by $\sigma_i(Y)$ the $i$th elementary symmetric polynomial in the variables $y_1, \ldots, y_f$, that is,

$$\sigma_i(Y) = \sum_{I \in \binom{\{1, \ldots, f\}}{i}} \prod_{v \in I} y_v.$$

In particular $\sigma_0(Y) = 1$ and $\sigma_i(Y) = 0$ for $i > f$ and $i < 0$. For any fixed $Y$ and $l$ let us define an $l$ by $l$ matrix $D_l(Y)$. Let the general entry of $D_l(Y)$ be $d_{ij}$, where $d_{ij} = \sigma_{2i-j}(Y)$.

THEOREM 1.4. *Suppose* $k = 3t + 1$, $t \geqslant 1$, $c_2, c_4, \ldots, c_{2t}$ *are arbitrary but fixed elements of* $K$ *and* $\mathscr{F} \subset \binom{K}{k}$ *consists of those k-tuples* $A = \{x_1, \ldots, x_k\}$ *for which*

$$\sigma_{2i}(A) = c_{2i} \text{ holds}, \qquad i = 1, \ldots, t;$$

*moreover, for every subset* $Y \subset A$ *we have*

$$\det D_l(Y) \neq 0, \qquad l = 1, \ldots, |Y| - 1.$$

*Then* $\mathscr{F}$ *contains no four distinct sets* $A$, $B$, $A'$, $B'$ *satisfying*

$$A \cup B = A' \cup B', \qquad A \cap B = A' \cap B'.$$

This theorem and the next three propositions give the lower bound of (3).

PROPOSITION 1.5. *Suppose* $|Y| > l$. *Then the polynomial* $\det D_l(Y)$ *is not the zero polynomial.*

PROPOSITION 1.6. *Suppose* $|K| = q$, *in particular that* $K$ *is finite. Then the number of k-tuples* $A \in \binom{K}{k}$ *for which* $\det D_l(Y) = 0$ *holds for some* $l$ *and some* $Y \subset A$, $l < |Y|$, *is bounded from above by* $2^k k^2 \binom{q}{k-1}$.

PROPOSITION 1.7. *There exist constants* $c_2, c_4, \ldots, c_{2t} \in K$ *such that the family* $\mathscr{F}$ *defined in Theorem* 1.4 *satisfies*

$$|\mathscr{F}| \geqslant \frac{\binom{q}{k}}{q^t} - 2^k k^2 \binom{q}{k-1} \Big/ q^t.$$

Now for $n$ a prime power the lower bound in [3] is immediate from Theorem 1.4 and Proposition 1.7. Note that $H_k(n)$ is a monotone increasing function of $n$. As for every $n$ there exists a prime $q$ satisfying $n \geqslant q \geqslant n - o(n)$ (see [6]) the lower bound in [3] holds for all $n$.

## 2. The Proof of Theorem 1.4

Suppose that for some $A, A', B, B'$ the assumptions $A \cup B = A' \cup B'$, $A \cap B = A' \cap B'$ hold. We have to prove $\{A, B\} = \{A', B'\}$. Let us define $C = A \cap A'$, $D = A \cap (B' - A')$, $C' = B \cap B'$, $D' = B \cap (A' - B')$. Then these four sets satisfy $C \cap D = C \cap D' = C' \cap D = C' \cap D' = \emptyset$, $C \cup D = A$, $C \cup D' = A'$, $C' \cup D = B'$, $C' \cup D' = B$. Consequently, we have that

$$\sigma_{2i}(C \cup D) = \sigma_{2i}(C \cup D') = \sigma_{2i}(C' \cup D) = \sigma_{i2}(C' \cup D') = c_{2i}$$

holds for $0 \leqslant i \leqslant t$.

Proposition 2.1. *There exist non-negative integers $a, b$ such that $a + b = t$ and $\sigma_i(C) = \sigma_i(C')$, $\sigma_j(D) = \sigma_j(D')$ hold for $0 \leqslant i \leqslant 2a$, $0 \leqslant j \leqslant 2b$.*

*Proof.* Let $a$ and $b$ be the largest integers for which $\sigma_i(C) = \sigma_i(C')$ and $\sigma_j(D) = \sigma_j(D')$ hold for $0 \leqslant i \leqslant 2a$, $0 \leqslant j \leqslant 2b$, respectively. Assume $a + n < t$. Let us write out the equation for $\sigma_{2a+2b+2}$:

$$\sigma_{2a+2b+2}(C \cup D) = \sum_{0 \leqslant i \leqslant 2a+2b+2} \sigma_i(C)\, \sigma_{2a+2b+2-i}(D) = c_{2a+2b+2}.$$

Denoting this equation by $e_{11}$ and the analogous equations for $C' \cup D'$, $C \cup D'$, $C' \cup D$ by $e_{22}, e_{12}, e_{21}$, respectively, $e_{11} + e_{22} - e_{12} - e_{21}$ reads

$$\sum_{0 \leqslant i \leqslant 2a+2b+2} (\sigma_i(C) - \sigma_i(C'))(\sigma_{2a+2b+2-i}(D) - \sigma_{2a+2b+2-i}(D')) = 0.$$

Since $\sigma_i(C) = \sigma_i(C')$ for $0 \leqslant i \leqslant 2a$ and $\sigma_j(D) = \sigma_j(D')$ for $0 \leqslant j \leqslant 2b$, this equation reduces to

$$(\sigma_{2a+1}(C) - \sigma_{2a+1}(C'))(\sigma_{2b+1}(D) - \sigma_{2b+1}(D')) = 0.$$

Assume by symmetry $\sigma_{2a+1}(C) = \sigma_{2a+1}(C')$. Now write the equations involving $\sigma_{2a+2}$:

$$\sigma_{2a+2}(C \cup D) = \sum_{0 \leqslant i \leqslant 2a+2} \sigma_i(C)\, \sigma_{2a+2-i}(D) = c_{2a+2}.$$

Subtracting from this the corresponding equation for $C'$ and $D$ we obtain $(\sigma_0(D) = 1)$

$$\begin{aligned} \sigma_{2a+2}(C) - \sigma_{2a+2}(C') &+ \sum_{0 \leqslant i \leqslant 2a+1} (\sigma_i(C) - \sigma_i(C'))\, \sigma_{2a+2-i}(D) \\ &= \sigma_{2a+2}(C \cup D) - \sigma_{2a+2}(C' \cup D) = 0. \end{aligned}$$

Since $\sigma_i(C) = \sigma_i(C')$ for $0 \leqslant i \leqslant 2a+1$, $\sigma_{2a+2}(C) = \sigma_{2a+2}(C')$, contradicting the maximal choice of $a$. ∎

If $|C| \leqslant 2a$, then $C = C'$ and thus $A = B'$, $A' = B$, i.e., $\{A, B\} = \{A', B'\}$ follows. Similarly, in the case $|D| \leqslant 2b$. Thus we may assume $|C| > 2a$, $|D| > 2b$.

PROPOSITION 2.2. *Either* $|C| - 2a \leqslant t - a$ *or* $|D| - 2b \leqslant t - b$ *holds.*

*Proof.* Suppose the contrary. Then we infer—using $|C| + |D| = 3t + 1$—

$$3t + 1 - 2a - 2b \geqslant 2t - a - b + 2 \qquad \text{or equivalently } t \geqslant a + b + 1,$$

contradicting the choice of $a$, $b$. ∎

Assume, by symmetry: $0 < |D| - 2b \leqslant t - b$ holds.

PROPOSITION 2.3. $\sigma_j(D) = \sigma_j(D')$ *holds for all* $j \geqslant 0$.

Note that this proposition yields $D = D'$ and consequently $\{A, B\} = \{A', B'\}$, i.e., it concludes the proof of Theorem 1.4.

*Proof of the Proposition.* For $i = 1, ..., t - b$ let $e_i$ ($e_i'$) denote the equation $\sigma_{2b+2i}(C \cup D) = c_{2b+2i}$ ($\sigma_{2b+2i}(C \cup D') = c_{2b+2i}$), respectively. Now $e_i - e_i'$ reads (using $\sigma_j(D) = \sigma_j(D')$ for $0 \leqslant j \leqslant 2b$)

$$\sum_{0 \leqslant j < 2i} \sigma_j(C)(\sigma_{2b+2i-j}(D) - \sigma_{2b+2i-j}(D')) = 0.$$

This is a homogeneous system of linear equations in the variables $\sigma_{2b+v}(D) - \sigma_{2b+v}(D')$ for $v = 1, ..., t - b$, because $|D| \leqslant t + b$ by our assumptions, i.e., $\sigma_j(D) = 0$ for $j > t + b$. By the assumptions of Theorem 1.4 the determinant of this system $\det D_{t-b}(C)$ is non-zero; consequently $\sigma_{2b+v}(D) = \sigma_{2b+v}(D')$ holds for $v = 1, ..., t - b$ and consequently $\sigma_j(D) = \sigma_j(D')$ for all $j \geqslant 0$. ∎

## 3. THE PROOF OF PROPOSITIONS 1.5, 1.6, AND 1.7

*The Proof of Proposition* 1.5. Suppose first $l$ odd. Let $L$ be an arbitrary extension field of $K$ over which $x^l - 1$ factorizes (splits?) into linear factors, i.e., $(x^l - 1) = \prod_{i=1}^{l} (x - \varepsilon_i)$ holds with $\varepsilon_i \in L$. Note that the $\varepsilon_i$ are not necessarily distinct but $\sigma_j(\varepsilon_1, ..., \varepsilon_l) = 0$ holds for all $j > 0$ except for $j = l$, $\sigma_l(\varepsilon_1, ..., \varepsilon_l) = 1$. Let us count $D_l(\varepsilon_1, ..., \varepsilon_l, 0, 0, ..., 0)$. Denoting by $d_{ij}$ the $(i, j)$th entry of $D_l(\varepsilon_1, ..., \varepsilon_l, 0, ..., 0)$ we see that

$$\begin{aligned} d_{ij} &= 1 \qquad \text{if } j \equiv 2i \pmod{l} \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Consequently

$$\det D_l(\varepsilon_1, ..., \varepsilon_l, 0, ..., 0) = (-1)^{\binom{(l+1)/2}{2}} \neq 0,$$

thus $\det D_l(Y)$ is not the zero-polynomial over $K$ either.

If $l$ is even, we argue in the same way but for an extension field $L'$ containing all the roots of $x^{l+1}-1$. ∎

Note that it might happen that $D_l(Y)$ assumes only the 0 value over $K$ if $|K|$ is small $(|K| \leqslant (l/2))$.

*The Proof of Proposition* 1.6. Fix a subset $N = \{n_1, n_2, ..., n_j\} \subset \{1, 2, ..., k\}$, $N = \varnothing$. Let us suppose $X = \{x_1, ..., x_k\} \in \binom{K}{k}$, such that $Y = \{x_i \in X,\ i \in N\}$, $\det D_l(Y) = 0$ $(|Y| = |N| > l)$. By the preceding proposition we may find $n_i \in N$ for which $\det D_l(x_{n_1}, ..., x_{n_{i-1}}, y, x_{n_{i+1}}, ..., x_{n_j})$ considered as a polynomial $p(y)$ of $y$ is not identically zero. Moreover $\deg p(y) \leqslant l$. For any choice of $\{x_1, x_2, ..., x_{n_i-1}, x_{n_i+1}, ..., x_k\}$ there are at most $l$ values of $y$ satisfying $\det D_l(Y) = p(y) = 0$. Since there are only $2^k - 2$ choices for $N$, and less than $k$ choices for $l$, the statement follows. ∎

*The Proof of Proposition* 1.7. Let us look at the values of $\sigma_{2i}(X)$ for $1 \leqslant i \leqslant t$, $|X| = k$ and $X$ such that $\det D_l(Y) \neq 0$ holds for all $Y \subset X$. The number of possible value sequences is bounded by $q^t$. On the other hand, by Proposition 1.6 we have at least $\binom{q}{k} - 2^k k^2 \binom{q}{k-1}$ choices for $X$; thus there is a particular value sequence, say $c_2, c_4, ..., c_{2t}$, which occurs at least $\binom{q}{k}/q^t - 2^k k^2 \binom{q}{k-1}/q^t$ times. ∎

## 4. The Proof of the Lower Bounds of (4) and (5)

First we prove (4). Let $\mathscr{F} \subset \binom{X}{3t+1}$ be a family of maximal size and without four distinct sets $A, B, A', B'$ satisfying $A \cup B = A' \cup B'$, $A \cap B = A' \cap B'$. Define $\mathscr{F}(x) = \{F - \{x\}\colon x \in F \in \mathscr{F}\}$, for every $x \in X$. Obviously $\sum_{x \in X} |\mathscr{F}(x)| = (3t+1)\,|\mathscr{F}| = (3t+1)\,H_{3t+1}(n)$ holds. Thus we may choose some $x \in X$ satisfying $|\mathscr{F}(x)| \geqslant ((3t+1)/n)\,H_{3t+1}(n)$. Moreover, $\mathscr{F}(x)$ satisfies the assumptions: therefore are no four distinct sets $A, A', B, B' \in \mathscr{F}(x)$ satisfying $A \cup B = A' \cup B'$, $A \cap B = A' \cap B'$, $\mathscr{F}(x) \subset \binom{X - \{x\}}{3t}$. Using (3), we infer

$$H_{3t}(n) \geqslant |\mathscr{F}(x)| \geqslant \frac{3t+1}{n}\left(\frac{1}{(3t+1)!} - o(1)\right) n^{nt+1}$$

$$= \left(\frac{1}{(3t)!} - o(1)\right) n^{2t}.$$

To prove (5) consider again a family $\mathscr{F}$ realizing the maximum size but this time for $k' = 2k = 6t + 4 = 3(2t+1) + 1$. For each $F \in \mathscr{F}$ let us fix a partition of $F$ into two-element sets: $F = P_1 \cup \cdots \cup P_{3t+2}$. Let us define

$$\tilde{\mathscr{F}} = \left\{ \{P_1, ..., P_{3t+2}\} : P_i \in \binom{X}{2}, \qquad 1 \leqslant i \leqslant 3t+2 \; (P_1 \cup \cdots \cup P_{3t+2}) \in \mathscr{F} \right\}.$$

Obviously $|\tilde{\mathscr{F}}| = |\mathscr{F}|$, and $\tilde{\mathscr{F}}$ also satisfies that for any four fistinct $A, B, A', B' \in \tilde{\mathscr{F}}$ either $A \cup B \neq A' \cup B'$ or $A \cap B \neq A' \cap B'$. Thus, we have

$$H_{3t+2}\left(\binom{n}{2}\right) \geqslant \frac{1 - o(1)}{(6t+4)!} \left(2 \binom{n}{2}\right)^{(4t+3)/2}.$$

Using (3) and setting $3t + 2 = k$ we infer

$$H_k(n) \geqslant \left(\frac{1}{2k!} - o(1)\right) (2n)^{(4t+3)/2}.$$

## 5. The Proof of the Upper Bounds

Let us set $s = \lceil (2k-1)/3 \rceil$. Suppose $\mathscr{F} \subset \binom{X}{k}$, $\mathscr{F}$ contains no four distinct sets $A, B, A', B'$ satisfying $A \cup B = A' \cup B'$, $A \cap B = A' \cap B'$. For $S \in \binom{X}{s}$ let us set $\mathscr{F}(S) = \{F - S : S \subset F \in \mathscr{F}\}$ and $d_{\mathscr{F}}(S) = |\mathscr{F}(S)|$. Obviously,

$$\sum_{S \in \binom{X}{s}} d_{\mathscr{F}}(S) = \sum_{F \in \mathscr{F}} \binom{|F|}{s} = \binom{k}{s} |\mathscr{F}|. \tag{6}$$

Using the inequality between the arithmetic and quadratic means:

$$\sum_{S \in \binom{X}{s}} (d_{\mathscr{F}}(S))^2 \geqslant \left(\binom{k}{s} |\mathscr{F}|\right)^2 \Big/ \binom{n}{s}.$$

Combining this with (6) we obtain

$$\sum_{S \in \binom{X}{s}} \binom{d_{\mathscr{F}}(S)}{2} \geqslant \frac{1}{2} \binom{k}{s} |\mathscr{F}| \left(\binom{k}{s} |\mathscr{F}| \Big/ \binom{n}{s} - 1\right). \tag{7}$$

On the other hand for $S, S' \in \binom{X}{s}$, $T, T' \in (\mathscr{F}(S) \cap \mathscr{F}(S'))$ implies that the four distinct sets $A = S \cup T$, $B = S' \cup T'$, $A' = S \cup T'$, $B' = S' \cup T$ satisfy $A \cup B = A' \cup B'$, $A \cap B = (S \cap S') \cup (T \cap T') = A' \cap B'$. Consequently the sets $\binom{\mathscr{F}(S)}{2}$ are pairwise disjoint for $S \in \binom{X}{s}$, yielding

$$\sum_{S \in \binom{X}{s}} \binom{d_{\mathscr{F}}(S)}{2} \leqslant \binom{\binom{n}{k-s}}{2}. \tag{8}$$

Let us set $a=|\mathscr{F}|\binom{k}{s}/\binom{n}{s}$ and note $\binom{b}{2}<b^2/2$. Combining (7) and (8) we obtain

$$\binom{n}{s}a(a-1)\leqslant\binom{n}{k-s}^2. \tag{9}$$

If $k=3t+1$ then $s=2t+1$ and (9) yields

$$|\mathscr{F}|\leqslant(1+o(1))\binom{n}{2t+1}\Big/\binom{k}{2t+1},$$

as desired.

If $k=3t$ then $s=2t$, (9) yields $a(a-1)\leqslant\binom{2t}{t}$ and consequently $a<\sqrt{\binom{2t}{t}}+1$, that is, $|\mathscr{F}|<\binom{n}{2t}(\sqrt{\binom{2t}{t}}+1)/\binom{3t}{t}$, yielding (4).

If $k=3t+2$ then $s=2t+1$. Now (9) yields

$$a(a-1)\leqslant(1+o(1))\frac{n}{t+1}\binom{2t+1}{t+1},$$

consequently

$$|\mathscr{F}|\leqslant n^{(4t+3)/2}\frac{\sqrt{\binom{2t+1}{t+1}/(t+1)}}{\binom{3t+2}{2t+1}(2t+1)!}(1+o(1)). \blacksquare$$

To complete the proof of Theorem 1.1 we must prove Proposition 1.2. The first two inequalities are trivial; to prove the third let us take $\mathscr{F}\subset\binom{X}{k}$, satisfying $|\mathscr{F}|=H_k(n)$ and for any four distinct members $A, B, A', B'\in\mathscr{F}$ either $A\cup B\neq A'\cup B'$ or $A\cap B\neq A'\cap B'$. Let $X=Y_0\cup Y_1\cup\cdots\cup Y_{k-1}$ be a random equipartition of $X$, that is, $|Y_i|=\lceil(n-i)/k\rceil$. Let us define $\mathscr{F}(Y_0,\dots,Y_{k-1})=\{F\in\mathscr{F}\colon |F\cap Y_i|=1,\ 0\leqslant i\leqslant k-1\}$. The expected number of members of $\mathscr{F}(Y_0,\dots,Y_{k-1})$ is

$$\frac{|Y_0|\,|Y_1|\cdot\cdots\cdot|Y_{k-1}|}{\binom{n}{k}}|\mathscr{F}|=|\mathscr{F}|\,k!\prod_{i=0}^{k-1}\frac{\lceil(n-i)/k\rceil}{n-i}\geqslant\frac{k!}{k^k}|\mathscr{F}|. \tag{10}$$

Thus there exists a choice of $Y_0,\dots,Y_{k-1}$, forming an equipartition of $X$ and satisfying

$$|\mathscr{F}(Y_0, Y_1,\dots, Y_{k-1})|\geqslant H_k(n)\,k!/k^k.$$

(The theorem "every $k$-uniform $\mathscr{F}$ contains a $k$-partite $\tilde{\mathscr{F}}\subset\mathscr{F}$ with $|\tilde{\mathscr{F}}|\geqslant|\mathscr{F}|\,k!/k^k$" is due to Erdős and Kleitman [3].)

Now the validity of the third inequality is proved if we show that $\tilde{\mathscr{F}}=\mathscr{F}(Y_0,\dots,Y_{k-1})$ is union-free. Suppose the contrary, i.e., there exist $A, B, A', B'\in\tilde{\mathscr{F}}$, $\{A, B\}\neq\{A', B'\}$ with $A\cup B=A'\cup B'$. Then

$(A \cup B) \cap Y_i = (A' \cup B') \cap Y_i$ holds for $i = 0, \ldots, k-1$. Since for $F \in \mathscr{F}$ we have $|F \cap Y_i| = 1$, $(A \cap B) \cap Y_i = (A' \cap B') \cap Y_i$. Taking the union of these sets for $i = 0, \ldots, k-1$ we infer $A \cap B = A' \cap B'$. As $\{A, B\} \neq \{A', B'\}$, $A, B, A', B'$ must all be distinct members of $\tilde{\mathscr{F}}$ and then of $\mathscr{F}$, contradicting the choice of $\mathscr{F}$. ∎

## 6. Concluding Remarks

It would be nice to find out what other, more algebraic properties are possessed by the family $\mathscr{F}$, constructed in Theorem 1.4.

Let us note that for $k \leqslant 4$ and $k = 6$ the family $\mathscr{F}$ is weakly union-free. In particular

$$F_4(n) = \left(\frac{1}{24} + o(1)\right) n^3.$$

## References

1. P. Erdős, On sequences of integers no one of which divides the product of two others and some related problems, *Mitt. Forsch. Math. Mech.* **2** (1938), 74–32.
2. P. Erdős, Problems and results in combinatorial analysis, *in* "Proceedings, 8th Southeastern Conf. on Combinatorics, Graph Theory and computing, Baton Rouge, 1977," pp. 3–12, Louisiana State University Congressus Numerantium XIX.
3. P. Erdős and D. J. Kleitman, On coloring graphs to maximize the proportion of multicolored $k$-edges, *J. Combin. Theory* **5** (1968), 164–169.
4. P. Frankl and Z. Füredi, A new extremal property of Steiner triple-systems, *Discr. Math.* **48** (1984), 205–212.
5. Z. Füredi, Graphs without quadrilaterals, *J. Combin. Theory Sec. B* **34** (1983), 187–190.
6. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, p. 371, formula (22.19.2), Oxford Univ. Press (Clarendon), London/New York, 1958.