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COMMUNICATION

WHAT MUST BE CONTAINED IN EVERY ORIENTED k-UNIFORM HYPERGRAPH

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1. Introduction

Let $X = \{1, 2, \ldots, n\}$ be the set of the first n integers. By a k-uniform hypergraph (or simply k-graph) on n vertices we mean a family of distinct k-subsets of X—called edges. If the vertices of each edge are linearly ordered, then the k-graph is called oriented. For $k = 2$ it is simply a directed, asymmetric graph. Further, if the 2-graph is complete, then the corresponding complete oriented 2-graph is just a tournament. With every edge $e = \{x_1, \ldots, x_k\}$, $x_1 < \cdots < x_k$ associate a permutation ρ_e, where $\rho_e(i)$ is the position of x_i in the linear order.

Definition 1. An oriented k-graph H is called *n-unavoidable* if H is contained as a subhypergraph in every orientation of the complete k-graph on n vertices. Also, H is called *unavoidable* if it is n-unavoidable for some n.

The first unavoidability result was obtained by Rèdei [5] who showed that every tournament on n vertices contains a Hamiltonian path, i.e., an oriented path of length n.

Linial, Saks and Vera Sós [3, 6] proved various interesting theorems on the unavoidability of certain oriented graphs in tournaments.

In [6] Saks and Sós raised the problem of determining the unavoidable oriented k-graphs. Note that for $k = 2$ this problem was solved by Erdös and Moser who showed that the transitive tournament on $\lfloor \log_2 n \rfloor + 1$ vertices is n-unavoidable (see [2]). The fact that every unavoidable digraph is transitive follows by observing that every subgraph of a transitive tournament is transitive.

Let H be an oriented k-graph on X, i.e. $H \subset \binom{X}{k}$, and let π be an arbitrary permutation of $\{1, 2, \ldots, k\}$. We define a directed graph H_π by putting an edge from x to y if for some i, j there exists an edge $h = \{x_1, \ldots, x_k\}$, $1 \leq x_1 < \cdots < x_k$, in H with $x_i = x$ and $x_j = y$ and $\pi^{-1}\rho_h(i) < \pi^{-1}\rho_h(j)$. In words, replace each edge by a transitive tournament ordered according to the permutation $\pi^{-1}\rho_h$ then take the union of these directed edges.

Definition 2. The k-graph H is called *strongly transitive* if for every permutation π the directed graph H_π is transitive (i.e., asymmetric and has no circuits).

The following statement gives the solution of the problem of Saks and Sós.

Theorem. *The oriented k-graph H is unavoidable if and only if it is strongly transitive.*

2. Proof of the Theorem

Suppose first that H is strongly transitive. Let m be such a large integer that in every partition of the k-subsets of $\{1,2,\ldots,m\}$ into $k!$ classes there is an n-element set Y, all of whose k-subsets are in the same class (such an m exists by Ramsey's theorem [4]).

Consider complete oriented k-graph on m vertices. With each k-set $A = \{a_1,\ldots,a_k\}$, $1 \leq a_1 < \cdots < a_k \leq m$ whose vertices are linearly ordered by $a_{\rho(1)} \to a_{\rho(2)} \to \cdots \to a_{\rho(k)}$ we associate the permutation ρ^{-1}. By choice of m there exists an n-element set $Y = \{y_1,\ldots,y_n\}$, $y_1 < \cdots < y_n$ so that all the k-subsets of Y are associated with the same permutation π.

Since H is strongly transitive, the directed graph H_π is transitive. Let $i_1, i_2,\ldots,i_n$ be a permutation of $1,\ldots,n$ so that all directed edges in H_π go from left to right.

Now it is easy to check that the mapping $i_j \to y_j$, $j = 1,\ldots,n$ embeds H into the complete oriented k-graph.

Suppose now that H is unavoidable. For each permutation π of $1,2,\ldots,k$ and every positive integer m define a complete oriented k-graph on m vertices, $K(m,\pi)$, by ordering every edge according to π. By definition for m sufficiently large, H can be embedded into $K(m,\pi)$. Since $K(m,\pi)_\pi$ is transitive, the statement follows. $\square$

3. Concluding remarks

The condition of strongly transitivity might look complicated, however, it can be checked fast for fixed k. An easy consequence of strongly transitivity is that whenever two edges intersect in two or more vertices, then these vertices have the same relative position in both edges. This necessary (but not sufficient) condition can be easily checked even if k is large.

An example of strongly transitive oriented k-graphs is given by complete

k-partite k-graphs: let $X = X_1 \cup X_2 \cdots \cup X_k$ be a partition, π a permutation of $1, \ldots, k$; and let the edges of $P(\pi)$ be all k-sets $\{x_1, \ldots, x_k\}$ with $x_i \in X_i$ and ordered according π, i.e., x_i is the $\pi(i)$th element in it.

One can add further edges to $P(\pi)$ maintaining its strong transitivity. Namely, add arbitrary strongly transitive k-graphs with vertex set X_i, $i = 1, \ldots, k$. This leads us to the following

Problem. What is the maximum number of edges in an unavoidable oriented k-graph on n vertices?

We could not answer this question even for $k = 3$. Then it is easy to see that any 4 vertices span at most 2 edges. In view of a result of de Caen [1] this gives an upper bound of the form $(\frac{1}{3} + o(1))\binom{n}{3}$.

The above, iterative construction with complete 3-partite graphs yields a lower bound of the form $(\frac{1}{4} + o(1))\binom{n}{3}$.

References

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