SECTIONS OF VARIETIES OVER FINITE FIELDS AS LARGE INTERSECTION FAMILIES

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Suppose that $v > k > 0$ and $L = \{l_1, \ldots, l_s\}$ with $0 \leqslant l_1 < \ldots < l_s < k$. A family $\mathscr{F}$ of k-element subsets of a v-element set is called a (v, L, k)-system if for all F, F' in $\mathscr{F}$ one has $|F \cap F'| \in L$. It is known that, for $v > v_0(k)$, then $|\mathscr{F}| \leqslant \prod_{1 \leqslant i \leqslant s}(v-l_i)/(k-l_i)$ and that equality corresponds to structures of great regularity, known as perfect matroid designs.

Here we consider the family of sections of each type of a given non-singular quadric or Hermitian variety in a projective space $\mathrm{PG}(n,q)$ as a (v, L, k)-system. The corresponding values of k and L are calculated. If such a family has size of order $v^{s'}$, then by the above bound we have that $s' \leqslant s$. All families for which $s - s' \leqslant 5$ are listed. For $s - s' \leqslant 3$, it is shown that these families have the largest possible order of magnitude apart from four families, all with $q = 2$, which are not optimal.

The case in which the sections by 3-dimensional subspaces are elliptic quadrics provides families with cv^4 members, $k = q^2+1$ and $L = \{0, 1, 2, q+1\}$. As q increases, one gets fairly close to perfect matroid designs, since $|\mathscr{F}|/[\prod(v-l_i)/(k-l_i)] \to 1$ as $q \to \infty$.

1. *Projective spaces: notation*

We use the following notation throughout:

$K = \mathrm{GF}(q)$;
$\mathrm{PG}(n,q)$ is the projective space of n dimensions over K;
$\mathbf{P}(X)$ is the point of $\mathrm{PG}(n,q)$ with coordinate vector $X = (x_0, \ldots, x_n)$;
$\mathbf{V}(F) = \{\mathbf{P}(X) \mid F(x_0, \ldots, x_n) = 0\}$ where F is a form in $K[X_0, \ldots, X_n]$;
Π_r is a subspace of dimension r, with $-1 \leqslant r \leqslant n$;
$\Pi_r \mathscr{V}$ is the cone with vertex Π_r and base $\mathscr{V}$ in a subspace Π_t skew to Π_r; it comprises the points in all subspaces $P\Pi_r$ for P in $\mathscr{V}$.

To simplify some numerical formulas, we define the following symbols:

$$[r,s]_+ = \prod_{i=r}^{i=s}(q^i+1) \quad \text{for } r \leqslant s,$$

$$[r,s]_- = \prod_{i=r}^{i=s}(q^i-1) \quad \text{for } r \leqslant s,$$

$$[r,s]_\varepsilon = \prod_{i=r}^{i=s}[(\sqrt{q})^i-(-1)^i] \quad \text{for } r \leqslant s;$$

for $r > s$, each of these symbols is 1. We also write

$$\theta_n = |\mathrm{PG}(n,q)| = (q^{n+1}-1)/(q-1), \quad \text{where } n \geqslant 0,$$

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and let $N(\Pi_r, \Pi_n)$ be the number of Π_r in $PG(n,q)$,

$$N(\Pi_r,\Pi_n) = [n-r+1,n+1]_-/[1,r+1]_- \sim q^{n(r+1)-r(r+1)} \quad \text{for large } q.$$

For any variety $\mathscr{V}$, the *projective index* g is the largest dimension of the subspaces lying on $\mathscr{V}$.

In the quadrics and Hermitian varieties, defined in §§2 and 3, each variety is given in canonical form and is unique up to projective equivalence.

For other background on projective spaces see [**9**].

2. *Quadrics in* PG(*n*,*q*)

First we list the canonical forms for quadrics and then give some of their basic properties, concentrating on numerical ones.

(i) *Non-singular quadrics:*

n even, $\mathscr{P}_n = V(X_0^2 + X_1X_2 + \dots + X_{n-1}X_n)$, *parabolic*,

$$n \text{ odd}, \begin{cases} \mathscr{H}_n = V(X_0X_1 + X_2X_3 + \dots + X_{n-1}X_n), & \textit{hyperbolic}, \\ \mathscr{E}_n = V(f(X_0,X_1) + X_2X_3 + \dots + X_{n-1}X_n), & \textit{elliptic}, \end{cases}$$

where f is irreducible over K.

(ii) *Singular quadrics:*

$$\left.\begin{array}{l} t \text{ even} \\ 0 \leqslant t \leqslant n-1 \end{array}\right\} \quad \Pi_{n-t-1}\mathscr{P}_t = V(X_0^2 + X_1X_2 + \dots + X_{t-1}X_t),$$

$$\left.\begin{array}{l} t \text{ odd} \\ 1 \leqslant t \leqslant n-1 \end{array}\right\} \quad \begin{array}{l} \Pi_{n-t-1}\mathscr{H}_t = V(X_0X_1 + X_2X_3 + \dots + X_{t-1}X_t), \\ \Pi_{n-t-1}\mathscr{E}_t = V(f(X_0,X_1) + X_2X_3 + \dots + X_{t-1}X_t). \end{array}$$

For any non-singular quadric $\mathscr{Q}_n$, the quadric $\Pi_{-1}\mathscr{Q}_n = \mathscr{Q}_n$.

The section of $\mathscr{Q}_n$ by Π_d is either Π_d itself or a quadric $\Pi_{d-t-1}\mathscr{Q}_t$. When $\Pi_d \subset \mathscr{Q}_n$, then $\Pi_d \cap \mathscr{Q}_n = \Pi_d\mathscr{H}_{-1}$, a hyperbolic section. To each quadric $\Pi_{n-t-1}\mathscr{Q}_t$ we attach the character $w = 0$, 1, or 2 according as $\mathscr{Q}_t = \mathscr{E}_t$, $\mathscr{P}_t$, or $\mathscr{H}_t$. From [**9**, p. 110], we have

$$|\mathscr{Q}_n| = \theta_{n-1} + (w-1)q^{(n-1)/2} \sim q^{n-1}$$

and

$$|\Pi_{n-t-1}\mathscr{Q}_t| = \theta_{n-1} + (w-1)q^{n-(t+1)/2}.$$

In particular,

$$|\mathscr{P}_n| = |\Pi_{n-t-1}\mathscr{P}_t| = \theta_{n-1}.$$

The projective index g of quadrics is as shown in Table 1.

As defined in §1, a quadric $\Pi_r\mathscr{Q}_t$ is a cone with vertex Π_r and base $\mathscr{Q}_t$. So the points of the quadric consist of the joins of all points of Π_r to all points of $\mathscr{Q}_t$. If, in $PG(n,q)$,

$$\Pi_r\mathscr{Q}_t = V(F(X_0, X_1, \dots, X_t))$$

with F irreducible, then $\mathscr{Q}_t$ lies in the t-space $\mathbf{V}(X_{t+1}, X_{t+2}, \ldots, X_n)$ and Π_r is the space $\mathbf{V}(X_0, X_1, \ldots, X_t)$; that is, $r = n-t-1$. For example, in PG(3, q), the cone

TABLE 1

$\mathscr{Q}_n$	$\mathscr{E}_n$	$\mathscr{P}_n$	$\mathscr{H}_n$
g	$\frac{1}{2}(n-3)$	$\frac{1}{2}(n-2)$	$\frac{1}{2}(n-1)$
$\Pi_{n-t-1}\mathscr{Q}_t$	$\Pi_{n-t-1}\mathscr{E}_t$	$\Pi_{n-t-1}\mathscr{P}_t$	$\Pi_{n-t-1}\mathscr{H}_t$
g	$\frac{1}{2}(2n-t-3)$	$\frac{1}{2}(2n-t-2)$	$\frac{1}{2}(2n-t-1)$

$\Pi_0\mathscr{P}_2 = \mathbf{V}(X_0{}^2 + X_1X_2)$ consists of the join of the vertex $\Pi_0 = \mathbf{P}(0,0,0,1)$ to the conic $\mathscr{P}_2 = \mathbf{V}(X_0{}^2 + X_1X_2) \cap \mathbf{V}(X_3)$ (Fig. 1). Put parametrically, in PG(3, q),

$$\mathscr{P}_2 = \{\mathbf{P}(st, -s^2, t^2, 0) \mid s, t \in \mathrm{GF}(q)\},$$

$$\Pi_0\mathscr{P}_2 = \{\mathbf{P}(st, -s^2, t^2, \lambda) \mid s, t, \lambda \in \mathrm{GF}(q)\}.$$

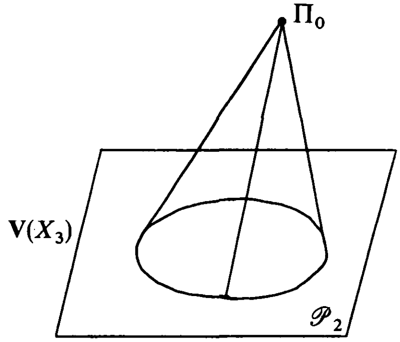

FIG. 1

For low dimensions, we list all quadrics $\mathscr{W}$ in Table 2.

TABLE 2

Space	$\mathscr{W}$	$\lvert\mathscr{W}\rvert$	g	Description
PG(0, q)	$\mathscr{P}_0$	0	-1	$\varnothing$
PG(1, q)	$\mathscr{H}_1$	2	0	point pair
	$\mathscr{E}_1$	0	-1	$\varnothing$
	$\Pi_0\mathscr{P}_0$	1	0	point
PG(2, q)	$\mathscr{P}_2$	$q+1$	0	conic; no three points collinear
	$\Pi_0\mathscr{H}_1$	$2q+1$	1	line pair
	$\Pi_0\mathscr{E}_1$	1	0	point
	$\Pi_1\mathscr{P}_0$	$q+1$	1	line
PG(3, q)	$\mathscr{H}_3$	$(q+1)^2$	1	hyperboloid; each point lies on two of its $2(q+1)$ lines
	$\mathscr{E}_3$	q^2+1	0	ellipsoid; no three points collinear
	$\Pi_0\mathscr{P}_2$	q^2+q+1	1	cone; $q+1$ lines through the vertex
	$\Pi_1\mathscr{H}_1$	$2q^2+q+1$	2	plane pair
	$\Pi_1\mathscr{E}_1$	$q+1$	1	line
	$\Pi_2\mathscr{P}_0$	q^2+q+1	2	plane

In PG(4, q), $\mathscr{W} = \mathscr{P}_4, \Pi_0\mathscr{H}_3, \Pi_0\mathscr{E}_3, \Pi_1\mathscr{P}_2, \Pi_2\mathscr{H}_1, \Pi_2\mathscr{E}_1, \Pi_3\mathscr{P}_0$.
In PG(5, q), $\mathscr{W} = \mathscr{H}_5, \mathscr{E}_5, \Pi_0\mathscr{P}_4, \Pi_1\mathscr{H}_3, \Pi_1\mathscr{E}_3, \Pi_2\mathscr{P}_2, \Pi_3\mathscr{H}_1, \Pi_3\mathscr{E}_1, \Pi_4\mathscr{P}_0$.

PROPOSITION 2.1. *Let $N(\Pi_{d-t-1}\mathscr{Q}_t, \mathscr{Q}_n)$ be the number of subspaces Π_d such that $\Pi_d \cap \mathscr{Q}_n$ is projectively equivalent to $\Pi_{d-t-1}\mathscr{Q}_t$, where $\mathscr{Q}_n$ has character w and $\mathscr{Q}_t$ has character u. Then, with $T = n+t-2d$,*

$$N(\Pi_{d-t-1}\mathscr{Q}_t, \mathscr{Q}_n) = q^{\frac{1}{4}\{T[t+1+uw(2-u)(2-w)]-u(2-u)(w-1)^2\}}$$
$$\times[\tfrac{1}{2}\{T+u+(1+3u-2u^2)w-u(2-u)w^2\}, \tfrac{1}{2}(n+1-w)]_+$$
$$\times[\tfrac{1}{2}\{T+2-u-(1-5u+2u^2)w-u(2-u)w^2\}, \tfrac{1}{2}(n-1+w)]_-$$
$$\div\{[u(2-u), \tfrac{1}{2}(t+1-u)]_+[1, \tfrac{1}{2}(t-1+u)]_-[1, d-t]_-\}$$
$$\sim q^{n(d+1)-\frac{1}{2}\{3d^2+d(3-2t)+t(t-1)\}}.$$

This is known as the *big formula for quadrics*, [**10**]. It is an accumulation of several formulas proved geometrically by Segre [**11**] and algebraically by Dai and Feng [**1, 5**].

PROPOSITION 2.2. *For a fixed q, the only cases for which two quadrics or their sections have the same number of points are as follows:*

(a) $|\Pi_n\mathscr{H}_{-1}| = |\Pi_n\mathscr{E}_1| = |\Pi_{n-t}\mathscr{P}_t| = \theta_n$;
(b) $q = 2$, $|\Pi_{n-2}\mathscr{H}_1| = |\Pi_{n-3}\mathscr{E}_3| = 3.2^{n-1}-1$.

Proof. We compare two quadrics $\mathscr{W} = \Pi_{n-t-1}\mathscr{Q}_t$ and $\mathscr{W}' = \Pi_{n'-t'-1}\mathscr{Q}_{t'}$ of characters w and w' respectively. From above, $|\mathscr{W}| = \theta_{n-1}+(w-1)q^{n-(t+1)/2}$.

When $w = 2$, we have $-1 \leqslant t \leqslant n$ and t is odd. When $w = 1$, we have $0 \leqslant t \leqslant n$ and t is even. When $w = 0$, we have $1 \leqslant t \leqslant n$ and t is odd. Now six separate cases are considered.

(i) $w = 2$, $w' = 2$. Here we have

$$|\mathscr{W}| = |\mathscr{W}'| \Rightarrow \theta_{n-1}+q^{n-(t+1)/2} = \theta_{n'-1}+q^{n'-(t'+1)/2}$$
$$\Rightarrow q^{n-1}+\ldots+q^{n'}+q^{n-(t+1)/2} = q^{n'-(t'+1)/2} \quad \text{when } n > n',$$

which gives a contradiction.

(ii) $w = 2$, $w' = 1$. Here,

$$|\mathscr{W}| = |\mathscr{W}'| \Rightarrow \theta_{n-1}+q^{n-(t+1)/2} = \theta_{n'-1}$$
$$\Rightarrow n' > n \text{ and } q^{n-(t+1)/2} = q^n+q^{n+1}+\ldots+q^{n'-1}$$
$$\Rightarrow n' = n+1 \text{ and } t = -1$$
$$\Rightarrow \mathscr{W} = \Pi_n\mathscr{H}_{-1} \text{ and } \mathscr{W}' = \Pi_{n-t'}\mathscr{P}_{t'};$$

this is part of Case (a).

(iii) $w = 2$, $w' = 0$. Here,

$$|\mathscr{W}| = |\mathscr{W}'| \Rightarrow \theta_{n-1}+q^{n-(t+1)/2} = \theta_{n'-1}-q^{n'-(t'+1)/2}$$
$$\Rightarrow n' > n \text{ and } q^{n-(t+1)/2}+q^{n'-(t'+1)/2} = q^n+q^{n+1}+\ldots+q^{n'-1}$$
$$\Rightarrow n' = n+2,\ t = -1, \text{ and } t' = 1 \text{ or}$$
$$n' = n+1 \text{ and } q^{n-(t+1)/2}+q^{n-(t'-1)/2} = q^n.$$

In the former case, $\mathscr{W} = \Pi_n\mathscr{H}_{-1}$ and $\mathscr{W}' = \Pi_n\mathscr{E}_1$; this is included in (a). In the latter case, $q = 2$ and $q^{n-(t+1)/2} = q^{n-(t'-1)/2} = \frac{1}{2}q^n$, whence $\frac{1}{2}(t+1) = \frac{1}{2}(t'-1) = 1$; so $q = 2$, $t = 1$, $t' = 3$. Thus $\mathscr{W} = \Pi_{n-2}\mathscr{H}_1$ and $\mathscr{W}' = \Pi_{n-3}\mathscr{E}_3$ with $q = 2$. This is Case (b).

(iv) $w = 1$, $w' = 1$. Here,

$$|\mathscr{W}| = |\mathscr{W}'| \Rightarrow \theta_{n-1} = \theta_{n'-1} \Rightarrow n = n',$$

which is again included in (a).

(v) $w = 1$, $w' = 0$. Here,

$$\begin{aligned}|\mathscr{W}| = |\mathscr{W}'| &\Rightarrow \theta_{n-1} = \theta_{n'-1} - q^{n'-(t'+1)/2}\\ &\Rightarrow n' > n \text{ and } q^{n'-(t'+1)/2} = q^n + q^{n+1} + \ldots + q^{n'-1}\\ &\Rightarrow n' = n+1 \text{ and } t' = 1\\ &\Rightarrow \mathscr{W} = \Pi_{n-t-1}\mathscr{P}_t \text{ and } \mathscr{W}' = \Pi_{n-1}\mathscr{E}_1;\end{aligned}$$

this is the remaining part of (a).

(vi) $w = 0$, $w' = 0$. Here,

$$\begin{aligned}|\mathscr{W}| = |\mathscr{W}'| &\Rightarrow \theta_{n-1} - q^{n-(t+1)/2} = \theta_{n'-1} - q^{n'-(t'+1)/2}\\ &\Rightarrow q^{n-1} + \ldots + q^{n'} + q^{n'-(t'+1)/2} = q^{n-(t+1)/2} \quad \text{when } n > n',\end{aligned}$$

which gives a contradiction.

3. *Hermitian varieties in* PG(n, q), *with* q *square*

In a similar fashion to the previous section, we list for Hermitian varieties the canonical forms and basic numerical properties.

(i) *Non-singular Hermitian varieties:*

$$\mathscr{U}_n = \mathbf{V}(X_0\bar{X}_0 + X_1\bar{X}_1 + \ldots + X_n\bar{X}_n) \quad \text{where } \bar{X} = X^{\sqrt{q}}.$$

(ii) *Singular Hermitian varieties:*

$$\Pi_{n-t-1}\mathscr{U}_t = \mathbf{V}(X_0\bar{X}_0 + X_1\bar{X}_1 + \ldots + X_t\bar{X}_t) \quad \text{where } 0 \leqslant t \leqslant n-1.$$

As for quadrics, $\Pi_{-1}\mathscr{U}_n = \mathscr{U}_n$.

The section of $\mathscr{U}_n$ by Π_d is either Π_d itself or a Hermitian variety $\Pi_{d-t-1}\mathscr{U}_t$. When $\Pi_d \subset \mathscr{U}_n$, then $\Pi_d \cap \mathscr{U}_n = \Pi_d\mathscr{U}_{-1}$. From [9, p. 102],

$$|\mathscr{U}_n| = \theta_{n-1} + (q^n - (-1)^n q^{n/2})/(\sqrt{q}+1) \sim q^{n-\frac{1}{2}}$$

and

$$|\Pi_{n-t-1}\mathscr{U}_t| = \theta_{n-1} + (q^n - (-1)^t q^{n-t/2})/(\sqrt{q}+1).$$

The projective index g of Hermitian varieties is shown in Table 3.

TABLE 3

$\mathscr{U}_n$			$\Pi_{n-t-1}\mathscr{U}_t$		
n	odd	even	t	odd	even
g	$\frac{1}{2}(n-1)$	$\frac{1}{2}(n-2)$	g	$\frac{1}{2}(2n-t-1)$	$\frac{1}{2}(2n-t-2)$

For low dimensions, the Hermitian varieties $\mathscr{W}$ are listed in Table 4.

TABLE 4

Space	$\mathscr{W}$	$\lvert\mathscr{W}\rvert$	g	Description
PG(0, q)	$\mathscr{U}_0$	0	-1	$\varnothing$
PG(1, q)	$\mathscr{U}_1$	$\sqrt{q}+1$	0	subline PG(1, $\sqrt{q}$)
	$\Pi_0\mathscr{U}_0$	1	0	point
PG(2, q)	$\mathscr{U}_2$	$q\sqrt{q}+1$	0	unital; each section by a line is $\mathscr{U}_1$ or $\Pi_0\mathscr{U}_0$
	$\Pi_0\mathscr{U}_1$	$q\sqrt{q}+q+1$	1	$\sqrt{q}+1$ concurrent lines
	$\Pi_1\mathscr{U}_0$	$q+1$	1	line
PG(3, q)	$\mathscr{U}_3$	$(q\sqrt{q}+1)(q+1)$	1	Hermitian surface containing $(q\sqrt{q}+1)(\sqrt{q}+1)$ lines; each plane section is $\mathscr{U}_2$ or $\Pi_0\mathscr{U}_1$
	$\Pi_0\mathscr{U}_2$	$q^2\sqrt{q}+q+1$	1	$q\sqrt{q}+1$ lines through the vertex
	$\Pi_1\mathscr{U}_1$	$q^2\sqrt{q}+q^2+q+1$	2	$\sqrt{q}+1$ collinear planes
	$\Pi_2\mathscr{U}_0$	q^2+q+1	2	plane

In PG(4, q), $\mathscr{W} = \mathscr{U}_4, \Pi_0\mathscr{U}_3, \Pi_1\mathscr{U}_2, \Pi_2\mathscr{U}_1, \Pi_3\mathscr{U}_0$.
In PG(5, q), $\mathscr{W} = \mathscr{U}_5, \Pi_0\mathscr{U}_4, \Pi_1\mathscr{U}_3, \Pi_2\mathscr{U}_2, \Pi_3\mathscr{U}_1, \Pi_4\mathscr{U}_0$.

PROPOSITION 3.1. *Let $N(\Pi_{d-t-1}\mathscr{U}_t, \mathscr{U}_n)$ be the number of subspaces Π_d such that $\Pi_d \cap \mathscr{U}_n$ is projectively equivalent to $\Pi_{d-t-1}\mathscr{U}_t$. Then, with $T = n+t-2d$,*

$$N(\Pi_{d-t-1}\mathscr{U}_t, \mathscr{U}_n) = q^{\frac{1}{4}T(t+1)}[t+2, n+1]_\varepsilon/\{[1, T]_\varepsilon[1, d-t]_-\}$$
$$\sim q^{n(d+1)-\frac{1}{4}\{3d^2+2d(1-t)+t^2\}}.$$

Proof. See Wan and Yang [12].

PROPOSITION 3.2. *For a fixed q, the only cases in which two Hermitian varieties or their sections have the same number of points are the following:*

$$\lvert\Pi_n\mathscr{U}_{-1}\rvert = \lvert\Pi_n\mathscr{U}_0\rvert = \theta_n.$$

Proof. We have

$$\lvert\Pi_{n-t-1}\mathscr{U}_t\rvert = \lvert\Pi_{n'-t'-1}\mathscr{U}_{t'}\rvert$$
$$\Rightarrow\ \theta_{n-1}+(q^n-(-1)^t q^{n-t/2})/(\sqrt{q}+1) = \theta_{n'-1}+(q^{n'}-(-1)^{t'} q^{n'-t'/2})/(\sqrt{q}+1)$$
$$\Rightarrow\ (\sqrt{q}+1)(q^{n-1}+\ldots+q^{n'})+q^n-(-1)^t q^{n-t/2} = q^{n'}-(-1)^{t'} q^{n'-t'/2} \quad \text{when } n > n'$$
$$\Rightarrow\ n = n'+1 \text{ and } \sqrt{q}.q^{n-1}+q^n = (-1)^t q^{n-t/2}-(-1)^{t'} q^{n'-t'/2}$$
$$\Rightarrow\ t = 0 \text{ and } t' = -1$$
$$\Rightarrow\ \Pi_{n-t-1}\mathscr{U}_t = \Pi_{n-1}\mathscr{U}_0 \text{ and } \Pi_{n'-t'-1}\mathscr{U}_{t'} = \Pi_{n-1}\mathscr{U}_{-1}.$$

4. *Large intersection families of sets and perfect matroid designs*

Suppose that v and k are integers such that $v > k > 0$. Fix a subset $L = \{l_1, l_2, \ldots, l_s\}$, with $l_1 < l_2 < \ldots < l_s$, of $\{0, 1, \ldots, k-1\}$ and a set X with $\lvert X\rvert = v$. A family of sets $\mathscr{A} = \{A_i\}$ is a *(v, L, k)-family* and is denoted $\mathscr{A}(v, L, k)$ if $A_i \subset X$, $\lvert A_i\rvert = k$, and $\lvert A_i \cap A_j\rvert \in L$ for $i \neq j$. The maximum cardinality of a (v, L, k)-family is denoted $m(v, L, k)$. Deza, Erdös, and Frankl [3] have proved the following.

THEOREM 4.1. *For $v \geqslant v_0(L,k)$,*

$$m(v,L,k) \leqslant \prod_{1 \leqslant i \leqslant s} (v-l_i)/(k-l_i). \qquad (*)$$

Further, either

$$(l_2-l_1)\,|\,(l_3-l_2)\,|\ldots|\,(l_s-l_{s-1})\,|\,(k-l_s)$$

or

$$m(v,L,k) \leqslant c(L,k)v^{s-1}$$

for a suitable constant $c(L,k)$.

The aim is to seek large (v,L,k)-families. Examples of such families are *perfect matroid designs*, PMD's for short. A *matroid*, or more exactly, the *hyperplane family of a matroid* is a family $\{H_j\}$ of subsets of X such that

(i) $H_1 \not\subset H_2$ if $H_1 \neq H_2$,

(ii) for any H_1, H_2 with $H_1 \neq H_2$ and x in $X\backslash(H_1 \cup H_2)$, there exists a unique subset H_3 with $(H_1 \cap H_2) \cup \{x\} \subset H_3$.

Subsets of X which are intersections of the sets H_j (*hyperplanes*) are *flats* of the matroid. Each subset Y of X has a well-defined *rank* and the rank r of X is the *rank* of the matroid. For any flat F of rank i and an element x in $X\backslash F$, there is a unique flat of rank $i+1$ containing $F \cup \{x\}$, providing $i < r$.

A PMD(v,L,k) is a matroid of rank $r = s+1$ such that all flats of rank i, with $0 \leqslant i \leqslant r$, have the same cardinality l_{i+1}. Here we also use the notation that $k = l_{s+1}$ and $v = l_{s+2}$. Without loss of generality, we may consider only *simple* PMD's, namely those with $l_1 = 0$, $l_2 = 1$.

Every known example of a PMD(v,L,k) belongs to one of the following four classes.

(1) $X = \mathrm{PG}(n,q)$ and $\{H_j\}$ is the set of all $(s-1)$-dimensional subspaces for a fixed s such that $1 < s \leqslant \frac{1}{2}n$; so $l_1 = 0$, $l_i = \theta_{i-2}$ for $2 \leqslant i \leqslant s+1$, and $l_{s+2} = v = \theta_n$.

(2) $X = \mathrm{AG}(n,q)$, affine space over GF(q), and $\{H_j\}$ is the set of all $(s-1)$-dimensional subspaces for a fixed s such that $1 < s \leqslant \frac{1}{2}n$; then $l_1 = 0$, $l_i = q^{i-2}$ for $2 \leqslant i \leqslant s+1$, and $l_{s+2} = v = q^n$.

(3) $X = S(t,k,v)$, a Steiner system, and $\{H_j\}$ is the set of blocks; so $l_i = i-1$ for $1 \leqslant i \leqslant t$, $l_{t+1} = l_{s+1} = k$, and $l_{t+2} = l_{s+2} = v$.

(4) $X = \mathrm{ATS}(m)$, an affine triple system and $\{H_j\}$ is the set of blocks. Then X is a PMD$(3^m, \{0,1,3\}, 9)$ of rank 4.

The examples of type (3) with $t = k$ are truncated Boolean algebras. Those of type (4) can be defined as Steiner systems $S(2,3,v)$ such that any triangle generates an affine plane AG$(2,3)$.

For further information on PMD's, see Deza and Singhi [4].

The hyperplane family of a PMD(v,L,k) is an $\mathscr{A}(v,L,k)$ with

$$|\mathscr{A}(v,L,k)| = \textstyle\prod_{l\in L}(v-l)/(k-l).$$

Deza [2] showed that, when $v \geqslant v_0(L,k)$, any family $\mathscr{A}(v,L,k)$ for which this equality holds is necessarily a PMD(v,L,k). The upper bound (*) can be regarded as

$$m(v,L,k) \leqslant c(L,k)v^s.$$

Frankl [6] showed that if there does not exist a PMD$(k, L\backslash\{l_s\}, l_s)$, then

$$m(v,L,k) \leqslant c'(L,k)v^{s-1}.$$

A family $\mathscr{A}(v, L, k) = \{A_i\}$ is *maximal* if there exists no k-subset B of X such that $|B \cap A_i| \in L$ for all A_i. Deza and Singhi [4] showed that a PMD(v, L, k) in one of the above four classes is maximal, and conjectured that the result holds for every PMD.

There is another general bound on $m(v, L, k)$ due to Frankl and Wilson [7].

THEOREM 4.2. *Suppose that p is a prime and $\mu_1, \mu_2, \ldots, \mu_r$ are integers such that*
(a) $0 \leqslant \mu_1 < \mu_2 < \ldots < \mu_r$,
(b) $l \equiv \mu_1$ *or* μ_2 *or* ... *or* $\mu_r \pmod p$, *for all l in L,*
(c) $k \not\equiv \mu_i \pmod p$, *for* $i = 1, 2, \ldots, r$.
Then

$$m(v, L, k) \leqslant \binom{v}{r}.$$

5. *Sections of quadrics and Hermitian varieties as intersection families*

Here we examine $\mathscr{Q}_n$ and $\mathscr{U}_n$ for families $\mathscr{A}(v, L, k)$, with $|L| = s$, where $m = |\mathscr{A}(v, L, k)|$ is as large as possible. From § 4, m cannot be of order greater than v^s; in other words, if $m \sim cv^{s'}$, then $s \geqslant s'$. So we look at families for which $s - s'$ is small.

For quadrics and Hermitian varieties $\mathscr{W}_n$, we define the families

$$\mathscr{F} = \mathscr{F}(\Pi_{d-t-1}\mathscr{W}_t, \mathscr{W}_n)$$

to consist of all $\Pi_d \cap \mathscr{W}_n$ projectively equivalent to $\Pi_{d-t-1}\mathscr{W}_t$. So the families considered are, for a sufficiently large fixed n,

$$\mathscr{F}(\Pi_{d-t-1}\mathscr{E}_t, \mathscr{Q}_n), \quad \mathscr{F}(\Pi_{d-t-1}\mathscr{P}_t, \mathscr{Q}_n), \quad \mathscr{F}(\Pi_{d-t-1}\mathscr{H}_t, \mathscr{Q}_n), \quad \mathscr{F}(\Pi_{d-t-1}\mathscr{U}_t, \mathscr{U}_n).$$

In the first three cases, $\mathscr{Q}_n$ can be elliptic, parabolic, or hyperbolic. A family $\mathscr{F}(\Pi_d, \mathscr{W}_n)$ is simply a subfamily of the PMD formed by all subspaces of PG(n, q) and these are not considered.

There follows a list of all families for which $s - s' = 0, 1, 2, 3, 4, 5$ in Tables 5 and 6. Table 5 is for $q > 2$ and Table 6 is for $q = 2$. The latter case must be considered separately because of Proposition 2.2 (b).

The parameters v and m are given asymptotically; exact values are in §§ 2 and 3. We recall the parameters of the family $\mathscr{F} = \mathscr{F}(\Pi_{d-t-1}\mathscr{W}_t, \mathscr{W}_n)$:

$|\mathscr{F}| = m$,
$|\mathscr{W}_n| = v$,
$|\Pi_{d-t-1}\mathscr{W}_t| = k$,
$\mathscr{L}$ is the set of projectively distinct $A_i \cap A_j$,
$L = \{|A| \mid A \in \mathscr{L}\}$,
$s = |L|$,
d is the dimension of the space containing an A_i,
t is the dimension of the space containing the non-singular part of A_i.

THEOREM 5.1. *The intersection families for which $D = s - s' \leqslant 5$ are exactly those of Tables 5 and 6. The corresponding k-sets are as follows.*
(a) $q > 2$.
$D = 0$: $\mathscr{P}_2, \mathscr{U}_2, \mathscr{E}_3$;
$D = 1$: $\Pi_0\mathscr{H}_1, \Pi_0\mathscr{U}_1, \mathscr{H}_3, \Pi_0\mathscr{P}_2$;
$D = 2$: $\Pi_1\mathscr{H}_1, \mathscr{U}_3, \Pi_0\mathscr{U}_2, \Pi_1\mathscr{U}_1, \Pi_0\mathscr{E}_3, \Pi_1\mathscr{P}_2$;
$D = 3$: $\mathscr{P}_4, \Pi_0\mathscr{H}_3, \Pi_2\mathscr{H}_1, \mathscr{U}_4, \Pi_2\mathscr{U}_1, \Pi_2\mathscr{P}_2$;
$D = 4$: $\mathscr{E}_5, \Pi_1\mathscr{E}_3, \Pi_3\mathscr{H}_1, \Pi_3\mathscr{P}_2, \Pi_1\mathscr{U}_2, \Pi_3\mathscr{U}_1$;
$D = 5$: $\mathscr{H}_5, \Pi_1\mathscr{H}_3, \Pi_4\mathscr{H}_1, \Pi_4\mathscr{P}_2, \Pi_0\mathscr{U}_3, \Pi_4\mathscr{U}_1$.

TABLE 5. $q > 2$

$s-s'$	d	t	$\Pi_{d-t-1}\mathscr{W}_t$	$\mathscr{L}$	L	k	s	v	m	$(v/k)^s/m$
0	2	2	$\mathscr{P}_2$	{∅, point, point pair}	$\{0,1,2\}$	$q+1$	3	q^{n-1}	q^{3n-6}	1
0	2	2	$\mathscr{U}_2$	{∅, point, subline}	$\{0,1,\sqrt{q}+1\}$	$q\sqrt{q}+1$	3	$q^{n-\frac12}$	q^{3n-6}	1
0	3	3	$\mathscr{E}_3$	{∅, point, point pair, conic}	$\{0,1,2,q+1\}$	q^2+1	4	q^{n-1}	q^{4n-12}	1
1	2	1	$\Pi_0\mathscr{H}_1$	{∅, point, point pair, line}	$\{0,1,2,q+1\}$	$2q+1$	4	q^{n-1}	q^{3n-7}	v
1	2	1	$\Pi_0\mathscr{U}_1$	{∅, point, subline, line}	$\{0,1,\sqrt{q}+1,q+1\}$	$q\sqrt{q}+q+1$	4	$q^{n-\frac12}$	$q^{3n-13/2}$	$q^{-1}v$
1	3	3	$\mathscr{H}_3$	{∅, point, point pair, line, conic, line pair}	$\{0,1,2,q+1,2q+1\}$	$(q+1)^2$	5	q^{n-1}	q^{4n-12}	$q^{-2}v$
1	3	2	$\Pi_0\mathscr{P}_2$	{∅, point, point pair, line, conic, line pair}	$\{0,1,2,q+1,2q+1\}$	q^2+q+1	5	q^{n-1}	q^{4n-13}	$q^{-1}v$
2	3	1	$\Pi_1\mathscr{H}_1$	{∅, point, point pair, line, line pair, plane}	$\{0,1,2,q+1,2q+1,q^2+q+1\}$	$2q^2+q+1$	6	q^{n-1}	q^{4n-15}	$q^{-1}v^2$
2	3	3	$\mathscr{U}_3$	{∅, point, subline, line, unital, concurrent lines}	$\{0,1,\sqrt{q}+1,q+1,q\sqrt{q}+1,q\sqrt{q}+q+1\}$	$(q\sqrt{q}+1)(q+1)$	6	$q^{n-\frac12}$	q^{4n-12}	$q^{-5}v^2$
2	3	2	$\Pi_0\mathscr{U}_2$	{∅, point, subline, line, unital, concurrent lines}	$\{0,1,\sqrt{q}+1,q+1,q\sqrt{q}+1,q\sqrt{q}+q+1\}$	$q^2\sqrt{q}+q+1$	6	$q^{n-\frac12}$	$q^{4n-25/2}$	$q^{-9/2}v^2$
2	3	1	$\Pi_1\mathscr{U}_1$	{∅, point, subline, line, concurrent lines, plane}	$\{0,1,\sqrt{q}+1,q+1,q\sqrt{q}+q+1,q^2+q+1\}$	$q^2\sqrt{q}+q^2+q+1$	6	$q^{n-\frac12}$	q^{4n-14}	$q^{-3}v^2$
2	4	3	$\Pi_0\mathscr{E}_3$	{∅, point, point pair, line, conic, line pair, ellipsoid, cone}	$\{0,1,2,q+1,2q+1,q^2+1,q^2+q+1\}$	q^3+q+1	7	q^{n-1}	q^{5n-21}	$q^{-5}v^2$
2	4	2	$\Pi_1\mathscr{P}_2$	{∅, point, point pair, line, conic, line pair, plane, cone, plane pair}	$\{0,1,2,q+1,2q+1,q^2+q+1,2q^2+q+1\}$	$(q^2+1)(q+1)$	7	q^{n-1}	q^{5n-23}	$q^{-3}v^2$
3	4	4	$\mathscr{P}_4$	{∅, point, point pair, line, conic, line pair, hyperboloid, ellipsoid, cone}	$\{0,1,2,q+1,2q+1,(q+1)^2,q^2+1,q^2+q+1\}$	$(q^2+1)(q+1)$	8	q^{n-1}	q^{5n-20}	$q^{-9}v^3$
3	4	3	$\Pi_0\mathscr{H}_3$	{∅, point, point pair, line, conic, line pair, hyperboloid, cone, plane, plane pair}	$\{0,1,2,q+1,2q+1,(q+1)^2,q^2+q+1,2q^2+q+1\}$	q^3+2q^2+q+1	8	q^{n-1}	q^{5n-21}	$q^{-8}v^3$

TABLE 5 (continued)

$s-s'$	d	t	$\Pi_{d-t-1}\mathscr{W}_t$	$\mathscr{L}$	L	k	s	v		$(v/k)^s/m$
3	4	1	$\Pi_2\mathscr{H}_1$	{∅, point, point pair, line, line pair, plane, plane pair, solid}	$\{0,1,2,q+1,$ $2q+1,q^2+q+1,$ $2q^2+q+1,$ $(q^2+1)(q+1)\}$	$2q^3+q^2+q+1$	8	q^{n-1}	q^{5n}	$q^{-3}v^3$
3	4	4	$\mathscr{U}_4$	{∅, point, subline, line, unital, concurrent lines, $\mathscr{U}_3, \Pi_0\mathscr{U}_2$}	$\{0,1,\sqrt{q}+1,$ $q+1,q\sqrt{q}+1,$ $q\sqrt{q}+q+1,$ $(q\sqrt{q}+1)(q+1),$ $q^2\sqrt{q}+q+1\}$	$(q^2\sqrt{q}+1)(q+1)$	8	$q^{n-\frac{1}{2}}$	q^{5n}	$q^{-21/2}v^3$
3	4	1	$\Pi_2\mathscr{U}_1$	{∅, point, subline, line, concurrent lines, plane, solid, collinear planes}	$\{0,1,\sqrt{q}+1,$ $q+1,q\sqrt{q}+q+1,$ $q^2+q+1,$ $(q^2+1)(q+1),$ $q^2\sqrt{q}+q^2+q+1\}$	$q^3\sqrt{q}+q^3+q^2+q+1$	8	$q^{n-\frac{1}{2}}$	q^{5n}	$q^{-6}v^3$
3	5	2	$\Pi_2\mathscr{P}_2$	{∅, point, point pair, line, conic, line pair, plane, cone, plane pair, solid, solid pair, $\Pi_1\mathscr{P}_2$}	$\{0,1,2,q+1,$ $2q+1,q^2+q+1,$ $2q^2+q+1,$ $q^3+q^2+q+1,$ $2q^3+q^2+q+1\}$	$q^4+q^3+q^2+q+1$	9	q^{n-1}	q^{6n}	$q^{-8}v^3$
4	4	2	$\Pi_1\mathscr{U}_2$	$\{\varnothing,\Pi_0,\mathscr{U}_1,\Pi_1,\mathscr{U}_2,\Pi_0\mathscr{U}_1,\Pi_2,$ $\Pi_0\mathscr{U}_2,\Pi_1\mathscr{U}_1\}$	$\{0,1,\sqrt{q}+1,\theta_1,q\sqrt{q}+1,$ $q\sqrt{q}+\theta_1,\theta_2,q^2\sqrt{q}+\theta_1,$ $q^2\sqrt{q}+\theta_2\}$	$q^3\sqrt{q}+\theta_2$	9	$q^{n-\frac{1}{2}}$	q^{5n}	$q^{-12}v^4$
4	5	1	$\Pi_3\mathscr{U}_1$	$\{\varnothing,\Pi_0,\mathscr{U}_1,\Pi_1,\Pi_0\mathscr{U}_1,\Pi_2,\Pi_3,$ $\Pi_1\mathscr{U}_1,\Pi_4,\Pi_2\mathscr{U}_1\}$	$\{0,1,\sqrt{q}+1,\theta_1,q\sqrt{q}+\theta_1,$ $\theta_2,q^2\sqrt{q}+\theta_2,\theta_3,$ $q^3\sqrt{q}+\theta_3,\theta_4\}$	$q^4\sqrt{q}+\theta_4$	10	$q^{n-\frac{1}{2}}$	q^{6n}	$q^{-10}v^4$
4	5	5	$\mathscr{E}_5$	$\{\varnothing,\Pi_0,\mathscr{H}_1,\Pi_1,\mathscr{P}_2,\Pi_0\mathscr{H}_1,\mathscr{H}_3,$ $\mathscr{E}_3,\Pi_0\mathscr{P}_2,\mathscr{P}_4,\Pi_0\mathscr{E}_3\}$	$\{0,1,2,\theta_1,2q+1,q^2+1,$ $\theta_2,(q+1)^2,q^3+q+1,\theta_3\}$	$(q+1)(q^3+1)$	10	q^{n-1}	q^{6n}	$q^{-16}v^4$
4	5	3	$\Pi_1\mathscr{E}_3$	$\{\varnothing,\Pi_0,\mathscr{H}_1,\Pi_1,\mathscr{P}_2,\Pi_0\mathscr{H}_1,\Pi_2,$ $\mathscr{E}_3,\Pi_0\mathscr{P}_2,\Pi_1\mathscr{H}_1,\Pi_0\mathscr{E}_3,\Pi_1\mathscr{P}_2\}$	$\{0,1,2,\theta_1,2q+1,q^2+1,$ $\theta_2,2q^2+q+1,q^3+q+1,\theta_3\}$	q^4+q^2+q+1	10	q^{n-1}	q^{6n}	$q^{-13}v^4$
4	5	1	$\Pi_3\mathscr{H}_1$	$\{\varnothing,\Pi_0,\mathscr{H}_1,\Pi_1,\Pi_0\mathscr{H}_1,\Pi_2,$ $\Pi_1\mathscr{H}_1,\Pi_3,\Pi_2\mathscr{H}_1,\Pi_4\}$	$\{0,1,2,\theta_1,2q+1,\theta_2,$ $2q^2+q+1,\theta_3,2q^3+\theta_2,\theta_4\}$	$2q^4+\theta_3$	10	q^{n-1}	q^{6n}	$q^{-6}v^4$

4	6	2	$\Pi_3\mathscr{P}_2$	$\{\varnothing, \Pi_0, \mathscr{H}_1, \Pi_1, \mathscr{P}_2, \Pi_0\mathscr{H}_1, \Pi_2, \Pi_1\mathscr{H}_1, \Pi_0\mathscr{P}_2, \Pi_3, \Pi_2\mathscr{H}_1, \Pi_4, \Pi_3\mathscr{H}_1, \Pi_2\mathscr{P}_2\}$	$\{0, 1, 2, \theta_1, 2q+1, 2q^2+\theta_1, \theta_3, 2q^3+\theta_2, \theta_4, 2q^4+\theta_3\}$	θ_5	11	q^{n-1}	q^{7n-52}	$q^{-10}v^4$
5	4	3	$\Pi_0\mathscr{U}_3$	$\{\varnothing, \Pi_0, \mathscr{U}_1, \Pi_1, \mathscr{U}_2, \Pi_0\mathscr{U}_1, \Pi_2, \mathscr{U}_3, \Pi_0\mathscr{U}_2, \Pi_1\mathscr{U}_1\}$	$\{0, 1, \sqrt{q}+1, \theta_1, q\sqrt{q}+1, q\sqrt{q}+\theta_1, \theta_2, (q+1)(q\sqrt{q}+1), q^2\sqrt{q}+\theta_1, q^2\sqrt{q}+\theta_2\}$	$q^2\sqrt{q}\theta_1+\theta_2$	10	$q^{n-\frac{1}{2}}$	q^{5n-32}	$q^{-3}v^5$
5	6	1	$\Pi_4\mathscr{U}_1$	$\{\varnothing, \Pi_0, \mathscr{U}_1, \Pi_1, \Pi_0\mathscr{U}_1, \Pi_2, \Pi_1\mathscr{U}_1, \Pi_3, \Pi_2\mathscr{U}_1, \Pi_4, \Pi_3\mathscr{U}_1, \Pi_5\}$	$\{0, 1, \sqrt{q}+1, \theta_1, q\sqrt{q}+\theta_1, \theta_2, q^2\sqrt{q}+\theta_2, \theta_3, q^3\sqrt{q}+\theta_3, \theta_4, q^4\sqrt{q}+\theta_4, \theta_5\}$	$q^5\sqrt{q}+\theta_5$	12	$q^{n-\frac{1}{2}}$	$q^{7n-109/2}$	$q^{-15}v^5$
5	5	5	$\mathscr{H}_5$	$\{\varnothing, \Pi_0, \mathscr{H}_1, \Pi_1, \mathscr{P}_2, \Pi_0\mathscr{H}_1, \Pi_2, \mathscr{H}_3, \mathscr{E}_3, \Pi_0\mathscr{P}_2, \Pi_1\mathscr{H}_1, \mathscr{P}_4, \Pi_0\mathscr{H}_3\}$	$\{0, 1, 2, \theta_1, 2q+1, \theta_2, (q+1)^2, q^2+1, 2q^2+q+1, \theta_3, q^3+2q^2+q+1\}$	$(q^2+1)\theta_2$	11	q^{n-1}	q^{6n-30}	$q^{-20}v^5$
5	5	3	$\Pi_1\mathscr{H}_3$	$\{\varnothing, \Pi_0, \mathscr{H}_1, \Pi_1, \mathscr{P}_2, \Pi_0\mathscr{H}_1, \Pi_2, \mathscr{H}_3, \Pi_0\mathscr{P}_2, \Pi_1\mathscr{H}_1, \Pi_0\mathscr{H}_3, \Pi_1\mathscr{P}_2, \Pi_3, \Pi_2\mathscr{H}_1\}$	$\{0, 1, 2, \theta_1, 2q+1, \theta_2, (q+1)^2, 2q+\theta_1, q^3+2q^2+q+1, \theta_3, 2q^3+\theta_2\}$	θ_4+q^3	11	q^{n-1}	q^{6n-33}	$q^{-17}v^5$
5	6	1	$\Pi_4\mathscr{H}_1$	$\{\varnothing, \Pi_0, \mathscr{H}_1, \Pi_1, \Pi_0\mathscr{H}_1, \Pi_2, \Pi_1\mathscr{H}_1, \Pi_3, \Pi_2\mathscr{H}_1, \Pi_4, \Pi_3\mathscr{H}_1, \Pi_5\}$	$\{0, 1, 2, \theta_1, 2q+1, \theta_2, 2q^2+\theta_1, \theta_3, 2q^3+\theta_2, \theta_4, 2q^4+\theta_3, \theta_5\}$	$2q^5+\theta_4$	12	q^{n-1}	q^{7n-57}	$q^{-10}v^5$
5	7	2	$\Pi_4\mathscr{P}_2$	$\{\varnothing, \Pi_0, \mathscr{H}_1, \Pi_1, \mathscr{P}_2, \Pi_0\mathscr{H}_1, \Pi_2, \Pi_1\mathscr{H}_1, \Pi_0\mathscr{P}_2, \Pi_3, \Pi_2\mathscr{H}_1, \Pi_4, \Pi_3\mathscr{H}_1, \Pi_2\mathscr{P}_2, \Pi_5, \Pi_4\mathscr{H}_1, \Pi_3\mathscr{P}_2\}$	$\{0, 1, 2, \theta_1, 2q+1, \theta_2, 2q^2+\theta_1, \theta_3, 2q^3+\theta_2, \theta_4, 2q^4+\theta_3, \theta_5, 2q^5+\theta_4\}$	θ_6	13	q^{n-1}	q^{8n-71}	$q^{-15}v^5$

TABLE 6. $q = 2$

$s-s'$	d	t	$\Pi_{d-t-1}\mathscr{W}_t$	L	k	s	v	m
0	2	2	$\mathscr{P}_2$	$\{0,1,2\}$	3	3	2^{n-1}	2^{3n-6}
0	3	3	$\mathscr{E}_3$	$\{0,1,2,3\}$	5	4	2^{n-1}	2^{4n-12}
1	2	1	$\Pi_0\mathscr{H}_1$	$\{0,1,2,3\}$	5	4	2^{n-1}	2^{3n-7}
1	3	3	$\mathscr{H}_3$	$\{0,1,2,3,5\}$	9	5	2^{n-1}	2^{4n-12}
1	3	2	$\Pi_0\mathscr{P}_2$	$\{0,1,2,3,5\}$	7	5	2^{n-1}	2^{4n-13}
1	4	3	$\Pi_0\mathscr{E}_3$	$\{0,1,2,3,5,7\}$	11	6	2^{n-1}	2^{5n-21}
2	3	1	$\Pi_1\mathscr{H}_1$	$\{0,1,2,3,5,7\}$	11	6	2^{n-1}	2^{4n-15}
2	4	2	$\Pi_1\mathscr{P}_2$	$\{0,1,2,3,5,7,11\}$	15	7	2^{n-1}	2^{5n-23}
2	4	4	$\mathscr{P}_4$	$\{0,1,2,3,5,7,9\}$	15	7	2^{n-1}	2^{5n-20}
2	5	3	$\Pi_1\mathscr{E}_3$	$\{0,1,2,3,5,7,11,15\}$	23	8	2^{n-1}	2^{6n-33}
3	4	1	$\Pi_2\mathscr{H}_1$	$\{0,1,2,3,5,7,11,15\}$	23	8	2^{n-1}	2^{5n-26}
3	4	3	$\Pi_0\mathscr{H}_3$	$\{0,1,2,3,5,7,9,11\}$	19	8	2^{n-1}	2^{5n-21}
3	5	2	$\Pi_2\mathscr{P}_2$	$\{0,1,2,3,5,7,11,15,23\}$	31	9	2^{n-1}	2^{6n-36}
3	5	5	$\mathscr{E}_5$	$\{0,1,2,3,5,7,9,11,15\}$	27	9	2^{n-1}	2^{6n-30}
3	6	3	$\Pi_2\mathscr{E}_3$	$\{0,1,2,3,5,7,11,15,23,31\}$	47	10	2^{n-1}	2^{7n-48}
4	5	5	$\mathscr{H}_5$	$\{0,1,2,3,5,7,9,11,15,19\}$	35	10	2^{n-1}	2^{6n-30}
4	5	1	$\Pi_3\mathscr{H}_1$	$\{0,1,2,3,5,7,11,15,23,31\}$	47	10	2^{n-1}	2^{6n-40}
4	5	4	$\Pi_0\mathscr{P}_4$	$\{0,1,2,3,5,7,9,11,15,19\}$	31	10	2^{n-1}	2^{6n-31}
4	6	2	$\Pi_3\mathscr{P}_2$	$\{0,1,2,3,5,7,11,15,23,31,47\}$	63	11	2^{n-1}	2^{7n-52}
4	7	3	$\Pi_3\mathscr{E}_3$	$\{0,1,2,3,5,7,11,15,23,31,47,63\}$	95	12	2^{n-1}	2^{8n-66}
5	5	3	$\Pi_1\mathscr{H}_3$	$\{0,1,2,3,5,7,9,11,15,19,23\}$	39	11	2^{n-1}	2^{6n-33}
5	6	1	$\Pi_4\mathscr{H}_1$	$\{0,1,2,3,5,7,11,15,23,31,47,63\}$	95	12	2^{n-1}	2^{7n-57}
5	7	2	$\Pi_4\mathscr{P}_2$	$\{0,1,2,3,5,7,11,15,23,31,47,63,95\}$	127	13	2^{n-1}	2^{8n-71}
5	8	3	$\Pi_4\mathscr{E}_3$	$\{0,1,2,3,5,7,11,15,23,31,47,63,95,127\}$	191	14	2^{n-1}	2^{9n-87}

(b) $q = 2$.

$D = 0$: $\mathscr{P}_2, \mathscr{E}_3$;
$D = 1$: $\Pi_0\mathscr{H}_1, \mathscr{H}_3, \Pi_0\mathscr{P}_2, \Pi_0\mathscr{E}_3$;
$D = 2$: $\Pi_1\mathscr{H}_1, \Pi_1\mathscr{P}_2, \mathscr{P}_4, \Pi_1\mathscr{E}_3$;
$D = 3$: $\Pi_2\mathscr{H}_1, \Pi_0\mathscr{H}_3, \Pi_2\mathscr{P}_2, \mathscr{E}_5, \Pi_2\mathscr{E}_3$;
$D = 4$: $\mathscr{H}_5, \Pi_3\mathscr{H}_1, \Pi_0\mathscr{P}_4, \Pi_3\mathscr{P}_2, \Pi_3\mathscr{E}_3$;
$D = 5$: $\Pi_1\mathscr{H}_3, \Pi_4\mathscr{H}_1, \Pi_4\mathscr{P}_2, \Pi_4\mathscr{E}_3$.

Proof. The necessary facts are contained in Tables 7 and 9.

The first three rows of Table 7 give the number of projectively distinct hyperbolic, elliptic, and parabolic sections of each type of non-singular quadric; these numbers

TABLE 7. Sections of quadrics

		$\mathcal{H}_n$	$\mathcal{E}_n$	$\mathcal{P}_n$
(1)	Number of hyperbolic sections	$\frac{1}{8}(n^2-1)+n$	$\frac{1}{8}(n-1)(n+5)$	$\frac{1}{8}n(n+6)$
(2)	Number of elliptic sections	$\frac{1}{8}(n^2-1)$	$\frac{1}{8}(n-1)(n+5)$	$\frac{1}{8}n(n+2)$
(3)	Number of parabolic sections	$\frac{1}{8}(n+1)(n+3)$	$\frac{1}{8}(n+1)(n+3)$	$\frac{1}{8}n(n+6)$
(4)	Number of elliptic sections of different order to a hyperbolic or parabolic section			
	(a) $q>2$	$\frac{1}{8}(n-1)(n-3)$	$\frac{1}{8}(n^2-9)$	$\frac{1}{8}n(n-2)$
	(b) $q=2$	$\begin{cases}0, n=1\\ \frac{1}{8}(n-3)(n-5), n>\end{cases}$	$\frac{1}{8}(n+1)(n-5)$	$\frac{1}{8}(n-2)(n-4)$
(5)	Number of parabolic sections of different order to a hyperbolic section	$\begin{cases}1, n=1\\ \frac{1}{2}(n-1), n>1\end{cases}$	$\frac{1}{2}(n+1)$	$\frac{1}{2}n$
(6)	Total number of distinct sections ((1)+(2)+(3))	$\frac{1}{8}(3n+1)(n+1)+n$	$\frac{1}{8}(3n+7)(n-1)+n$	$\frac{1}{8}n(3n+14)$
(7)	Total number of distinct cardinalities of sections			
	(a) $q>2$: (1)+(4)(a)+(5)	$\begin{cases}2, n=1\\ \frac{1}{4}(n^2-1)+n, n>1\end{cases}$	$\frac{1}{4}(n+5)(n-1)$	$\frac{1}{4}n(n+4)$
	(b) $q=2$: (1)+(4)(b)+(5)	$\begin{cases}2, n=1\\ \frac{1}{4}(n+1)^2+1\end{cases}$	$\frac{1}{4}(n+1)^2$	$\frac{1}{4}n(n+2)+1$

TABLE 7 (continued)

	$\Pi_e\mathscr{H}_t$	$\Pi_e\mathscr{E}_t$	$\Pi_e\mathscr{P}_t$
(8) Total number of distinct cardinalities of sections, s			
(a) $q > 2$	$\begin{cases} 2e+4, t=1 \\ et+\frac{1}{4}(t-1)(t+9)+2, t>1 \end{cases}$	$et+\frac{1}{4}(t-1)(t+9)+1$	$et+\frac{1}{4}t(t+8)$
(b) $q = 2$	$\begin{cases} 2e+4, t=1 \\ 3e+9, t=3 \\ e(t-1)+\frac{1}{4}(t+1)(t+5)-1, t>3 \end{cases}$	$e(t-1)+\frac{1}{4}(t+1)(t+5)-2$	$2e+5, t=2$ $e(t-1)+\frac{1}{4}t(t+6), t>2$
(9) Dimension of family, s'	$e+t+2$	$e+t+2$	$e+t+2$
(10) Defect of family, $D = s - s'$			
(a) $q > 2$	$\begin{cases} e+1, t=1 \\ e(t-1)+\frac{1}{4}(t-1)(t+5)-1, t>1 \end{cases}$	$e(t-1)+\frac{1}{4}(t-1)(t+5)-2$	$e(t-1)+\frac{1}{4}t(t+4)-2$
(b) $q = 2$	$\begin{cases} e+1, t=1 \\ 2e+4, t=3 \\ e(t-2)+\frac{1}{4}(t-1)(t+3)-1, t>3 \end{cases}$	$e(t-2)+\frac{1}{4}(t-1)(t+3)-2$	$\begin{cases} e+1, t=2 \\ e(t-2)+\frac{1}{4}t(t+2)-2, t>2 \end{cases}$

come from [10]. To calculate s, we need to consider Proposition 2.2, which explains when the cardinalities of projectively distinct sections coincide. For $q > 2$, we throw away all sections $\Pi_i\mathscr{E}_1$ and count one parabolic section for each dimension two larger than the projective index, as well as the empty section $\mathscr{P}_0$. For $q = 2$, it is necessary also to discard sections $\Pi_i\mathscr{E}_3$ where a section $\Pi_{i+1}\mathscr{H}_1$ is present. Hence row (7) of the table gives s for the quadric $\mathscr{Q}_n$.

To find s for an arbitrary quadric $\Pi_{d-t-1}\mathscr{Q}_t = \Pi_e\mathscr{Q}_t$, we consider the sections of $\Pi_{e+1}\mathscr{Q}_t$ that are not sections of $\Pi_e\mathscr{Q}_t$. If, for a fixed s, the quadric $\Pi_e\mathscr{Q}_t$ has a section $\Pi_r\mathscr{Q}_s$ with r a maximum and the character of $\mathscr{Q}_s$ determined, then $\Pi_{e+1}\mathscr{Q}_t$ has a section $\Pi_{r+1}\mathscr{Q}_s$ of the same character with $r+1$ the maximum value possible.

For the quadric $\Pi_e\mathscr{Q}_t$, denote s by $s_{e,t}$. Then an easy count gives Table 8.

TABLE 8

$\mathscr{Q}_t$	$s_{e+1,t}-s_{e,t}$ $q>2$	$q=2$
$\mathscr{H}_t$	$2, t=1$	$2, t=1$
	$t, t>1$	$3, t=3$
		$t-1, t>3$
$\mathscr{E}_t$	t	$t-1$
$\mathscr{P}_t$	t	$2, t=2$
		$t-1, t>2$

Row (7) of Table 7 gives the numbers $s_{-1,n}$. Hence $s_{e,t}$ may be calculated as in row (8). Finally, $s' = e+t+2$, and $D = s-s'$ is given in row (10).

Table 9 gives a similar analysis for $\Pi_e\mathscr{U}_t$. Here, if we write $\mu_{e,t}$ for the value of s for a Hermitian variety $\Pi_e\mathscr{U}_t$, then $\mu_{e+1,t}-\mu_{e,t} = t+1$.

TABLE 9. Sections of Hermitian varieties

	$\mathscr{U}_n$ n odd	n even
(1) Number of distinct sections	$\frac{1}{4}(n+1)(n+5)-1$	$\frac{1}{4}n(n+6)$
(2) Number of distinct cardinalities of sections	$\frac{1}{4}(n+1)(n+3)$	$\frac{1}{4}n(n+4)$
	$\Pi_e\mathscr{U}_t$ t odd	t even
(3) Number of distinct cardinalities of sections, s	$e(t+1)+\frac{1}{4}(t+1)(t+7)$	$e(t+1)+\frac{1}{4}t(t+8)+1$
(4) Dimension of family, s'	$e+t+2$	$e+t+2$
(5) $D = s-s'$	$et-1+\frac{1}{4}(t+1)(t+3)$	$et-1+\frac{1}{4}t(t+4)$

6. *More upper bounds on the size of intersection families*

The aim of this section is to present some general methods for bounding $m(v, L, k)$. These methods can be applied to show that all families in Table 5 of defect $s-s' \leqslant 3$ with $q > 2$ have the greatest possible order of magnitude. For $q = 2$, there are four exceptions, namely the families with sections $\Pi_i\mathscr{H}_1$ and $\Pi_0\mathscr{H}_3$. We recall that $f(v)$ is of *order* v^α if there exist positive constants c and d such that $cv^\alpha \leqslant f(v) \leqslant dv^\alpha$ for all $v > v_0$.

Suppose that $\mathscr{F} = \{F_1, ..., F_m\}$ is a (v, L, k)-system. For F in $\mathscr{F}$, the *trace of* $\mathscr{F}$ *on* F is

$$\mathscr{T}_{\mathscr{F}}(F) = \{F \cap F' \mid F' \in \mathscr{F},\ F' \neq F\}.$$

Fundamental for our investigations is the following theorem, which was conjectured by Frankl and proved by Füredi [8].

THEOREM 6.1. *There exists a positive constant c_k such that any (v, L, k)-system $\mathscr{F}$ has a subsystem $\mathscr{F}^* \subset \mathscr{F}$ satisfying the following conditions:*

(i) $|\mathscr{F}^*| \geqslant c_k |\mathscr{F}|$;

(ii) *the families $\mathscr{T}_{\mathscr{F}^*}(F)$ are all isomorphic for F in $\mathscr{F}^*$;*

(iii) *$\mathscr{T}_{\mathscr{F}^*}(F)$ is closed under intersection, that is*

$$T_1, T_2 \in \mathscr{T}_{\mathscr{F}^*}(F) \Rightarrow T_1 \cap T_2 \in \mathscr{T}_{\mathscr{F}^*}(F);$$

(iv) $|T| \in L$ *for all T in $\mathscr{T}_{\mathscr{F}^*}(F)$.*

From (i), we have $c_k \leqslant |\mathscr{F}^*|/|\mathscr{F}| \leqslant 1$; so $|\mathscr{F}^*|$ and $|\mathscr{F}|$ have the same order as functions of v. Therefore, if we are only interested in the order of magnitude of $m(v, L, k)$, we may replace $\mathscr{F}$ by $\mathscr{F}^*$. Thus, in this section, we now assume that $\mathscr{F} = \mathscr{F}^*$ and write $\mathscr{T}(F)$ instead of $\mathscr{T}_{\mathscr{F}^*}(F)$.

DEFINITION 6.2. A set $G \subseteq F$ is *free* with respect to $\mathscr{T}(F)$ if there is no T in $\mathscr{T}(F)$ with $G \subseteq T$.

Note that F itself is always free.

PROPOSITION 6.3. *If F in $\mathscr{F}$ has a free subset G of size l, then*

$$|\mathscr{F}| \leqslant \binom{v}{l}.$$

Proof. Since, for F and F' in $\mathscr{F}$, the sets $\mathscr{T}(F)$ and $\mathscr{T}(F')$ are isomorphic, all F' in $\mathscr{F}$ have a free subset $G(F')$. By the definition of free subset and of $\mathscr{T}(F)$, we have $G(F') \neq G(F'')$ for F', F'' in $\mathscr{F}$. Thus

$$|\mathscr{F}| \leqslant \left|\binom{X}{l}\right| = \binom{v}{l}.$$

Our method for proving upper bounds on $|\mathscr{F}|$ will consist of establishing that, for every $\mathscr{T} \subset 2^{\{1,2,...,k\}}$ satisfying (iii) and (iv) of Theorem 6.1, there exists a relatively small free subset.

Let us introduce the notation

$$a(k, L) = \max\{\min\{|G|: G \subset \{1,2,...,k\},\ G \text{ is free with respect to } \mathscr{T}\}: \mathscr{T} \subset 2^{\{1,2,...,k\}}, \mathscr{T} \text{ satisfies (iii), (iv)}\}.$$

Theorem 6.1 and Proposition 6.3 imply the following.

PROPOSITION 6.4.

$$m(v,L,k) \leqslant c_k^{-1}\binom{v}{a(k,L)} = O(v^{a(k,L)}).$$

The good thing about $a(k,L)$ is that it is independent of v; it can also be bounded by looking at subsets of $\{1,2,...,k\}$ only. The bad thing about it is that it cannot be calculated easily; for example, $a(111,\{0,1,11\}) = 3$ if and only if a projective plane of order 10 exists.

However, we do have the following inequality.

PROPOSITION 6.5. *For each i with $1 \leqslant i \leqslant s$,*

$$a(k,L) \leqslant \max\{a(k,L\backslash\{l_i\}), a(l_i,\{l_1,l_2,...,l_{i-1}\}) + a(k-l_i,\{0,l_{i+1}-l_i,...,l_s-l_i\})\}.$$

Proof. Let $\mathcal{T} \subset 2^{\{1,...,k\}}$ be a family for which the value of $a(k,L)$ as $\min|G|$ is realized. If $|T| \neq l_i$ for all T in $\mathcal{T}$, then by definition there exists a free subset of size $a(k,L\backslash\{l_i\})$ with respect to $\mathcal{T}$; thus $a(k,L) \leqslant a(k,L\backslash\{l_i\})$.

Hence we may assume that $|T_0| = l_i$ for some T_0 in $\mathcal{T}$. Define

$$\mathcal{T}_0 = \{T \in \mathcal{T} \mid T \subset T_0\} \quad\text{and}\quad \mathcal{T}_1 = \{T\backslash T_0 \mid T_0 \subseteq T \in \mathcal{T}\}.$$

Clearly $\mathcal{T}_0$ and $\mathcal{T}_1$ satisfy (iii) and (iv) with $L_0 = \{l_1,...,l_{i-1}\}$ and

$$L_1 = \{0,l_{i+1}-l_i,...,l_s-l_i\}$$

respectively. Thus we may choose free subsets G_0, G_1 such that

$$|G_0| = a(l_i,\{l_1,...,l_{i-1}\}), \quad |G_1| = a(k-l_i,\{0,l_{i+1}-l_i,...,l_s-l_i\}),$$

where $G_0 \subset T_0$ and $G_1 \subseteq (\{1,...,k\}\backslash T_0)$.

It is sufficient to show that $G_0 \cup G_1$ is a free subset with respect to $\mathcal{T}$.

Suppose that $(G_0 \cup G_1) \subseteq T \in \mathcal{T}$. Since $G_0 \subseteq T_0$, we have $G_0 \subseteq (T_0 \cap T)$. As $T_0 \cap T \in \mathcal{T}$ and G_0 is free with respect to $\mathcal{T}_0$, we have $T_0 \cap T = T_0$; that is, $T_0 \subseteq T$. Also $G_1 \subset T\backslash T_0$ and G_1 is free with respect to $\mathcal{T}_1$. Thus $T\backslash T_0 = \{1,2,...,k\}\backslash T_0$. Consequently, $T = \{1,2,...,k\}$; that is, $G_0 \cup G_1$ is free.

A more indirect way of bounding $a(k,L)$ is provided by the following.

PROPOSITION 6.6. *If $a(k,L) \geqslant b$, where b is a non-negative integer, then there exists a family*

$$\mathcal{B} = \{B_1,B_2,...,B_b\} \subset 2^{\{1,2,...,k-b\}}$$

such that, for every $1 \leqslant j \leqslant b$ and $1 \leqslant i_1 < i_2 < ... < i_j \leqslant b$,

$$(|B_{i_1} \cap ... \cap B_{i_j}| + b - j) \in L.$$

Proof. Let $\mathcal{T} \subset 2^{\{1,...,k\}}$ be a family for which the value of $a(k,L)$ as $\min|G|$ is realized. Let G be a minimal free subset for $\mathcal{T}$. Then $|G| \geqslant b$. By symmetry, assume that $F = \{1,2,...,k\}$ and $\{1,2,...,b\} \subseteq G$. By the minimal choice of G, for $1 \leqslant i \leqslant b$ there exists T_i in $\mathcal{T}$ with $G\backslash\{i\} \subset T_i$. Define $B_i = T_i \cap \{b+1,...,k\}$.

Since $\mathcal{T}$ satisfies (iii) and (iv), for $1 \leqslant j \leqslant b$ and $1 \leqslant i_1 < i_2 < ... < i_j \leqslant b$, we have that $|T_{i_1} \cap ... \cap T_{i_j}| = |B_{1_1} \cap ... \cap B_{i_j}| + b - j \in L$.

Now let us turn to the specific values of L given in Tables 5 and 6. First note that

$$m(v,\{0,1,2,3,5,7,9,11\},19)\geqslant m(v,\{1,3,5,7,9,11\},19).$$

Now,

$$m(v,\{1,3,5,7,9,11\},19)\geqslant m(v-1,\{0,2,4,6,8,10\},18),$$

since we may just add a point to all sets in a family realizing the right-hand side. Also

$$m(v-1,\{0,2,4,6,8,10\},18)\geqslant m(\tfrac{1}{2}(v-1),\{0,1,2,3,4,5\},9),$$

since we may double any point in a family realizing this right-hand side. For large v,

$$m(\tfrac{1}{2}(v-1),\{0,1,2,3,4,5\},9)\geqslant cv^6,$$

where c is a positive constant; see [**6**]. Hence,

$$m(v,\{0,1,2,3,5,7,9,11\},19)\geqslant cv^6,$$

where c is a positive constant.

Thus, for $q=2$, the section $\Pi_0\mathscr{H}_3$ does not give a family of best-possible order of magnitude, *optimal* for short. The same holds for $\Pi_i\mathscr{H}_1$ with $i\geqslant 0$, since the corresponding values of k and L are the same as for $\Pi_{i-1}\mathscr{E}_3$. However, the latter family has greater order of magnitude; compare Proposition 2.2.

In view of Theorem 4.1, all families with $s-s'=0$ are optimal. Also those for which $s-s'=1$ and the divisibility condition in Theorem 4.1 is not satisfied are optimal. This covers almost all cases with $s-s'=1$ and $q\neq 2$. The only exception is the case where $q=4$, $k=13$, $L=\{0,1,3,5\}$: the section is $\Pi_0\mathscr{U}_1$. Applying Proposition 6.5 with $l_4=5$ gives

$$\begin{aligned}a(k,L)&\leqslant \max\{a(13,\{0,1,3\}),a(5,\{0,1,3\})+a(8,\{0\})\}\\&=\max\{3,a(5,\{0,1,3\})+1\}.\end{aligned}$$

Thus it is sufficient to have $a(5,\{0,1,3\})=2$, which can be checked directly.

All the cases for $q=2$ with $s-s'=1$ and for $q>2$ with $s-s'=2$ or 3 can be handled in a similar way, that is by repeated application of Proposition 6.5. Therefore we pick out only one case which illustrates the general procedure. We show that the family with section $\Pi_1\mathscr{H}_1$ is optimal for $q>2$. We have

$$\begin{aligned}&a(2q^2+q+1,\{0,1,2,q+1,2q+1,q^2+q+1\})\\&\quad\leqslant \max\{a(2q^2+q+1,\{0,1,2,q+1,q^2+q+1\}),a(2q+1,\{0,1,2,q+1\})\\&\qquad\qquad +a(2q^2-q,\{0,q^2-q\})\}\\&\quad=\max\{a(2q^2+q+1,\{0,1,2,q+1,q^2+q+1\}),3+1\}\\&\quad\leqslant \max\{a(2q^2+q+1,\{0,1,2,q+1\}),a(q^2+q+1,\{0,1,2,q+1\})\\&\qquad\qquad +a(q^2,\{0\}),4\}\\&\quad=\max\{a(2q^2+q+1,\{0,1,2,q+1\}),3+1,4\}\\&\quad=4.\end{aligned}$$

Use has also been made of Theorem 4.1: if the divisibility condition is not satisfied, then $a(k,L)\leqslant s-1$.

The remaining cases can be solved by applying Proposition 6.6. As an example, we take the most complicated case, that with section $\Pi_2\mathscr{E}_3$.

Suppose that, on the contrary, $m(v,L,k) \not\leqslant O(v^7)$. Then, in view of Proposition 6.4, Proposition 6.6 implies the existence of eight sets $B_1,\dots,B_8 \subset \{1,2,\dots,39\}$ such that

(i) $|B_i| \in \{0,4,8,16,24\}$, for $1 \leqslant i \leqslant 8$;
(ii) $|B_{i_1} \cap B_{i_2}| \in \{1,5,9,17,25\}$, for $1 \leqslant i_1 < i_2 \leqslant 8$;
(iii) $|B_{i_1} \cap B_{i_2} \cap B_{i_3}| \in \{0,2,6,10,18,26\}$, for $1 \leqslant i_1 < i_2 < i_3 \leqslant 8$;
(iv) $|B_{i_1} \cap B_{i_2} \cap B_{i_3} \cap B_{i_4}| \in \{1,3,7,11,19,27\}$, for $1 \leqslant i_1 < i_2 < i_3 < i_4 \leqslant 8$;
(v) $|B_{i_1} \cap \dots \cap B_{i_5}| \in \{0,2,4,8,12,20,28\}$, for $1 \leqslant i_1 < \dots < i_5 \leqslant 8$.

Since the 4-wise intersections are non-empty, the 3-wise intersections are also non-empty. Thus we may leave out 0 from the possible sizes in (iii). Hence $|B_{i_1} \cap B_{i_2} \cap B_{i_3}| \geqslant 2$ and we deduce that $|B_{i_1} \cap B_{i_2}| \geqslant 5$ and $|B_i| \geqslant 8$ in the same way. Similarly, $|B_i| \leqslant 24$ implies $|B_{i_1} \cap B_{i_2}| \leqslant 24$ and thus $|B_{i_1} \cap B_{i_2}| \leqslant 17$. This in its turn yields $|B_{i_1} \cap B_{i_2} \cap B_{i_3}| \leqslant 10$ and $|B_{i_1} \cap B_{i_2} \cap B_{i_3} \cap B_{i_4}| \leqslant 7$. Let us rewrite the conditions:

(i) $|B_i| \in \{8,16,24\}$;
(ii) $|B_{i_1} \cap B_{i_2}| \in \{5,9,17\}$;
(iii) $|B_{i_1} \cap B_{i_2} \cap B_{i_3}| \in \{2,6,10\}$;
(iv) $|B_{i_1} \cap B_{i_2} \cap B_{i_3} \cap B_{i_4}| \in \{1,3,7\}$;
(v) $|B_{i_1} \cap B_{i_2} \cap B_{i_3} \cap B_{i_4} \cap B_{i_5}| \in \{0,2,4\}$.

Suppose first that for some $1 \leqslant i_1 < i_2 < i_3 \leqslant 8$, we have $|B_{i_1} \cap B_{i_2} \cap B_{i_3}| = 2$; assume by symmetry that $|B_6 \cap B_7 \cap B_8| = 2$. Set $A_i = B_i \cap B_6 \cap B_7 \cap B_8$ for $i = 1,2,3,4,5$. In view of (iv), $|A_i| = 1$ and, in view of (v), $A_i \cap A_j = \varnothing$ for $1 \leqslant i \neq j \leqslant 5$. However this is impossible as $2 < 5$. Thus $|B_{i_1} \cap B_{i_2} \cap B_{i_3}| \geqslant 6$. Consequently, $|B_{i_1} \cap B_{i_2}| \geqslant 9$ and $|B_i| \geqslant 16$.

Suppose that, for some $1 \leqslant i_1 < i_2 \leqslant 8$, we have $|B_{i_1} \cap B_{i_2}| = 9$. Assume that $|B_7 \cap B_8| = 9$ and define $D_i = B_i \cap B_7 \cap B_8$ for $1 \leqslant i \leqslant 6$. Thus $|D_i| = 6$, and we deduce that $|D_{i_1} \cap D_{i_2}| = 3$, for $1 \leqslant i_1 < i_2 \leqslant 6$. However, one cannot take more than three 6-element subsets of a 9-set with pairwise intersections exactly 3, a contradiction.

We are left with the case that $|B_{i_1} \cap B_{i_2}| = 17$ for $1 \leqslant i_1 < i_2 \leqslant 8$. Consequently, $|B_i| = 24$ for $1 \leqslant i \leqslant 8$. Define $C_i = B_i \cap B_8$ and $\bar{C}_i = B_8 \backslash C_i$, for $1 \leqslant i \leqslant 7$. Then $|C_i| = 17$ and $|\bar{C}_i| = 7$. For $1 \leqslant i < j \leqslant 7$ we have

$$|C_i \cap C_j| = |C_i| + |C_j| - |C_i \cup C_j| \geqslant |C_i| + |C_j| - |B_8| = 10.$$

Thus (iii) yields $|C_i \cap C_j| = 10$ or equivalently $\bar{C}_i \cap \bar{C}_j = \varnothing$ But this is impossible as there is room only for three pairwise disjoint 7-sets in a 24-element set, establishing the final contradiction. Thus the proof that $a(47,\{0,1,2,3,5,7,11,15,23,31\}) \leqslant 7$ is complete.

7. *Maximal intersection families*

We consider which of the families $\mathscr{F}$ of §5 are maximal in the sense of §4; that is, can we add k-sets to $\mathscr{F}$ without increasing the size of the set L? The following proposition shows that some of the families $\mathscr{F}$ are not maximal.

THEOREM 7.1. (a) *$\mathscr{F}(\Pi_{d-t-1}\mathscr{P}_t, \mathscr{Q}_n) \cup \mathscr{F}(\Pi_{d-1}, \mathscr{Q}_n)$ is an intersection family of the same asymptotic size as the first component.*

(b) *For $q = 2$, $\mathscr{F}(\Pi_{d-2}\mathscr{H}_1, \mathscr{Q}_n) \cup \mathscr{F}(\Pi_{d-3}\mathscr{E}_3, \mathscr{Q}_n)$ is an intersection family of the same asymptotic size as the second component.*

Proof. (a) From §§1 and 2,

$$|\Pi_{d-t-1}\mathscr{P}_t| = \theta_{d-1} = |\Pi_{d-1}|.$$

From Proposition 2.1,

$$N(\Pi_{d-1}, \mathscr{Q}_n) = N(\Pi_{d-1}\mathscr{H}_{-1}, \mathscr{Q}_n) \sim q^{nd-\frac{1}{2}d(3d-1)},$$

$$N(\Pi_{d-t-1}\mathscr{P}_t, \mathscr{Q}_n) \sim q^{n(d+1)-\frac{1}{2}\{3d^2+d(3-2t)+t(t-1)\}}.$$

The intersections of spaces Π_{d-1} have size θ_i, for $0 \leqslant i \leqslant d-2$, or zero; all these numbers are in the set L for $\mathscr{F}(\Pi_{d-t-1}\mathscr{P}_t, \mathscr{Q}_n)$.

(b) By Proposition 2.2(b),

$$|\Pi_{d-2}\mathscr{H}_1| = |\Pi_{d-3}\mathscr{E}_3|.$$

The members of $\mathscr{L}$ for the first family are all of the form Π_i, for $-1 \leqslant i \leqslant d-2$, or $\Pi_j\mathscr{H}_1$, for $-1 \leqslant j \leqslant d-3$. All are contained in the set $\mathscr{L}$ for the second family. We have

$$N(\Pi_{d-2}\mathscr{H}_1, \mathscr{Q}_n) \sim 2^{n(d+1)-\frac{1}{2}d(3d+1)},$$

$$N(\Pi_{d-3}\mathscr{E}_3, \mathscr{Q}_n) \sim 2^{n(d+2)-3(d^2+d+2)/2}.$$

For both families, the set L is in fact the same.

8. *Further properties of intersection families on quadrics*

Let $\mathscr{A}$ be any of the (v, L, k)-families defined in §5 as sections of a quadric X. Let $\mathscr{B} = \{A_i \cap A_j \mid A_i, A_j \in \mathscr{A}, i \neq j\}$. We will denote by F^i any element of $\mathscr{B}$ of size l_i, for $0 \leqslant i \leqslant s-1$; for consistency, let F^s be any element of $\mathscr{A}$ and let $F^{s+1} = X$. The following properties are satisfied.

(1) $\mathscr{B}$ contains all subsets of X of size at most 2

(2) The set $\mathscr{B} \cup \mathscr{A} \cup \{X\}$ is a *partial perfect matroid design* PPMD(v, L, k); that is, for any F^i, with $0 \leqslant i \leqslant s$, and any F^1 (a point of X) with $F^1 \notin F^i$, there exists *at most one* F^{i+1} with $F^i \cup \{F^1\} \subset F^{i+1}$.

The term partial PMD is used since, if 'at most one' is replaced by 'exactly one', then we do have a PMD. A further reason is that, when $s = 2$, a partial PMD is a partial linear space.

(3) $\mathscr{B}$ is the set of r-wise intersections of elements of $\mathscr{A}$.

Deza, Erdös, and Frankl [3] showed that any family of k-sets of a given v-set with L as the set of sizes of r-wise intersections has cardinality at most

$$(r-1)\prod_{l\in L}(v-l)/(k-l).$$

So this bound holds for the family $\mathscr{A}$.

Another type of partial matroid, familiar to geometers but perhaps less so to combinatorialists, is constructed in the following way. Let $\mathscr{H}_{2e+1}$ be a hyperbolic quadric and let $\mathscr{A}, \mathscr{A}'$ be its two systems of generators, where a generator is a subspace of largest dimension lying on $\mathscr{H}_{2e+1}$. The dimension of a generator, the projective index in the language of §1, is e. In the terminology of §5, $\mathscr{A} \cup \mathscr{A}' = \mathscr{F}(\Pi_e, \mathscr{H}_{2e+1})$. Consider one system, say $\mathscr{A}$. Then

(a) $\mathscr{A}$ is a (v, L, k)-family;

(b) any two members of $\mathscr{A}$ intersect in a subspace of dimension d, where $d \equiv e \pmod 2$; that is,

$$L = \{0, \theta_1, \theta_3, \ldots, \theta_{e-2}\} \quad \text{when } e \text{ is odd,}$$

and

$$L = \{\theta_0, \theta_2, \theta_4, \ldots, \theta_{e-2}\} \quad \text{when } e \text{ is even;}$$

(c) property (2) holds when $\mathscr{B}$ is defined as above.

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